

TURAEV GENUS, KNOT SIGNATURE, AND THE KNOT HOMOLOGY CONCORDANCE INVARIANTS

OLIVER T. DASBACH AND ADAM M. LOWRANCE

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ABSTRACT. We give bounds on knot signature, the Ozsváth-Szabó τ invariant, and the Rasmussen s invariant in terms of the Turaev genus of the knot.

1. INTRODUCTION

Alternating knots have particularly simple reduced Khovanov homology and knot Floer homology. Lee [Lee05] showed that the reduced Khovanov homology of an alternating knot K is fully determined by its Jones polynomial $V_K(q)$ and its signature $\sigma(K)$. Analogously, Ozsváth and Szabó [OS03a] proved that the knot Floer homology of an alternating knot K is determined by its Alexander polynomial $\Delta_K(t)$ and its signature $\sigma(K)$. Furthermore, if K is an alternating knot, then the Ozsváth-Szabó τ invariant [OS03b], the Rasmussen s invariant [Ras04], and the signature of K satisfy

$$2\tau(K) = s(K) = -\sigma(K).$$

Note that it took some effort to show that in general $2\tau(K)$ and $s(K)$ are not equal [HO08].

To compute the signature, if D is a reduced alternating diagram of a knot K , Traczyk [Tra04] proved that

$$(1.1) \quad \sigma(K) = s_A(D) - n_+(D) - 1$$

$$(1.2) \quad = 1 + n_-(D) - s_B(D),$$

where $s_A(D)$ and $s_B(D)$ are the number of components in the all- A and all- B Kauffman resolutions of D respectively, and $n_+(D)$ and $n_-(D)$ are the number of positive and negative crossings in D respectively. Throughout this paper we choose our sign convention for the signature such that the signature of the positive trefoil is -2 .

Our goal is to generalize the relationships between knot signature, the Ozsváth-Szabó τ invariant, the Rasmussen s invariant, and Traczyk's combinatorial knot diagram data from equations (1.1) and (1.2) to non-alternating knots. These relationships lead to new lower bounds for the Turaev genus of a knot.

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For a given knot diagram in the plane, Turaev [Tur87] constructed an embedded oriented surface Σ_D on which the knot projects. In [DFK⁺08] it is pointed out that the knot projection is alternating on the Turaev surface and that the Turaev surface is a Heegaard surface for S^3 . The precise construction of the Turaev surface is given in Section 4. The Turaev genus of a knot $g_T(K)$ is the minimum genus of Σ_D over all diagrams of the knot. It is worth noting that the Turaev genus of a knot is zero if and only if the knot is alternating [DFK⁺08]. We will relate the Turaev genus of a knot K with $\sigma(K)$, $\tau(K)$ and $s(K)$ in the following:

Theorem 1.1. *Let K be a knot. Then*

$$\begin{aligned} \left| \tau(K) + \frac{\sigma(K)}{2} \right| &\leq g_T(K), \\ \frac{|s(K) + \sigma(K)|}{2} &\leq g_T(K), \text{ and} \\ \left| \tau(K) - \frac{s(K)}{2} \right| &\leq g_T(K). \end{aligned}$$

For alternating knots, i.e. when $g_T(K) = 0$, those inequalities reflect the results of Ozsváth, Szabó, and Rasmussen.

Abe [Abe09], using work of Livingston [Liv04], has shown that the three quantities on the left in Theorem 1.1 are also lower bounds for the alternation number of a knot, which is the minimum Gordian distance between a given knot and any alternating knot. Examining how the Turaev genus of a knot compares to its alternation number remains an interesting open problem.

The paper is organized as follows. In Section 2, we review the constructions of the Ozsváth-Szabó τ invariant and the Rasmussen s invariant. In Section 3, we show a relationship between the spanning tree complexes for reduced Khovanov homology and knot Floer homology. Section 4 is a review of the construction of the Turaev surface and its relationship to the spanning tree complexes. Finally, we show how knot signature fits into the picture in Section 5. In Section 6, we compute the bounds of Theorem 1.1 for knots obtained as the closure of 3-braids.

2. KNOT HOMOLOGY CONCORDANCE INVARIANTS

In this section, we recall the definitions of the Ozsváth-Szabó τ invariant [OS03b] and the Rasmussen s invariant [Ras04].

2.1. Ozsváth-Szabó τ invariant. Heegaard Floer homology is an invariant for closed 3-manifolds defined by Ozsváth and Szabó in [OS04c] and [OS04b]. The Heegaard Floer package gives rise to a concordance invariant, called the Ozsváth-Szabó τ invariant, whose construction is given below.

Suppose $(\Sigma, \alpha, \beta, w, z)$ is a Heegaard diagram subordinate to the knot K in S^3 . This means Σ is a genus g surface and both $\alpha = \{\alpha_1, \dots, \alpha_g\}$ and $\beta = \{\beta_1, \dots, \beta_g\}$ are g -tuples of homologically linearly independent, pairwise disjoint, simple closed curves in Σ . Also, w and z are points in the complement of the α and β curves in Σ lying in a neighborhood of the curve β_1 and situated on opposite sides of β_1 . The two sets of curves α and β are boundaries of attaching disks and specify handlebodies U_α and U_β both with boundary Σ and $U_\alpha \cup_\Sigma U_\beta \cong S^3$. The knot K can be isotoped onto Σ such that it is disjoint from β_2, \dots, β_g , an arc of K runs from the basepoint w to the basepoint z , and this arc intersects β_1 once transversely.

