

## POLAR ACTIONS ON CERTAIN PRINCIPAL BUNDLES OVER SYMMETRIC SPACES OF COMPACT TYPE

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ABSTRACT. We study polar actions with horizontal sections on the total space of certain principal bundles  $G/K \rightarrow G/H$  with base a symmetric space of compact type. We classify such actions up to orbit equivalence in many cases. In particular, we exhibit examples of hyperpolar actions with cohomogeneity greater than one on locally irreducible homogeneous spaces with nonnegative curvature which are not homeomorphic to symmetric spaces.

### 1. INTRODUCTION

An isometric action of a compact Lie group on a Riemannian manifold  $M$  is called *polar* if there exists an isometrically immersed connected complete submanifold  $i : \Sigma \rightarrow M$  which meets every orbit and always orthogonally; any such  $\Sigma$  is called a *section*, and if the section is flat, the action is called *hyperpolar*. Note that the immersion  $i$  might not be injective. For an introduction to polar actions, see [1, 10]. Most of the known examples of polar actions are polar representations [2], or actions on symmetric spaces [4, 7, 8, 11]. In this paper, we propose to consider polar actions on a certain class of nonsymmetric, homogeneous spaces.

Let  $G/H$  be a symmetric space of compact type endowed with a Riemannian metric induced from some  $\text{Ad}(G)$ -invariant inner product  $g$  on the Lie algebra  $\mathfrak{g}$  of  $G$ . Write  $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}_1$  for the Cartan decomposition. We assume that  $H$  is connected, and  $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_2$  (direct sum of ideals of  $\mathfrak{h}$ ). Let  $K$  denote the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{k}$ . The natural projection  $\pi : G/K \rightarrow G/H$  is an equivariant principal  $H/K$ -bundle with respect to the natural projection  $\pi_G : G \times H/K \rightarrow G$ . Define  $\mathfrak{q} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$  and consider the  $\text{Ad}(K)$ -invariant splitting  $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$ . For each  $s > 0$ , we define an  $\text{Ad}(K)$ -invariant inner product  $g_s$  on  $\mathfrak{q}$  by  $g_s = g|_{\mathfrak{p}_1 \times \mathfrak{p}_1} + s^2 g|_{\mathfrak{p}_2 \times \mathfrak{p}_2}$ . We also denote by  $g_s$  the induced  $G$ -invariant metric on  $G/K$ . It is easy to show that the action of  $G \times H/K$  on  $(G/K, g_s)$  defined by  $(g, hK).g'K = gg'h^{-1}K$  is almost effective and isometric. If  $L$  is a Lie subgroup of  $G$ , then the induced action of  $L \times H/K$  on  $(G/K, g_s)$  is called the *natural lifting* of the action of  $L$  on  $G/H$ . Our main result can be stated as follows. (Recall that two isometric actions are called *orbit equivalent* if there exists an isometry between the target spaces which maps orbits onto orbits.)

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**Theorem 1.1.** *Let  $L$  be a closed Lie subgroup of  $G$  which acts polarly on  $G/H$ , and let  $\Sigma$  be a section containing  $1H$ . We identify  $T_{1H}\Sigma = \mathfrak{m} \subset \mathfrak{p}_1$  as usual. If  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ , then  $L \times H/K$  acts polarly on  $(G/K, g_s)$  with sections horizontal with respect to  $\pi$ . Conversely, every polar action on  $(G/K, g_s)$  of a connected closed Lie subgroup of  $G \times H/K$  with horizontal sections is orbit equivalent to the natural lifting of a polar action on  $G/H$ .*

We remark that there exist many polar actions on  $G/H$  satisfying the assumption on  $\mathfrak{m}$  in Theorem 1.1 (for some  $\mathfrak{k}$ ). For example, this assumption is satisfied by hyperpolar actions (here  $[\mathfrak{m}, \mathfrak{m}] = 0$ ), and arbitrary polar actions if  $G/H$  is either an irreducible Hermitian symmetric space or a Wolf space. In fact, sections of polar actions on the last two classes of spaces are totally real with respect to the complex and quaternionic structure respectively; see [11, 14].

