NILPOTENCY OF NORMAL SUBGROUPS HAVING TWO G-CLASS SIZES

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Abstract. Let $G$ be a finite group. If $N$ is a normal subgroup which has exactly two $G$-conjugacy class sizes, then $N$ is nilpotent. In particular, we show that $N$ is abelian or is the product of a $p$-group $P$ by a central subgroup of $G$. Furthermore, when $P$ is not abelian, $P/(Z(G) \cap P)$ has exponent $p$.

1. Introduction

Let $G$ be a finite group and $N$ a normal subgroup of $G$. Since $N$ is a union of $G$-conjugacy classes, it is natural to wonder what information on the structure of $N$ can be obtained from its $G$-class sizes. One result of N. Ito claims that any finite group having exactly two conjugacy class sizes is nilpotent [8]. If every $G$-conjugacy class contained in $N$ has only two possible sizes, 1 or $m$, then is $N$ contained in $F(G)$, the Fitting subgroup of $G$? We remark that the fact that $N$ could have two $N$-conjugacy class sizes cannot be deduced from the property that $N$ has exactly two $G$-conjugacy class sizes. Some effort has been made in this direction, and in [3] the nilpotency of $N$ is shown under the additional hypothesis that $N$ contains some Sylow $p$-subgroup of $G$ for some prime $p$. In this paper we extend the result with complete generality.

Theorem A. If $N$ is a normal subgroup of a group $G$ and the size of any $G$-conjugacy class contained in $N$ is 1 or $m$, for some integer $m$, then $N$ is nilpotent. More precisely, $N$ is abelian or $N$ is the direct product of a nonabelian $p$-group $P$ by a central subgroup of $G$. In this case, $P/(Z(G) \cap P)$ has exponent $p$.

The proof of the nilpotency is clearly divided into two parts. The first part is simpler and deals with the case in which $N$ is solvable (in fact, it is enough to suppose that $F(N) > Z(N)$). The second part, when $N$ is not solvable, relies on the classification of the finite simple groups by means of a result on $CP$-groups, that is, on groups having all elements of prime power order.

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2. Proofs

If \( x \) is any element of a group \( G \), we denote by \( x^G \) the conjugacy class of \( x \) in \( G \) and \( |x^G| \) is called the index or the class size of \( x \) in \( G \). The rest of the notation is standard. We begin by showing the following result whose techniques are elementary.

**Theorem 1.** Let \( N \) be a normal subgroup of a group \( G \) such that the size of any \( G \)-conjugacy class contained in \( N \) is 1 or \( m \), for some integer \( m \). Then either \( N \) is abelian or \( N/(N \cap Z(G)) \) is a \( CP \)-group.

**Proof.** We will denote by \( \overline{G} = G/(Z(G) \cap N) \) and we will use bars to work in this factor group. Suppose that there exists \( g \in N \) such that the order of \( \overline{g} \) is not a prime power and we will work to get a contradiction. Then, there exist at least two distinct primes \( p \) and \( q \) such that \( g_p \) and \( g_q \), that is, the \( p \)-part and \( q \)-part of \( q \) respectively, are non-central elements. By hypothesis, \( C_G(g) = C_G(g_p) = C_G(g_q) \).

We continue by following the steps.

**Step 1.** \( C_N(g) \leq Z(C_G(g)) \).

Let \( w \) be a \( q' \)-element of \( C_N(g) = C_N(g_q) \) and suppose that it is noncentral in \( G \). Then 

\[
C_G(g_qw) = C_G(g_q) \cap C_G(w) = C_G(g_q) = C_G(w)
\]

by the hypothesis of the theorem. Hence, \( w \in Z(C_G(g_q)) = Z(C_G(g)) \). Similarly, one can obtain that if \( t \) is a \( p' \)-element of \( C_N(g) = C_N(g_p) \), then \( t \in Z(C_G(g)) \). As a consequence, we conclude that any element \( z \in C_N(g) \) belongs to \( Z(C_G(g)) \).

**Step 2.** If \( z \in N - Z(G) \) is such that \( C_G(g) \neq C_G(z) \), then \( C_N(g) \cap C_N(z) = Z(G) \cap N \). Furthermore, either \( N \) is abelian or \( Z(N) = Z(G) \cap N \).