Denote the g -fold symmetric product of Σ by $\text{Sym}^g(\Sigma)$ and consider the two embedded tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$. Let $\widehat{CF}(S^3)$ denote the \mathbb{Z} -module generated by the intersection points of \mathbb{T}_α and \mathbb{T}_β . The complex $\widehat{CF}(S^3)$ can be endowed with a differential that counts pseudo-holomorphic disks in $\text{Sym}^g(\Sigma)$ between intersection points of \mathbb{T}_α and \mathbb{T}_β . The homology of $\widehat{CF}(S^3)$ is denoted $\widehat{HF}(S^3)$ and is isomorphic to \mathbb{Z} (appearing in homological grading zero).

Ozsváth and Szabó [OS04a] and independently Rasmussen [Ras03] proved that a knot K induces a filtration on the chain complex $\widehat{CF}(S^3)$. Define $\mathcal{F}(K, m) \subset \widehat{CF}(S^3)$ to be the subcomplex generated by intersection points with filtration level less than or equal to m . There is an induced sequence of maps

$$i_K^m : H_*(\mathcal{F}(K, m)) \rightarrow H_*(\widehat{CF}(S^3)) = \widehat{HF}(S^3) \cong \mathbb{Z}$$

that are isomorphisms for all sufficiently large integers m . The Ozsváth-Szabó τ invariant is defined as

$$\tau(K) = \min\{m \in \mathbb{Z} \mid i_K^m \text{ is non-trivial}\}.$$

By construction $\tau(K)$ is a knot invariant, and Ozsváth and Szabó [OS03b] showed that $\tau(K)$ depends only on the concordance class of K .

Also, recall that one can use the filtration $\mathcal{F}(K, m)$ to define the knot Floer homology of K , denoted $\widehat{HFK}(K)$, as follows. Define

$$\widehat{HFK}(K) = \bigoplus_{m \in \mathbb{Z}} H_*(\mathcal{F}(K, m)/\mathcal{F}(K, m-1)).$$

Thus $\widehat{HFK}(K)$ is the homology of the complex $\widehat{CFK}(K)$, where $\widehat{CFK}(K)$ is generated by intersection points of \mathbb{T}_α and \mathbb{T}_β , but unlike in $\widehat{CF}(S^3)$, the differential in $\widehat{CFK}(K)$ must preserve filtration level.

2.2. The Rasmussen s invariant. Khovanov homology [Kho00] is a knot invariant that categorifies the Jones polynomial. Rasmussen [Ras04] used Lee’s deformation of Khovanov homology [Lee05] to define a concordance invariant, known as the Rasmussen s invariant, whose construction is described below.

Let D be a diagram of a knot K with crossings labelled 1 through k . Each crossing of D has an A -smoothing and a B -smoothing, as shown in Figure 1. Associate to each vertex I of the cube $\{A, B\}^k$ the collection of simple closed curves in the plane D_I obtained by smoothing the i -th crossing of D according to the i -th coordinate of I . To each D_I associate the \mathbb{Q} -vector space $V^{\otimes |I|}$ where V is free on two generators v_+ and v_- and $|I|$ is the number of components in D_I . Define a bigraded \mathbb{Q} -vector space, known as the cube of resolutions, by

$$CKh(D) = \bigoplus_{I \in \{A, B\}^k} V^{\otimes |I|}.$$

The homological grading of each summand $V^{\otimes |I|}$ is the number of B -smoothings in I minus the number of negative crossings in D (as in Figure 2).

We will investigate two different differentials on $CKh(D)$. The first, ∂_{Kh} , is Khovanov’s differential. The homology $H_*(CKh(D), \partial_{Kh})$ is denoted $Kh(K)$. The vector space $Kh(K)$ has a homological and Jones grading, and its filtered Euler characteristic is $(q^{1/2} + q^{-1/2})V_K(q)$ where $V_K(q)$ is the Jones polynomial of K . (Note that we normalize the Jones grading to be half the usual grading.) The

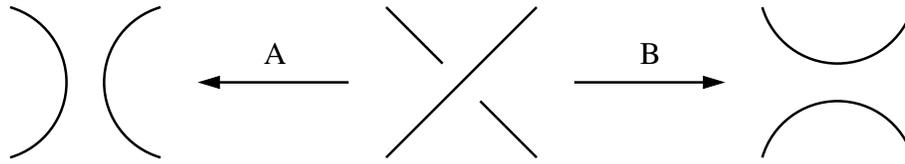


FIGURE 1. The A and B smoothings of a crossing.



FIGURE 2. The crossing on the left is positive, and the crossing on the right is negative.

second, ∂_{Lee} , is Lee’s differential. The homology $Kh'(K) := H_*(CKh(D), \partial_{\text{Lee}})$ is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$. Lee’s differential can be written as $\partial_{\text{Lee}} = \partial_{Kh} + \Phi$ where Φ increases Jones grading. The following theorem is implicit in Lee [Lee05] and explicitly stated in Rasmussen [Ras04].

Theorem 2.1 (Rasmussen [Ras04]). *Let K be a knot. There is a spectral sequence with E_2 term $Kh(K)$ that converges to $Kh'(K)$. The E_2 and higher terms of this spectral sequence are invariants of K .*

Lee identifies elements of $CKh(D)$ that represent the homology classes $\mathbb{Q} \oplus \mathbb{Q}$. These cycles are elements of $V^{\otimes |I|}$ where I is the vertex obtained by smoothing each crossing according the orientation of the knot; i.e. if a crossing is positive, then one chooses the A -smoothing, and if a crossing is negative, then one chooses the B -smoothing. Therefore, the homological gradings of both of these cycles must be zero.