In [15], Ziller shows (in a more general context) that the metric  $g_s$  is  $G \times H/K$ -naturally reductive for all  $s > 0$ , and  $G \times H/K$ -normal homogeneous if  $G$  is simple and  $s < 1$ . On the other hand, we have that  $(G/K, g_s)$  is locally irreducible if  $G$  is simple; see section 4. Hence, we get examples of hyperpolar actions with cohomogeneity greater than one on locally irreducible naturally reductive spaces with nonnegative curvature which are not homeomorphic to symmetric spaces; see section 5.

In many cases, we can use results of [9, 13] to find the identity component of the isometry group of  $(G/K, g_s)$ ; see Propositions 4.4 and 4.5. In particular we have:

**Corollary 1.2.** *Let  $G$  be a simple Lie group. We assume that either  $s \neq 1$ , or  $s = 1$  and  $K$  is nontrivial. If  $(G/K, g_s)$  is neither isometric to a round sphere, nor to a real projective space, then every polar action on  $(G/K, g_s)$  with horizontal sections is orbit equivalent to the natural lifting of a polar action on  $G/H$ .*

In section 2, we recall some properties of  $(G/K, g_s)$ . The main theorem is proved in section 3. Section 4 is devoted to proving that  $(G/K, g_s)$  is locally irreducible and to computing the identity component of its isometry group. Finally, we exhibit some examples in section 5.

## 2. PRELIMINARIES

In this paper, we always refer to the notation in the introduction. Next, we recall some results of Ziller [15].

Another presentation of  $G/K$  is obtained from the transitive action of  $\bar{G} := G \times H/K$  on  $G/K$ . Let  $\bar{K}$  be the corresponding isotropy group at the basepoint. If  $\bar{\mathfrak{g}}$  and  $\bar{\mathfrak{k}}$  are the Lie algebras of  $\bar{G}$  and  $\bar{K}$  respectively, then

$$\begin{aligned}\bar{\mathfrak{g}} &= \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{h}/\mathfrak{k}, \\ \bar{\mathfrak{k}} &= \mathfrak{k} \oplus \{ (0, X, X + \mathfrak{k}) \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{h}/\mathfrak{k} \mid X \in \mathfrak{p}_2 \}.\end{aligned}$$

As a reductive complement, we can take  $\bar{\mathfrak{q}} = \mathfrak{p}_1 \oplus \{ (0, s^2X, (s^2 - 1)X + \mathfrak{k}) \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{h}/\mathfrak{k} \mid X \in \mathfrak{p}_2 \}$ . The isomorphism between  $\bar{G}/\bar{K}$  and  $G/K$  on the Lie algebra level sends  $\mathfrak{p}_1$  to  $\mathfrak{p}_1$  as *id* and  $(0, s^2X, (s^2 - 1)X + \mathfrak{k})$  to  $X$ , so the metric  $g_s$

looks as follows on  $\bar{\mathfrak{q}}$ :

$$\begin{aligned} g_s|_{\mathfrak{p}_1 \times \mathfrak{p}_1} &= \text{as before,} \\ g_s(\mathfrak{p}_1, (0, s^2 X, (s^2 - 1)X + \mathfrak{k})) &= 0, \\ g_s((0, s^2 X, (s^2 - 1)X), (0, s^2 Y, (s^2 - 1)Y + \mathfrak{k})) &= s^2 g(X, Y). \end{aligned}$$

The metric  $g_s$  is  $G \times H/K$ -naturally reductive with respect to the decomposition  $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{q}}$ ; see [15, Theorem 3]. Moreover, if  $G$  is simple,  $g_s$  is not  $G$ -normal homogeneous for  $s \neq 1$ , and  $g_s$  is  $G \times H/K$ -normal homogeneous iff  $s < 1$ . In particular  $g_s$  is nonnegatively curved if  $G$  is simple and  $s < 1$ .