Let \( z \in N - Z(G) \) be such that \( C_G(g) \neq C_G(z) \). Suppose that there exists \( a \in C_N(g) \cap C_N(z) - Z(G) \). By Step 1, we have \( a \in Z(C_G(g)) \). Therefore, \( C_G(g) = C_G(a) \) by our assumptions. Thus, \( z \in C_N(a) = C_N(g) \leq Z(C_G(g)) \), using Step 1 again. Consequently, \( C_G(g) \leq C_G(z) \) and so \( C_G(g) = C_G(z) \), which is a contradiction. Hence, \( C_N(g) \cap C_N(z) = Z(G) \cap N \). Suppose now that \( C_G(g) = C_G(z) \) for every \( z \in N - Z(G) \). Then \( N = C_N(z) \) for all \( z \in N \); that is, \( N \) is abelian. Otherwise, there exists some \( z \in N - Z(G) \) such that \( C_G(g) \neq C_G(z) \), and then by the first assertion \( Z(N) \leq C_N(g) \cap C_N(z) = Z(G) \cap N \), and we deduce that \( Z(N) = Z(G) \cap N \).

In the rest of the proof, we may assume that \( N \) is not abelian (otherwise the theorem is proved), and thus we assume that \( Z(N) = Z(G) \cap N \).

**Step 3.** We have \( C_{\overline{G}}(\overline{g}) = \overline{C_G(g)} \). In particular, \( C_{\overline{G}}(\overline{g}) = \overline{C_N(g)} \).

We clearly have \( \overline{C_G(g)} \leq \overline{C_G(\overline{g})} \leq \overline{C_G(\overline{g_q})} \) and hence \( |C_{\overline{G}}(\overline{g})| \) divides \( |C_{\overline{G}}(\overline{g_q})| \). On the other hand, if \( \overline{g} \) is an \( r \)-element of \( \overline{C_{\overline{G}}(\overline{g_q})} \), with \( r \neq q \), then \( [y,g_q] \subseteq Z(G) \). If \( o(y) = k \), then \( 1 = [y^k,g_q] = [y,g_q^k] \). Hence, \( y \in C_G(g_q^k) = C_G(g_q) = C_G(g) \) and \( \overline{g} \in \overline{C_G(g)} \). Consequently, 

\[
|C_{\overline{G}}(\overline{g_q})|_r \leq |\overline{C_G(g)}|_r.
\]

As \( |\overline{C_G(g)}|_r \) divides \( |C_{\overline{G}}(\overline{g})|_r \), we conclude that \( |C_{\overline{G}}(\overline{g})|_r = |\overline{C_G(g)}|_r \) for each prime \( r \neq q \). Arguing similarly for the prime \( p \), we obtain \( |C_{\overline{G}}(\overline{g})|_r = |\overline{C_G(g)}|_r \) for every
prime $r \neq p$. Hence, $|C_{\pi}(g)| = |C_G(g)|$ and $C_{\pi}(g) = C_G(g)$. The second assertion is an immediate consequence.

We remark that the above three steps hold for every conjugate of $g$ in $G$.

**Step 4.** Conclusion.

First, we claim that there exists some $x \in N$ such that $\mathfrak{g}^{\mathfrak{N}} \cap \mathfrak{C}(x) = \emptyset$. Suppose that for every $x \in N - \mathfrak{Z}(G)$, there exists some $n \in N$ such that $g^n \in \mathfrak{C}(x)$. Then

$$N \subseteq \bigcup_{n \in N} C_N(g)^n,$$

and this implies that $N = C_N(g)$ and so $g \in \mathfrak{Z}(N) = \mathfrak{Z}(G) \cap N$, a contradiction. Thus the claim is proved.