Lee’s differential does not preserve the Jones grading. In order to obtain a well-defined Jones grading on Lee’s homology, one must minimize over all elements in a given homology class. More specifically, if $\alpha \in H_*(CKh(D), \partial_{\text{Lee}})$, then the Jones grading of α is the minimum Jones grading of any element a of $CKh(D)$ such that a represents the homology class α .

In [Ras04], Rasmussen showed that Lee’s homology is supported in two Jones gradings $s_{\min}(K)$ and $s_{\max}(K)$ depending only on K , and moreover $s_{\max}(K) = s_{\min}(K) + 1$. Since our Jones grading is half of Khovanov’s original Jones grading, both $s_{\min}(K)$ and $s_{\max}(K)$ are in $\mathbb{Z} + \frac{1}{2}$. The Rasmussen s invariant is defined as

$$s(K) = s_{\min}(K) + s_{\max}(K).$$

Therefore, $s(K)$ is an even integer, and Rasmussen showed that $s(K)$ depends only on the concordance class of K .

3. SPANNING TREE COMPLEXES

3.1. Construction of Tait’s checkerboard graph. Let D be a diagram of a knot K . Color regions of D white and black in a checkerboard fashion, i.e. so that if two regions are separated by an arc of D , then they are different colors. The checkerboard coloring gives rise to the two Tait checkerboard graphs G and G^* of D . The vertices of G are in one-to-one correspondence with the black regions, and the edges of G are in one-to-one correspondence with the crossings of D . Each edge in G is incident to the vertices that correspond to the black regions near the crossing. An edge in G is called an A -edge (respectively a B -edge) if the A -smoothing (respectively the B -smoothing) separates the black regions. The vertices of G^* are in one-to-one correspondence with the white regions, and the edges of G^* are in one-to-one correspondence with the crossings of D . Each edge in G^* is incident to the vertices that correspond to the white regions near the crossing. If an edge in G is an A -edge (respectively a B -edge), then the edge corresponding to the same crossing in G^* is a B -edge (respectively an A -edge). Observe that G^* is the planar dual of G . We choose the checkerboard coloring so that the number of B -edges in G is greater than or equal to the number of B -edges in G^* . Figure 3 shows an example of the Tait graphs for the 10_{124} knot. Let $\mathcal{T}(G)$ denote the set of spanning trees of G .

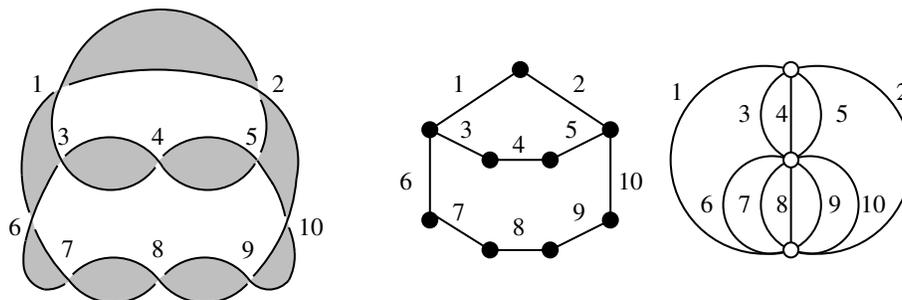


FIGURE 3. A diagram of the 10_{124} knot, along with its two Tait graphs. In the black graph (center), edges 1 and 2 are A -edges, while edges 3 through 10 are B -edges. Conversely, in the white graph (right), edges 1 and 2 are B -edges, while edges 3 through 10 are A -edges.

For any subgraph H of G , let $V(H)$ be the number of vertices in H . Each edge in G is associated to a crossing of D , and each crossing in D is either positive or negative (see Figure 2). Moreover, each edge in G is either an A -edge or a B -edge. For any subgraph H of G or G^* , let $E_A^+(H)$ denote the number of edges in H that are both A -edges and associated to a positive crossing. Similarly define $E_A^-(H)$, $E_B^+(H)$, and $E_B^-(H)$. Also, let $E^+(H)$ denote the number of edges in H associated to positive crossings in D and let $E^-(H)$ denote the number of edges in H associated to negative crossings in D . Note that $E^+(D) = n_+(D)$ and $E^-(D) = n_-(D)$. Since many of the subsequent arguments rely on graph-theoretic ideas, we favor using $E^\pm(G)$ over $n_\pm(D)$. Similarly, let $E_A(H)$ be the number of A -edges in H and let $E_B(H)$ be the number of B -edges in H . We alert the reader that in the literature A -edges are sometimes called negative edges and B -edges are called positive edges.

Since we have a different notion of positive and negative edges, we use the A and B notation instead.

If $M = \bigoplus M_{i,j}$ is a finitely generated, bigraded \mathbb{Z} -module, then define the δ -grading of M by $\delta = j - i$.

3.2. The knot Floer homology spanning tree complex. In [OS03a], Ozsváth and Szabó showed how to associate a Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ to a knot diagram D such that the intersection points of the tori \mathbb{T}_α and \mathbb{T}_β embedded into $\text{Sym}^g(\Sigma)$ are in one-to-one correspondence with the spanning trees of the Tait graph of D . Hence there exists a complex whose homology is knot Floer homology that is generated by the spanning trees of the Tait graph.