### 3. PROOF OF THE MAIN THEOREM

Let  $L$  be a closed Lie subgroup of  $G$  which acts polarly on  $G/H$ , and let  $\Sigma$  be a section containing  $1H$ . Let  $\mathfrak{s} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ , where  $T_{1H}\Sigma = \mathfrak{m} \subset \mathfrak{p}_1$ . Then  $\mathfrak{s}$  is a Lie subalgebra of  $\mathfrak{g}$ . Let  $S$  be the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{s}$ . Then  $\Sigma = S(1H)$  is a symmetric space and homogeneous under  $S$ . We prove that  $S(1K)$  is a section for the action of  $\bar{L} := L \times H/K$  on  $(G/K, g_s)$ . Since  $\pi^{-1}(gH) \subset \bar{L}(gK)$  and  $L(gH)$  meets  $S(1H)$ ,  $\bar{L}(gK)$  meets  $S(1K)$  for all  $gK \in G/K$ . Recall that  $G \times H/K$  preserves the vertical distribution (and then the horizontal distribution, with respect to  $\pi$ ) on  $(G/K, g_s)$ . The hypothesis  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$  implies that  $T_{1K}S(1K) = \mathfrak{m}$ , and hence  $S(1K)$  is horizontal. It follows that the  $\bar{L}$ -orbits in  $G/K$  meet  $S(1K)$  always orthogonally since  $\pi : (G/K, g_s) \rightarrow (G/H, g)$  is a Riemannian submersion.

Conversely, suppose that  $\bar{L}$  is a connected closed Lie subgroup of  $G \times H/K$  which acts polarly on  $(G/K, g_s)$  with a horizontal section  $\bar{\Sigma}$ .

*Claim 3.1.*  $1 \times H/K(x) \subset \bar{L}(x)$ , for any  $\bar{L}$ -regular point  $x \in G/K$ .

*Proof.* In fact, let  $M_r \subset G/K$  be the set of all the  $\bar{L}$ -regular points and let  $\mathcal{V}$  be the vertical distribution on  $G/K$ . Then  $\mathcal{V}$  is integrable with leaves given by the  $1 \times H/K$ -orbits. Let  $x \in M_r$ . Then there exists  $l \in \bar{L}$  such that  $lx \in \bar{\Sigma}$ . Since  $\bar{\Sigma}$  is horizontal and  $x$  is a regular point,  $\mathcal{V}_{lx} \subset T_{lx}\bar{L}(lx)$ . It follows that  $\mathcal{V}_x \subset T_x\bar{L}(x)$  since  $\mathcal{V}$  is  $\bar{L}$ -invariant. Now it is clear that  $1 \times H/K(x) \subset \bar{L}(x)$ .  $\square$

*Claim 3.2.* Let  $L' := \pi_G(\bar{L}) \times H/K$ . Then the actions of  $L'$  and  $\bar{L}$  on  $(G/K, g_s)$  are orbit-equivalent.

*Proof.* In fact, we have that  $\bar{L}(x) \subset L'(x)$  for all  $x \in G/K$  since  $\bar{L} \subset L'$ . Conversely, the previous claim implies that  $L'(x) \subset \bar{L}(x)$  for all  $\bar{L}$ -regular points  $x \in G/K$ . This already implies that the orbits of  $\bar{L}$  and  $L'$  in  $G/K$  coincide (see e.g. [5, Lemma 3.6] for the linear case).  $\square$

Now it is clear that the action of  $\pi_G(\bar{L})$  on  $G/H$  is polar with a section  $\pi(\bar{\Sigma})$ . This finishes the proof of Theorem 1.1.

### 4. THE IRREDUCIBILITY OF $(G/K, g_s)$ AND ITS ISOMETRY GROUP

The purpose of this section is to study the irreducibility of  $(G/K, g_s)$  and to compute the identity component of its isometry group. Some related results can be found in [3].

**Proposition 4.1.** *If  $G$  is a simple Lie group, then  $(G/K, g_s)$  is locally irreducible.*

*Proof.* Let  $\mathfrak{hol}(\bar{G}/\bar{K}, g_s)$  be the Lie algebra of the holonomy group of  $(\bar{G}/\bar{K}, g_s)$ . Since  $(\bar{G}/\bar{K}, g_s)$  is a compact naturally reductive space,  $\mathfrak{hol}(\bar{G}/\bar{K}, g_s)$  is the Lie subalgebra of  $\mathfrak{so}(\bar{\mathfrak{q}})$  generated by  $\{\Lambda(X) \in \mathfrak{so}(\bar{\mathfrak{q}}) \mid X \in \bar{\mathfrak{g}}\}$ , where