Now, the subgroup $\mathfrak{C}(x)$ operates on $\mathfrak{g}^{\mathfrak{N}}$ by conjugation. Moreover, no element in $\mathfrak{C}(x)$ distinct from 1 centralises any element in $\mathfrak{g}^{\mathfrak{N}}$. In fact, if there is some $1 \neq \mathfrak{h} \in \mathfrak{C}(x)$ which fixes some $g^t$ for some $t \in N$, it follows that $\mathfrak{h} \in \mathfrak{C}(g^t) \cap \mathfrak{C}(x)$, and, by applying Step 2 to $g^t$, we deduce that $C_G(g^t) = C_G(x)$, a contradiction. Hence, all orbits of $\mathfrak{C}(x)$ on $\mathfrak{g}^{\mathfrak{N}}$ have the same length, that is, $|\mathfrak{C}(x)|$, and this implies that $|\mathfrak{C}(x)|$ divides $|\mathfrak{g}^{\mathfrak{N}}| = |N : C_N(g)|$ by applying Step 3. Therefore, $|C_N(g)|$ divides $|N : C_N(x)| = |x^N|$, which implies that $|C_N(g)|$ divides $|x^G| = |g^G| = |\mathfrak{g}^G|/|\mathfrak{g}^{\mathfrak{N}}|$ by Step 3 again.

On the other hand, we claim that $\mathfrak{C}(g)$ operates without fixed points on $\mathfrak{g}^{\mathfrak{N}} - \mathfrak{g}^{\mathfrak{T}}$. If some $\mathfrak{w} \in \mathfrak{C}(g)$ distinct from 1 fixes some $\mathfrak{g}^{t}$ for some $t \in G$, then $\mathfrak{w} \in \mathfrak{C}(\mathfrak{g}^{t}) = C_G(g^{t})$ by applying Step 3 to $g^{t}$. Therefore, $\mathfrak{w} \in \mathfrak{C}(g) \cap C_N(g^{t})$, and $C_G(g^{t}) = C_G(g^{t})$ by Step 2. Thus, $\mathfrak{g}^{t} = \mathfrak{g}^{t} \in \mathfrak{C}(g) \cap \mathfrak{g}^{\mathfrak{T}}$. As a consequence, $\mathfrak{C}(g)$ divides $|\mathfrak{g}^{\mathfrak{N}}| - |\mathfrak{g}^{\mathfrak{T}} - \mathfrak{g}^{\mathfrak{N}}| = |\mathfrak{g}^{\mathfrak{N}}| - |\mathfrak{g}^{\mathfrak{T}} \cap \mathfrak{C}(g)|$. Finally, we conclude that $\mathfrak{C}(g)$ also divides $|\mathfrak{g}^{\mathfrak{T}} \cap \mathfrak{C}(g)|$, which is not possible because

$$0 < |\mathfrak{g}^{\mathfrak{T}} \cap \mathfrak{C}(g)| < |\mathfrak{C}(g)|.$$

This contradiction shows that any element of $\mathfrak{g}$ has prime power order. □

**Corollary 2.** Let $N$ be a normal subgroup of a group $G$ such that the size of any $G$-conjugacy class contained in $N$ is 1 or $m$, for some integer $m$. Then $N/\mathfrak{Z}(N)$ is a $CP$-group.

**Proof.** This is trivial from Theorem 1. □

In order to prove the nilpotency of $N$ in Theorem A, we will make use of the following results.

**Lemma 3.** Let $G$ be a $\pi$-separable group. The size of the conjugacy class of every $\pi$-element of $G$ is a $\pi$-number if and only if $G = H \times K$, where $H$ and $K$ are a Hall $\pi$-subgroup and a $\pi$-complement of $G$, respectively.

**Proof.** See Lemma 8 of [1] for instance. □

**Lemma 4.** Let $P \times Q$ be the direct product of a $p$-group $P$ and a $p'$-group $Q$ and suppose that $P \times Q$ acts on a $p$-group $G$. If $C_G(P) \subseteq C_G(Q)$, then $Q$ acts trivially on $G$.

**Proof.** This is Thompson’s $P \times Q$-Lemma. See for instance 8.2.8 of [9]. □
Lemma 5. Suppose that $G$ is a solvable group and that any $x \in O_q(G)$ has $q$-index in $G$ for every prime $q$. Then $G$ is nilpotent.