Proposition 3.1 (Ozsváth, Szabó [OS03a]). *Let D be a diagram of a knot K and let G be its Tait graph. There exists a complex $\widehat{CFK}(D)$ whose generators are in one-to-one correspondence with the spanning trees of G and whose homology is $\widehat{HFK}(K)$.*

Ozsváth and Szabó [OS03a] showed how to calculate the δ -grading of a generator by taking a certain sum over the crossings of the knot diagram. In [Low08], the second author interpreted the δ -grading in terms of information about the Tait graph of the knot diagram. The δ -grading corresponding to a spanning tree T is

$$\delta_{\widehat{HFK}}(T) = \frac{1}{2}(E_B^+(T) + E_A^+(G \setminus T) - E_A^-(T) - E_B^-(G \setminus T)).$$

3.3. The Khovanov homology spanning tree complex. In the cube of resolutions complex for Khovanov homology $CKh(D)$, one associates a two-dimensional vector space to each connected component of a Kauffman state. Wehrli [Weh08] and Champanerkar and Kofman [CK09] showed that the cube of resolutions $CKh(D)$ retracts onto a complex where one associates a two-dimensional vector space to each partial resolution of the knot diagram D that is a twisted unknot (a partial resolution of D that can be transformed into the trivial diagram of the unknot via Reidemeister one moves). The partial resolutions of D that are twisted unknots are in one-to-one correspondence with the spanning trees of the Tait graph of D . Similarly, there is a spanning tree complex for reduced Khovanov homology.

Let G be the Tait graph of a knot diagram D , and let $\mathcal{T}(G)$ the set of spanning trees of G . Define the spanning tree complex for Khovanov homology as

$$C(D) = \bigoplus_{T \in \mathcal{T}(G)} \mathbb{Z}[T_+, T_-]$$

and define the spanning tree complex for reduced Khovanov homology as

$$\tilde{C}(D) = \bigoplus_{T \in \mathcal{T}(G)} \mathbb{Z}[T].$$

Proposition 3.2 (Wehrli [Weh08], Champanerkar-Kofman [CK09]). *Let D be a diagram of a knot K .*

- (1) *There exists a spanning tree complex $C(D)$ whose homology is $Kh(K)$.*
- (2) *There exists a spanning tree complex $\tilde{C}(D)$ whose homology is $\widetilde{Kh}(K)$.*

Champanerkar and Kofman chose their gradings so that the bigraded Euler characteristic of $\widetilde{Kh}(K)$ is $q^{-1}V_K(q^2)$ where $V_K(q)$ is the Jones polynomial of K . We replace their j -grading by $\frac{j+1}{2}$ so that the bigraded Euler characteristic is $V_K(q)$.

The gradings between the Khovanov complex and the reduced Khovanov complex are related by

$$i_{Kh}(T_+) = i_{\widehat{Kh}}(T) = i_{Kh}(T_-) \text{ and}$$

$$j_{Kh}(T_+) - \frac{1}{2} = j_{\widehat{Kh}}(T) = j_{Kh}(T_-) + \frac{1}{2},$$

for any tree $T \in \mathcal{T}(G)$. The δ -grading corresponding to a spanning tree T in $\widetilde{C}(D)$ is

$$\delta_{\widehat{Kh}}(T) = E_B(T) + \frac{1}{4}(E^+(G) - E^-(G) - E_B(G) + E_A(G) - 2(V(G) - 1)).$$

For our convenience, we give two alternate formulations of $\delta_{\widehat{Kh}}(T)$. Since T is a spanning tree, $V(G) - 1 = E(T) = E_A(T) + E_B(T)$, and thus

$$2\delta_{\widehat{Kh}}(T) = 2E_B(T) + \frac{1}{2}(E^+(G) - E^-(G) - E_B(G) + E_A(G) - 2(E_A(T) + E_B(T)))$$

$$= E_B(T) - E_A(T) + \frac{1}{2}(E^+(G) - E^-(G) - E_B(G) + E_A(G)).$$

The number of crossings of D can be counted in two ways: by counting positive and negative crossings in D and by counting A -edges and B -edges in G . Therefore, $E^+(G) + E^-(G) = E_A(G) + E_B(G)$ or said another way $E^+(G) - E_B(G) = E_A(G) - E^-(G)$. This leads to our two new formulations of $\delta_{Kh}(T)$:

$$(3.1) \quad 2\delta_{\widehat{Kh}}(T) = E_B(T) - E_A(T) + E^+(G) - E_B(G), \text{ and}$$

$$(3.2) \quad 2\delta_{\widehat{Kh}}(T) = E_B(T) - E_A(T) - E^-(G) + E_A(G).$$

3.4. The δ -grading. The δ -grading of a spanning tree when considered in the reduced Khovanov complex is the same as the δ -grading of that spanning tree when considered in the knot Floer complex. We note that this is not true of either the homological or polynomial (Jones or Alexander) gradings individually.