$$\Lambda(X)Y = \begin{cases} \text{ad}(X)Y & \text{if } X \in \bar{\mathfrak{k}}, \\ \Lambda_{\bar{\mathfrak{q}}}(X)Y = \frac{1}{2}[X, Y]_{\bar{\mathfrak{q}}} & \text{if } X \in \bar{\mathfrak{q}}; \end{cases}$$

see [6, Theorem 4.7, p. 208]. Let  $X_0 \in \mathfrak{k}$ , and  $X_i, Y_i \in \mathfrak{p}_i$ , for  $i = 1, 2$ . Let  $X = X_0 + (0, X_2, X_2 + \mathfrak{k})$ ,  $Y = Y_1 + (0, s^2 Y_2, (s^2 - 1)Y_2 + \mathfrak{k})$ , and  $Z = (0, s^2 X_2, (s^2 - 1)X_2 + \mathfrak{k})$ . Then

$$(4.1) \quad \text{ad}(X)Y = [X_0, Y_1] + [X_2, Y_1] + (0, s^2[X_2, Y_2], (s^2 - 1)[X_2, Y_2] + \mathfrak{k}),$$

$$(4.2) \quad [X_1, Y]_{\bar{\mathfrak{q}}} = s^2[X_1, Y_2] + (0, s^2[X_1, Y_1]_{\mathfrak{p}_2}, (s^2 - 1)[X_1, Y_1]_{\mathfrak{p}_2} + \mathfrak{k}),$$

$$(4.3) \quad [Z, Y]_{\bar{\mathfrak{q}}} = s^2[X_2, Y_1] \\ + (0, s^2(2s^2 - 1)[X_2, Y_2], (s^2 - 1)(2s^2 - 1)[X_2, Y_2] + \mathfrak{k}).$$

Now let  $V$  be a  $\Lambda(\bar{\mathfrak{g}})$ -invariant subspace of  $\bar{\mathfrak{q}}$ . It is sufficient to prove that  $V = \bar{\mathfrak{q}}$  or  $0$ . Clearly  $V \cap \mathfrak{p}_1$  and the projection  $V_{\mathfrak{p}_1}$  of  $V$  on  $\mathfrak{p}_1$  are  $\text{ad}(\mathfrak{h})$ -invariant since  $V$  is  $\Lambda(\bar{\mathfrak{k}})$ -invariant. It is also clear that  $G/H$  is locally irreducible since  $G$  is simple. Then we have one of the following possibilities:  $V \cap \mathfrak{p}_1 = \mathfrak{p}_1$  or  $V \cap \mathfrak{p}_1 = 0$  and  $V_{\mathfrak{p}_1} = 0$  or  $V \cap \mathfrak{p}_1 = 0$  and  $V_{\mathfrak{p}_1} = \mathfrak{p}_1$ . The first possibility would imply via (4.2) and  $[\mathfrak{p}_1, \mathfrak{p}_1] = \mathfrak{h}$  that  $V = \bar{\mathfrak{q}}$ . The second possibility would imply that  $\{X_2 \in \mathfrak{p}_2 \mid (0, s^2 X_2, (s^2 - 1)X_2 + \mathfrak{k}) \in V\}$  is an ideal of  $\mathfrak{g}$ , and so  $V = 0$ . Finally we consider the last possibility. Here we have that  $\mathfrak{k}$  is an ideal of  $\mathfrak{g}$ , and hence  $\mathfrak{k} = 0$ . We also have that  $\dim \mathfrak{p}_1 \leq \dim V \leq \dim \mathfrak{p}_2$ . Since  $V^\perp$  is also  $\Lambda(\bar{\mathfrak{g}})$ -invariant,  $\dim \mathfrak{p}_1 \leq \dim V^\perp \leq \dim \mathfrak{p}_2$ . This implies that  $\dim V = \dim \mathfrak{p}_1 = \dim \mathfrak{p}_2$ . Therefore we can write  $V = \{X_1 + (0, s^2 \varphi(X_1), (s^2 - 1)\varphi(X_1) + \mathfrak{k}) \mid X_1 \in \mathfrak{p}_1\}$ , where  $\varphi : \mathfrak{p}_1 \rightarrow \mathfrak{p}_2$  is an  $\text{ad}(\mathfrak{p}_2)$ -invariant isomorphism. This implies that  $\mathfrak{p}_2$  is simple. Hence  $\mathfrak{p}_2 = [\mathfrak{p}_2, \mathfrak{p}_2]$ , and from (4.1) and (4.3) we have that  $\frac{2s^2-1}{s^2}\varphi(X_1) = \varphi(X_1)$ , for all  $X_1 \in \mathfrak{p}_1$ . Therefore  $s = 1$ . It follows that  $V$  cannot exist since  $g_1$  is a bi-invariant metric on  $G$ .  $\square$