Proof. We argue by induction on the order of $G$. The hypotheses are certainly inherited by normal subgroups of $G$, so every proper normal subgroup of $G$ is nilpotent. If $G$ is not nilpotent, this implies that $F(G)$ is a maximal normal subgroup of $G$. This means that $G/F(G)$ is a cyclic group of order $p$ for some prime $p$, so $|G:F(G)| = p$. Now, if $q \neq p$ is any prime dividing $|F(G)|$, we have that any $q$-element of $G$ has $q$-index. By Lemma 3, we have $G = Q \times H$, where $H$ is a $q$-complement of $G$, and by induction, we conclude that $H$, and accordingly, $G$, are nilpotent, which is a contradiction. ☐

We are going to make use of the structure of finite $CP$-groups. The structure of finite solvable $CP$-groups was given by H. Higman fifty years ago ([4]), and the structure of nonsolvable $CP$-groups and the classification of the simple $CP$-groups have been recently obtained by H. Heineken in [5].

Theorem 6. If $G$ is a finite, nonsolvable $CP$-group, then there are normal subgroups $B, C$ of $G$ such that $1 \leq B \subseteq C \subseteq G$ and $B$ is a $2$-group, $C/B$ is nonabelian and simple, and $G/C$ is a $p$-group for some prime $p$ and cyclic or generalized quaternion.

Proof. This is the main part of Proposition 2 of [5]. ☐

Theorem 7. If $G$ is a finite nonabelian simple $CP$-group, then $G$ is isomorphic to one of the following groups: $L_2(q)$, for $q = 5, 7, 8, 9, 17$, $L_5(4), Sz(8)$ or $Sz(32)$.

Proof. This is Proposition 3 of [5]. ☐

Theorem 8. Suppose that $N$ is a normal subgroup of a group $G$ and that the size of any $G$-conjugacy class contained in $N$ is 1 or $m$, for some integer $m$. Then $N$ is nilpotent.

Proof. We argue by induction on the order of $N$. Let $r$ and $q$ be any two primes dividing $|N|$. Let $x$ be any $r$-element of $N$ such that $x \notin Z(G)$ and take $Q$ to be a Sylow $q$-subgroup of $C_G(x)$. Let us consider the action of $Q \times (x)$ on $Q_0 = O_q(N)$. We claim that $C_{Q_0}(Q) \subseteq C_{Q_0}(x)$. In fact, if $z \in C_{Q_0}(Q)$ is noncentral in $G$, then $Q \cdot z \leq C_G(z) < G$. However, by hypothesis, $|C_G(z)|_q = |C_G(x)|_q = |Q|$, so in particular $z \in Q \cap Q_0 \subseteq C_{Q_0}(x)$. We can apply Lemma 4 and get $x \in C_N(O_q(N))$. This shows that for any prime $q$, we have that any element lying in $O_q(N)$ has $q$-index in $N$. Now, if $Z(N)_q < O_q(N)$ for some prime $q$, we take an element $w \in O_q(N) - Z(N)$ and we have $N = C_N(w)Q_w$ for some $q$-subgroup $Q_w$ of $N$. We show that $C_N(w)$ is nilpotent. For any $q'$-element $y \in C_N(w)$, by applying the hypothesis, we have

$$C_G(yw) = C_G(w) \cap C_G(y) = C_G(w) \subseteq C_G(y).$$

In particular $y \in Z(C_N(w))$, which means that $C_N(w)$ factorizes as the product of a $q'$-group by an abelian subgroup, and hence it is nilpotent as wanted. It follows that $N$ is solvable since it is a product of two nilpotent groups, and thus we can apply Lemma 5 to conclude that $N$ is nilpotent, so the theorem is proved.