Proposition 3.3. *Let G be the Tait graph of a knot diagram D . If T is a spanning tree of G , then $\delta_{\widehat{Kh}}(T) = \delta_{\widehat{HFK}}(T)$.*

Proof. From equation (3.1), we have

$$2\delta_{\widehat{Kh}}(T) = E_B(T) - E_A(T) + E^+(G) - E_B(G)$$

$$= E_B^+(T) + E_B^-(T) - E_A^+(T) - E_A^-(T) + E_A^+(G) + E_B^+(G) - E_B^+(G) - E_B^-(G)$$

$$= E_B^+(T) + E_B^-(T) - E_A^+(T) - E_A^-(T) + E_A^+(G) - E_B^-(G)$$

$$= E_B^+(T) - E_B^-(G \setminus T) - E_A^-(T) + E_A^+(G \setminus T)$$

$$= 2\delta_{\widehat{HFK}}(T). \quad \square$$

For the remainder of the paper, we use the notation $\delta(T)$ to equivalently mean $\delta_{\widehat{Kh}}(T)$ or $\delta_{\widehat{HFK}}(T)$. Define

$$\delta_{\min}(D) = \min\{\delta(T) \mid T \in \mathcal{T}(G)\} \text{ and } \delta_{\max}(D) = \max\{\delta(T) \mid T \in \mathcal{T}(G)\}.$$

Proposition 3.4. *Let D be a diagram of a knot K . Then $\delta_{\min}(D) \leq \tau(K) \leq \delta_{\max}(D)$.*

Proof. Proposition 3.1 implies there is a Heegaard diagram subordinate to K where the intersections points of \mathbb{T}_α and \mathbb{T}_β are in one-to-one correspondence with the spanning trees of the Tait graph G . One can use this Heegaard diagram to generate both the complexes $\widehat{CF}(S^3)$ and $\widehat{CFK}(K)$. By the definition of τ , there must be some spanning tree T in filtration level τ . Since the generator of $\widehat{HF}(S^3)$ is in homological grading 0, the tree T must also be in homological grading 0. Therefore, the tree T (viewed as a generator of $\widehat{CFK}(K)$) must satisfy $\delta(T) = \tau(K)$. \square

Proposition 3.5. *Let D be a diagram of a knot K . Then $2\delta_{\min}(D) \leq s(K) \leq 2\delta_{\max}(D)$.*

Proof. Since $C(D)$ is a deformation retract of $CKh(D)$, there exists a spectral sequence (analogous to the sequence of Theorem 2.1) whose E_1 page is $C(D)$, whose E_2 page is $Kh(K)$ and which converges to $\mathbb{Q} \oplus \mathbb{Q}$. Therefore, there exist two generators T_1 and T_2 of $C(D)$ with $i_{Kh}(T_1) = i_{Kh}(T_2) = 0$ and $j_{Kh}(T_1) = s_{\min}(K)$ and $j_{Kh}(T_2) = s_{\max}(K)$. Hence, there exists a spanning tree T such that $\delta_{\widehat{Kh}}(T) = s(K)/2$. \square

4. THE TURAEV SURFACE

The ideas discussed below involve ribbon graphs associated to a knot diagram. These ideas are developed by Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus (cf. [DFK⁺06] and [DFK⁺08]). The construction of the Turaev surface of a knot diagram is due to Turaev [Tur87].

Let D be a knot diagram and let Γ be the 4-valent plane graph obtained from D by forgetting the “over-under” information at each crossing. Regard Γ as embedded in \mathbb{R}^2 which is sitting inside \mathbb{R}^3 . Remove a neighborhood around each vertex of Γ , resulting in a collection of arcs in the plane. Replace each arc by a band which is perpendicular to the plane. In the neighborhoods removed earlier, place a saddle so that the circles obtained from choosing an A resolution at each crossing lie above the plane and so that the circles obtained from choosing a B resolution at each crossing lie below the plane. Such a saddle is shown in Figure 4. The boundary of

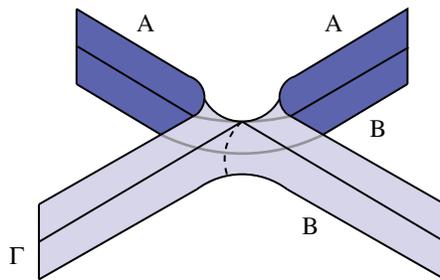


FIGURE 4. In a neighborhood of each vertex of Γ a saddle surface transitions between the A and B circles.

the resulting surface is a collection of disjoint circles, where circles corresponding to the all- A resolution lie above the plane and circles corresponding to the all- B resolution lie below the plane. Cap off each boundary circle with a disk to obtain Σ_D , the Turaev surface of D . The Turaev genus of a knot K is defined as

$$g_T(K) = \min\{g(\Sigma_D) \mid D \text{ is a diagram of } K\}.$$

A *ribbon graph* is a graph together with a cellular embedding into a surface. The genus $g(\mathbb{G})$ of a ribbon graph is the genus of the surface into which it embeds. Denote the number of vertices in a ribbon graph \mathbb{G} by $V(\mathbb{G})$. One can embed two ribbon graphs \mathbb{A} and \mathbb{B} into Σ_D as follows. The vertices of \mathbb{A} correspond to the disks used to cap off the A circles, and the edges of \mathbb{A} are the flowlines going from the vertices through the saddles. Similarly, the vertices of \mathbb{B} correspond to the disks used to cap off the B circles, and the edges of \mathbb{B} are the flowlines going from the vertices through the saddles. The ribbon graphs \mathbb{A} and \mathbb{B} are dual to one another on Σ_D , and therefore the Euler characteristic of Σ_D is determined by

$$\chi(\Sigma_D) = s_A(D) - c(D) + s_B(D),$$

where $c(D)$ is the number of crossings of D and $s_A(D)$ and $s_B(D)$ are the number of components in the all- A smoothing and all- B smoothing respectively.