Let  $M = G'/K'$  be a  $G'$ -naturally reductive space and let  $\nabla^c$  be the associated canonical connection. Let  $I_0(M)$  (respectively,  $\text{Aff}_0(\nabla^c)$ ) denote the identity component of the isometry group (respectively, group of  $\nabla^c$ -affine transformations) of  $M$ . The following result [9, p. 22] can be used to compute  $I_0(M)$ .

**Proposition 4.2.** *Let  $M$  be a compact locally irreducible naturally reductive space. If  $M$  is neither (globally) isometric to a round sphere, nor to a real projective space, then*

$$I_0(M) = \text{Aff}_0(\nabla^c) = \text{Tr}(\nabla^c)\hat{K},$$

where  $\text{Tr}(\nabla^c)$  denotes the group of transvections of the associated canonical connection, and  $\hat{K}$  denotes the connected Lie subgroup of  $\text{Aff}_0(\nabla^c)$  whose Lie algebra consists of  $\text{Tr}(\nabla^c)$ -invariant fields whose associated flows are  $\nabla^c$ -affine. Moreover,  $\text{Tr}(\nabla^c)$  commutes with  $\hat{K}$ .

Next we will use Propositions 4.1 and 4.2 to compute  $I_0(G/K, g_s)$ .

**Lemma 4.3.** *Let  $\text{Tr}(\nabla^c)$  be the group of transvections of the associated canonical connection to  $(G/K, g_s)$ . If  $G$  is simple, then  $\text{Tr}(\nabla^c)$  is equal to  $\bar{G}$  for all  $s \neq 1$ , and equal to  $G$  for  $s = 1$ .*

*Proof.* This follows from the fact that the Lie algebra of  $\text{Tr}(\nabla^c)$  is given by  $\mathfrak{tr}(\nabla^c) = \bar{\mathfrak{q}} + [\bar{\mathfrak{q}}, \bar{\mathfrak{q}}]$ .  $\square$

**Proposition 4.4.** *Let  $G$  be a simple Lie group. Assume that  $(G/K, g_s)$  is neither isometric to a round sphere, nor to a real projective space. If  $s^2 \neq 1$ , then  $G \times H/K$  (almost direct product) is the identity component of the isometry group of  $(G/K, g_s)$ .*

*Proof.* By Proposition 4.2 and Lemma 4.3 it is sufficient to prove that  $\hat{K} \subset \bar{G}$ , where  $\hat{K}$  denotes the connected Lie subgroup of the Lie group of  $\nabla^c$ -affine transformations of  $(\bar{G}/\bar{K}, g_s)$  whose Lie algebra consists of the  $\bar{G}$ -invariant fields whose associated flows are  $\nabla^c$ -affine. Let  $\mathfrak{z}(\bar{\mathfrak{g}})$  and  $\mathfrak{z}(\mathfrak{h}/\mathfrak{k})$  be the centers of  $\bar{\mathfrak{g}}$  and  $\mathfrak{h}/\mathfrak{k}$  respectively. It is easy to see that every  $X \in \mathfrak{z}(\bar{\mathfrak{g}}) = \mathfrak{z}(\mathfrak{h}/\mathfrak{k})$  induces a  $\bar{G}$ -invariant field on  $(\bar{G}/\bar{K}, g_s)$  (which coincides with the Killing vector field on  $(\bar{G}/\bar{K}, g_s)$  induced by  $X$ ). Conversely, let  $\hat{X}$  be a  $\bar{G}$ -invariant field, and let  $X \in \bar{\mathfrak{q}}$  such that  $X \cdot (1, 1K)\bar{K} = \hat{X}_{(1, 1K)\bar{K}}$ . Then  $X$  is a fixed vector of  $\text{Ad}(\bar{K})$ , and hence  $\text{ad}(\bar{\mathfrak{k}})X = 0$ . We write  $X = X_1 + (0, s^2X_2, (s^2 - 1)X_2 + \mathfrak{k})$ , where  $X_1 \in \mathfrak{p}_1$ ,  $X_2 \in \mathfrak{p}_2$ . Then, for all  $Z_1 \in \mathfrak{k}$  we have

$$\begin{aligned} 0 &= \text{ad}(Z_1)(X_1 + (0, s^2X_2, (s^2 - 1)X_2 + \mathfrak{k})) \\ &= [Z_1, X_1]. \end{aligned}$$

Hence  $\text{ad}(\mathfrak{k})X_1 = 0$ . Analogously, for all  $Z_2 \in \mathfrak{p}_2$  we have

$$\begin{aligned} 0 &= \text{ad}((0, Z_2, Z_2 + \mathfrak{k}))(X_1 + (0, s^2X_2, (s^2 - 1)X_2 + \mathfrak{k})) \\ &= [Z_2, X_1] + (0, s^2[Z_2, X_2], (s^2 - 1)[Z_2, X_2] + \mathfrak{k}). \end{aligned}$$

Hence  $\text{ad}(\mathfrak{p}_2)X_1 = 0$ , and  $X_2$  centralizes  $\mathfrak{p}_2$ . Therefore  $\text{ad}(\mathfrak{h})X_1 = 0$ . Since  $G/H$  is locally irreducible,  $X_1 = 0$ , and hence  $X = (0, s^2X_2, (s^2 - 1)X_2 + \mathfrak{k})$ , where  $X_2 + \mathfrak{k} \in \mathfrak{z}(\mathfrak{h}/\mathfrak{k})$ . Hence,  $X \cdot (1, 1K)\bar{K} = (0, s^2X_2, (s^2 - 1)X_2 + \mathfrak{k}) \cdot (1, 1K)\bar{K} = (0, 0, -X_2 + \mathfrak{k}) \cdot (1, 1K)\bar{K}$ , where  $X_2 + \mathfrak{k} \in \mathfrak{z}(\mathfrak{h}/\mathfrak{k})$ .  $\square$

**Proposition 4.5.** *Let  $G$  be a simple Lie group. Assume that  $(G/K, g_1)$  is neither isometric to a round sphere, nor to a real projective space. If  $K$  is nontrivial, then  $G \times H/K$  (almost direct product) is the identity component of the isometry group of  $(G/K, g_1)$ .*

*Proof.* We first observe that  $(G/K, g_1)$  is a normal homogeneous space, and by Lemma 4.3,  $\text{Tr}(\nabla^c) = G = G \times 1 \subset G \times H/K$ . By Proposition 4.2, it is sufficient to prove that  $\hat{K} = 1 \times H/K$ , where  $\hat{K}$  denotes the connected Lie subgroup of the group of  $\nabla^c$ -affine transformations of  $(G/K, g_1)$  whose Lie algebra consists of the  $G$ -invariant fields whose associated flows are  $\nabla^c$ -affine. Let  $F$  be the set of the fixed vectors of the action of  $\text{Ad}(K)$  on  $\mathfrak{q}$ . Let  $\hat{\mathfrak{k}}$  be the Lie algebra of  $G$ -invariant fields on  $(G/K, g_s)$ . Then  $\hat{\mathfrak{k}}$  is the Lie algebra of  $\hat{K}$  and this can be naturally identified with  $F$ ; see [13] for details. Since  $\text{Ad}(K)$  preserves the decomposition  $\mathfrak{q} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$  and  $\mathfrak{p}_2 \subset F$  (recall that  $[\mathfrak{k}, \mathfrak{p}_2] = 0$ ), we have that  $F = \mathfrak{p}_2 \oplus \{X \in \mathfrak{p}_1 \mid \text{Ad}(K)X = X\}$ . Assume that  $X \in \mathfrak{p}_1$  is a nontrivial fixed vector of  $\text{Ad}(K)$ . Since  $H$  normalizes  $K$ , it preserves  $F \cap \mathfrak{p}_1$ . By irreducibility of  $G/H$ ,  $F \cap \mathfrak{p}_1 = \mathfrak{p}_1$ . This implies that  $[\mathfrak{k}, \mathfrak{p}_1] = 0$ . It follows that  $\mathfrak{k} = 0$  since  $G$  is simple. This contradicts the hypothesis, and hence