Let \mathbb{G} be a ribbon graph. A *ribbon subgraph* \mathbb{H} of \mathbb{G} is a subgraph of \mathbb{G} such that the cyclic orientation of the edges in the embedding of \mathbb{H} is inherited from the embedding of \mathbb{G} . Note that the surfaces on which \mathbb{H} and \mathbb{G} are embedded are not necessarily the same. If \mathbb{G} is embedded on the surface Σ , then the connected components of $\Sigma \setminus \mathbb{G}$ are known as the faces of \mathbb{G} . A *spanning quasi-tree* \mathbb{T} of \mathbb{G} is a connected ribbon subgraph of \mathbb{G} such that $V(\mathbb{T}) = V(\mathbb{G})$ and such that \mathbb{T} has one face. Denote the set of spanning quasi-trees of \mathbb{G} by $\mathcal{Q}(\mathbb{G})$.

Recall that $\mathcal{T}(G)$ denotes the set of spanning trees of the Tait graph G . Champanerkar, Kofman, and Stoltzfus [CKS07] defined maps $q_{\mathbb{A}} : \mathcal{T}(G) \rightarrow \mathcal{Q}(\mathbb{A})$ and $q_{\mathbb{B}} : \mathcal{T}(G) \rightarrow \mathcal{Q}(\mathbb{B})$. Since the sets of edges of G , \mathbb{A} , and \mathbb{B} are each in one-to-one correspondence with the crossings of D , we identify all three sets. Because elements of $\mathcal{T}(G)$, $\mathcal{Q}(\mathbb{A})$, and $\mathcal{Q}(\mathbb{B})$ are spanning, it suffices to define $q_{\mathbb{A}}$ and $q_{\mathbb{B}}$ on the set of edges of G . Let T be a spanning tree of G . An A -edge of G is in the quasi-tree $q_{\mathbb{A}}(T)$ if and only if it is in T , and a B -edge of G is in the quasi-tree $q_{\mathbb{A}}(T)$ if and only if it is in $G \setminus T$. Similarly, an A -edge of G is in the quasi-tree $q_{\mathbb{B}}(T)$ if and only if it is in $G \setminus T$, and a B -edge of G is in $q_{\mathbb{B}}(T)$ if and only if it is in T .

Theorem 4.1 (Champanerkar, Kofman, Stoltzfus [CKS07]). *The maps*

$$q_{\mathbb{A}} : \mathcal{T}(G) \rightarrow \mathcal{Q}(\mathbb{A}) \text{ and } q_{\mathbb{B}} : \mathcal{T}(G) \rightarrow \mathcal{Q}(\mathbb{B})$$

are bijections. Moreover, the genera of $q_{\mathbb{A}}(T)$ and $q_{\mathbb{B}}(T)$ are determined by

$$\begin{aligned} g(q_{\mathbb{A}}(T)) + E_B(T) &= \frac{V(G) + E_B(G) - s_A(D)}{2} \text{ and} \\ g(q_{\mathbb{B}}(T)) + E_A(T) &= \frac{V(G) + E_A(G) - s_B(D)}{2}. \end{aligned}$$

In the spanning tree complex for reduced Khovanov homology, the δ -grading for each spanning tree T is the number of B -edges in T (up to some overall shift dependent on the diagram D). Champanerkar, Kofman, and Stoltzfus [CKS07] use Theorem 4.1 to show that for any knot diagram, the difference between the maximum and minimum δ -gradings in the spanning tree complex for reduced Khovanov homology is equal to the genus of the Turaev surface of that diagram. Using a different approach, the second author [Low08] proved that for any knot diagram, the difference between the maximum and minimum δ -gradings in the spanning tree complex for knot Floer homology is equal to the genus of the Turaev surface of that diagram. In light of Proposition 3.3, the second author's result can be shown

using the same approach as Champanerkar, Kofman, and Stoltzfus [CKS07], and both results can be encoded by the equation

$$(4.1) \quad g(\Sigma_D) = \delta_{\max}(D) - \delta_{\min}(D),$$

where D is a knot diagram and Σ_D is the Turaev surface of D .

The maximum and minimum δ -gradings are related to Traczyk’s combinatorial data coming from a diagram of the knot.

Corollary 4.2. *Let D be a knot diagram, and let G be its Tait graph. Then*

$$\begin{aligned} 2\delta_{\min}(D) &= s_B(D) - E^-(G) - 1 \text{ and} \\ 2\delta_{\max}(D) &= 1 + E^+(G) - s_A(D). \end{aligned}$$

Proof. Let T_{\min} be a spanning tree such that $\delta(T_{\min}) = \delta_{\min}(D)$. By the definition of $q_{\mathbb{B}}$, the number of edges in $q_{\mathbb{B}}(T_{\min})$ is $E_A(G \setminus T_{\min}) + E_B(T_{\min})$. Since $\delta(T_{\min}) = \delta_{\min}(D)$, the tree T_{\min} has the maximum number of A -edges possible, and thus Theorem 4.1 implies that $g(q_{\mathbb{B}}(T_{\min})) = 0$. Therefore, $q_{\mathbb{B}}(T_{\min})$ is a spanning tree of the underlying graph of \mathbb{B} and has $s_B(D) - 1$ edges.