$F = \mathfrak{p}_2$ . Moreover, if  $\hat{X}$  is the  $G$ -invariant field on  $G/K$  induced by  $X \in \mathfrak{p}_2$ , then the flow of  $\hat{X}$  is given by  $\{(1, \exp -tXK)\} \subset 1 \times H/K \subset I_0(G/K, g_1) = \text{Aff}_0(\nabla^c)$ .  $\square$

## 5. EXAMPLES

- (1) Consider the  $H$ -bundle  $G \rightarrow G/H$ , where  $G/H$  is a symmetric space of compact type. In this case we have natural liftings of hyperpolar actions on  $G/H$  to the compact Lie group  $G$  equipped with left-invariant metrics  $g_s$ ,  $s > 0$ .
- (2) Let  $G/H$  be a symmetric space of compact type, where  $H$  is connected. If  $H = K_1K_2$  (almost direct product) where  $K_1$  and  $K_2$  are connected, then we have two principal bundles  $G/K_1 \rightarrow G/H$  and  $G/K_2 \rightarrow G/H$ . In this case we have natural liftings of hyperpolar actions on  $G/H$  to  $(G/K_1, g_s)$  and  $(G/K_2, g_s)$ . It follows from Table 1 that we can exhibit examples of hyperpolar actions with cohomogeneity greater than one on locally irreducible homogeneous spaces with nonnegative curvature which are not homeomorphic to symmetric spaces.

TABLE 1

$G$	$K_1$	$K_2$
$SU(4)$	$SO(3)$	$SO(3)$
$SU(p+q)$	$SU(p)SU(q)$	$U(1)$
$SU(p+q)$	$SU(p)U(1)$	$SU(q)$
$SO(p+q)$	$SO(p)$	$SO(q)$
$SO(4+q)$	$SO(3)SO(q)$	$SO(3)$
$SO(2n)$	$SU(n)$	$U(1)$
$Sp(n)$	$SU(n)$	$U(1)$
$Sp(p+q)$	$Sp(p)$	$Sp(q)$
$E_6$	$SU(6)$	$SU(2)$
$E_6$	$Spin(10)$	$U(1)$
$E_7$	$Spin(12)$	$SU(2)$
$E_7$	$E_6$	$U(1)$
$E_8$	$E_7$	$SU(2)$
$F_4$	$Sp(3)$	$SU(2)$
$G_2$	$SU(2)$	$SU(2)$

- (3) Consider the  $U(1)$ -bundle  $SU(n+1)/SU(n) \rightarrow SU(n+1)/S(U(n)U(1))$  and the  $Sp(1)$ -bundle  $Sp(n+1)/Sp(n) \rightarrow Sp(n+1)/Sp(n)Sp(1)$ . By [12, Proposition 2.1] and [14, Proposition 4.16], sections of polar actions on  $\mathbb{C}P^n = SU(n+1)/S(U(n)U(1))$  and  $\mathbb{H}P^n = Sp(n+1)/Sp(n)Sp(1)$  are

totally real with respect to the complex and quaternionic structure respectively, and by [11, Theorems 2.A.1 and 2.A.2], these sections are isometric to real projective spaces. This implies that the condition  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$  (given in Theorem 1.1) is always satisfied for polar actions on  $\mathbb{C}P^n$  and  $\mathbb{H}P^n$ . Therefore, we get examples of polar actions with nonflat sections on  $(S^{2n+1} = SU(n+1)/SU(n), g_s)$  and  $(S^{4n+3} = Sp(n+1)/Sp(n), g_s)$ , in particular on Berger spheres.

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