Equation (3.1) implies

$$\begin{aligned} 2\delta(T_{\min}) &= E_B(T_{\min}) - E_A(T_{\min}) + E_A(G) - E^-(G) \\ &= E_A(G \setminus T_{\min}) + E_B(T_{\min}) - E^-(G) \\ &= s_B(D) - E^-(G) - 1. \end{aligned}$$

Similarly, let T_{\max} be a spanning tree such that $\delta(T_{\max}) = \delta_{\max}(D)$. By the definition of $q_{\mathbb{A}}$, the number of edges in $q_{\mathbb{A}}(T_{\max})$ is $E_A(T_{\max}) + E_B(G \setminus T_{\max})$. Since $\delta(T_{\max}) = \delta_{\max}(D)$, the tree T_{\max} has the maximum number of B -edges possible, and thus Theorem 4.1 implies that $g(q_{\mathbb{A}}(T_{\max})) = 0$. Therefore, $q_{\mathbb{A}}(T_{\max})$ is a spanning tree of the underlying graph of \mathbb{A} and has $s_A(D) - 1$ edges.

Equation (3.2) implies

$$\begin{aligned} 2\delta(T_{\max}) &= E_B(T_{\max}) - E_A(T_{\max}) + E^+(G) - E_B(G) \\ &= E^+(G) - E_B(G \setminus T_{\max}) - E_A(T_{\max}) \\ &= 1 + E^+(G) - s_A(D). \end{aligned} \quad \square$$

5. KNOT SIGNATURE

The signature of a knot $\sigma(K)$ was defined by Trotter in [Tro62] and was shown to be a concordance invariant by Murasugi in [Mur65] (see also [KT76]). In this section, we show that $\sigma(K)$ satisfies inequalities similar to the inequalities satisfied by $\tau(K)$ and $s(K)$. Consequently, one has new lower bounds for the Turaev genus of a knot.

Let G be the Tait graph of a diagram D of a knot K . Define G_A and G_B to be the graphs obtained from G by deleting all B -edges and A -edges, respectively. Furthermore, for any graph Γ define $\beta_0(\Gamma)$ and $\beta_1(\Gamma)$ to be the zeroth and first Betti numbers of Γ (thought of as a 1-complex).

One can use results of Murasugi [Mur89] and Thistlethwaite [Thi88] to obtain the desired inequality for knot signature. We warn the reader that the previous two references have opposite conventions for labeling edges of G positive or negative. Translated into the notation of this paper, equation (13.4) of Murasugi [Mur89] is the following proposition.

Proposition 5.1 (Murasugi [Mur89]). *Let G be the Tait graph of a diagram of the knot K . Then*

$$\beta_0(G_B) + \beta_1(G_B) - E^+(G) - 1 \leq \sigma(K) \leq E^-(G) - \beta_0(G_A) - \beta_1(G_A) + 1.$$

Also, the following proposition appears in Thistlethwaite [Thi88].

Proposition 5.2 (Thistlethwaite [Thi88]). *Let G be the Tait graph of a knot diagram D . Then*

$$\begin{aligned} s_A(D) &= \beta_0(G_B) + \beta_1(G_B) \text{ and} \\ s_B(D) &= \beta_0(G_A) + \beta_1(G_A). \end{aligned}$$

Propositions 5.1 and 5.2 imply the following result.

Proposition 5.3. *Let D be a diagram of a knot K . Then $2\delta_{\min}(D) \leq -\sigma(K) \leq 2\delta_{\max}(D)$.*

Proof. We have

$$\begin{aligned} 2\delta_{\min}(D) &= s_B(D) - E^-(G) - 1 \\ &= \beta_0(G_A) + \beta_1(G_A) - E^-(G) - 1 \\ &\leq -\sigma(K). \end{aligned}$$

Similarly,

$$\begin{aligned} 2\delta_{\max}(D) &= 1 + E^+(G) - s_A(D) \\ &= 1 + E^+(G) - \beta_0(G_B) - \beta_1(G_B) \\ &\geq -\sigma(K). \end{aligned} \quad \square$$

Proof of Theorem 1.1. Propositions 3.4, 3.5, and 5.3 give the following inequalities:

$$\begin{aligned} 2\delta_{\min}(D) &\leq 2\tau(K) \leq 2\delta_{\max}(D), \\ 2\delta_{\min}(D) &\leq s(K) \leq 2\delta_{\max}(D), \text{ and} \\ 2\delta_{\min}(D) &\leq -\sigma(K) \leq 2\delta_{\max}(D). \end{aligned}$$

The result now follows from equation (4.1). □

Lobb [Lob09] gave upper and lower bounds on the Rasmussen s invariant. Lobb's bounds also depend on the diagram of the knot. He used combinatorial data obtained from the oriented resolution of the diagram. Our results are similar in nature, but we use combinatorial data obtained from the all- A and all- B resolutions.

6. EXAMPLE: 3-BRAID KNOTS

In this section, we examine knots obtained as the closure of a 3-braid and compute the bounds of Theorem 1.1 for each such knot. In many cases, this leads to a computation of the Turaev genus of such knots.

Let B_3 denote the braid group on three strands, generated by elements σ_1 and σ_2 . Murasugi described the conjugacy classes of closed 3-braids.

Theorem 6.1 (Murasugi [Mur74]). *Any 3-braid is conjugate to exactly one braid of the form $(\sigma_1\sigma_2)^{3n} \cdot w$, where $n \in \mathbb{Z}$ and w is either*

- (1) *equal to $\sigma_1^{a_1}\sigma_2^{-b_1} \dots \sigma_1^{a_k}\sigma_2^{-b_k}$, where $a_i, b_i > 0$;*
- (2) *equal to σ_2^k for some $k \in \mathbb{Z}$; or*
- (3) *equal to $\sigma_1^m\sigma_2^{-1}$ where $m \in \{-1, -2, -3\}$.*

We say a 3-braid in one of the above forms is in Murasugi normal form. Closed 3-braids whose Murasugi normal form is of type (2) or type (3) with $m = -2$ are links. A closed 3-braid knot of type (3) is a $(3, q)$ torus knot.

6.1. Torus knots. Let $T(3, q)$ denote the $(3, q)$ torus knot. Throughout this subsection, we assume $q > 0$. The computations for $q < 0$ are similar. Ozsváth and Szabó [OS03b] and Rasmussen [Ras04] computed the value of the τ and s invariants for torus knots. In our case, we have

$$2\tau(T(3, q)) = s(T(3, q)) = 2q - 2.$$

Gordon, Litherland, and Murasugi [GLM81] showed that the signature of a $(3, q)$ torus knot is given by

$$\sigma(T(3, 3k + l)) = \begin{cases} -4k - 2l + 2 & \text{if } k \text{ is even,} \\ -4k - 2l & \text{if } k \text{ is odd,} \end{cases}$$

for $l = 1$ or 2 . If k is even, the bounds from Theorem 1.1 are

$$(6.1) \quad \begin{aligned} & \left| \tau(T(3, 3k + l)) + \frac{\sigma(T(3, 3k + l))}{2} \right| \\ &= \left| \frac{s(T(3, 3k + l)) + \sigma(T(3, 3k + l))}{2} \right| = k, \end{aligned}$$

for $l = 1$ or 2 . If k is odd, the bounds from Theorem 1.1 are

$$(6.2) \quad \begin{aligned} & \left| \tau(T(3, 3k + l)) + \frac{\sigma(T(3, 3k + l))}{2} \right| \\ &= \left| \frac{s(T(3, 3k + l)) + \sigma(T(3, 3k + l))}{2} \right| = k - 1, \end{aligned}$$

for $l = 1$ or 2 .

In [Low09], the second author found knot diagrams $D_{k,l}$ of the knots $T(3, 3k + l)$ such that the genus of the Turaev surface is given by

$$g(\Sigma_{D_{k,l}}) = k,$$

where $l = 1$ or 2 . Therefore equations (6.1) and (6.2) and Theorem 1.1 imply that

$$g_T(T(3, 3k + l)) = k,$$

for k even and $l = 1$ or 2 , and

$$g_T(T(3, 3k + l)) = k \text{ or } k - 1,$$

for k odd and $l = 1$ or 2 . However, using the lower bounds on Turaev genus given by reduced Khovanov width [CKS07], the second author [Low09] has shown that

$$g_T(T(3, 3k + l)) = k,$$

for all $k > 0$ and $l = 1$ or 2 . Thus, if k is odd, then the bounds of Theorem 1.1 for $T(3, 3k + l)$ are strictly less than the Turaev genus.

6.2. Non-torus closed 3-braids. We now turn our attention to closed 3-braid knots whose Murasugi normal form is of type (1). Throughout this subsection, we assume $n > 0$. The computations when $n < 0$ are similar.

Proposition 6.2. *Let K_n be the closure of $(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-b_1}\cdots\sigma_1^{a_k}\sigma_2^{-b_k}$. Then*

$$g_T(K) = n \text{ or } n - 1.$$

We will prove Proposition 6.2 using the bounds from Theorem 1.1, and so it is first necessary to compute $\tau(K_n)$, $s(K_n)$, and $\sigma(K_n)$. Erle calculated the signature of such a closed 3-braid knot.

Proposition 6.3 (Erle [Erl99]). *If K_n is the closure of $(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-b_1}\cdots\sigma_1^{a_k}\sigma_2^{-b_k}$, then*

$$\sigma(K_n) = -4n - \sum_{i=1}^k (a_i - b_i).$$

Using work of Van Cott [Cot08], Greene computed the Rasmussen s invariant for such closed 3-braids.

Proposition 6.4 (Greene [Gre09]). *Let K_n be a knot that is the closure of $(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-b_1}\cdots\sigma_1^{a_k}\sigma_2^{-b_k}$. Then*

$$(6.3) \quad s(K_n) = \begin{cases} 6n - 2 - \sigma(K_0), & \text{if } n > 0; \\ -\sigma(K_0), & \text{if } n = 0; \\ 6n + 2 - \sigma(K_0), & \text{if } n < 0. \end{cases}$$

Greene’s proof depends on the following facts:

- (1) For a quasi-alternating knot $s(K) = -\sigma(K)$.
- (2) s is a homomorphism from the smooth knot concordance group $\mathcal{C} \rightarrow \mathbb{Z}$.
- (3) $|s(K)| \leq 2g_4(K)$, where $g_4(K)$ is the 4-genus of K .
- (4) s of the (m, n) torus knot is $(m - 1)(n - 1)$.

Note that (2) – (4) above are the conditions appearing in Van Cott’s [Cot08] work.

Each of (1) – (4) also holds for 2τ , and so, using the notation of Proposition 6.4, we have

$$2\tau(K_n) = s(K_n).$$

Proof of Proposition 6.2. The bounds of Theorem 1.1 are given by

$$\left| \frac{s(K_n) + \sigma(K_n)}{2} \right| = \left| \tau(K_n) + \frac{\sigma(K_n)}{2} \right| = n - 1.$$

The second author [Low09] showed that $g_T(K_n) \leq n$. Hence

$$g_T(K_n) = n - 1 \text{ or } n.$$

□

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DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803-4918

E-mail address: `kasten@math.lsu.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242-1419

E-mail address: `alowrance@math.uiowa.edu`