Therefore, we can assume for the rest of the proof that $F(N) = Z(N)$, so $N$ is nonsolvable, and we will show that this leads to a contradiction. We know by Corollary 2 that $\overline{N} = N/Z(N)$ is a $CP$-group. By Theorem 6, there exist $\overline{B}$ and $\overline{C}$
subgroups of $\overline{N}$ such that $\overline{B}$ is a 2-group, $\overline{C}/\overline{B}$ is a nonabelian simple group, and $\overline{N}/\overline{C}$ is a $p$-group for some prime $p$. As $\Phi(N/\overline{Z}(N)) = 1$, we certainly have $\overline{B} = 1$. On the other hand, note that $\overline{C} = \Phi_N(N)$, so in particular $\overline{C}$ is characteristic in $\overline{N}$. Furthermore, since $\overline{Z}(N)$ is characteristic in $\overline{N}$, we conclude that $C$ is normal in $G$. Then, by the inductive hypothesis, we can assume that $\overline{N}$ is a nonabelian simple $CP$-group. If $N' < N$, then $N'$ would be nilpotent by induction, so $N$ would be solvable, a contradiction. Hence $N' = N$, and thus $N$ is a quasi-simple group.

Now, we claim that $|N|$ divides $m$. As any element of $\overline{N}$ has prime power order, we can take a noncentral $p$-element $x \in N$ for any prime $p$ dividing $|\overline{N}|$ and notice that we can factorize $C_N(x) = C_N(x)_p \times \overline{Z}(N)_p$, where $C_N(x)_p$ is a Sylow $p$-subgroup of $C_N(x)$. Then

$$|x^N|_{p'} = |N : C_N(x)|_{p'} = |\overline{N}|_{p'}.$$

On the other hand, $|x^N|_{p'}$ divides $m_{p'}$, so by considering all primes we have that $|\overline{N}|$ must divide $m$, as claimed.

Now, the fact that every element of $N$ is central in $G$ or lies in a $G$-conjugacy class of size $m$ implies that

$$|N| = |\overline{Z}(G) \cap N| + mk,$$

for some integer $k$. By the above paragraph, we deduce that $|\overline{N}|$ divides $|\overline{Z}(G) \cap N|$, so in particular, it divides $|\overline{Z}(N)|$. As $N$ is a quasi-simple group, if $S$ is the associated simple group to $N$, then it can be assumed that $\overline{Z}(N) \subseteq M(S)$, where $M(S)$ is the Schur multiplier of $S$. One can easily check (for instance in [2]) that $M(S)$ has order 1, 2, 6 or 48 for the simple groups $S$ appearing in the list of Theorem 7. In all cases, the fact that $|\overline{N}| = |S|$ divides $|M(S)|$ provides the final contradiction.

**Corollary 9.** Suppose that $N$ is a normal subgroup of a group $G$ such that the size of any $G$-conjugacy class contained in $N$ is 1 or $m$, for some integer $m$. Then $N$ is nilpotent or $N = P \times A$, with $P$ a $p$-group and $A$ central in $G$.

**Proof.** We know that $N$ is nilpotent by Theorem 8. If $N$ is not abelian, then by applying Theorem 1, we have that $N/(\overline{Z}(G) \cap N)$ is a $p$-group for some prime $p$, and then the result follows. □

Examples for the two cases appearing in the above corollary can be easily constructed. Let $N$ be an abelian group of odd order and let $\alpha$ be the involutory automorphism of $N$. Then $N$ is an abelian normal subgroup of $G = N\langle \alpha \rangle$ such that the $G$-classes contained in $N$ have size 1 or 2. On the other hand, let $Q$ be the quaternion group of order 8 and $\beta \in \text{Aut}(Q)$ of order 3. If $G = Q\langle \beta \rangle$, then the $G$-classes contained in $Q$ have size 1 or 6. This is an example of a nonabelian normal $p$-subgroup of $G$ with exactly two $G$-class sizes.

Several authors, first Isaacs ([7]) and later A. Mann ([10]) or L. Verardi ([11]), have independently proved that if $G$ is a $p$-group with exactly two class sizes, then the exponent of $G/\overline{Z}(G)$ is $p$. We are going to extend this result for a normal $p$-subgroup $P$ with two $G$-class sizes, and in particular we provide an alternative proof for the case $P = G$. The approach consists in defining an appropriate normal abelian subgroup contained in $P$ which satisfies certain properties. This construction is inspired by the proof of Proposition 2.2 in [8]. We will also need the following recent result due to Isaacs.
Lemma 10. Let $K \trianglelefteq G$, where $G$ is an arbitrary finite group and $K$ is abelian. Let $x$ be a noncentral element of $G$, and let $y = [t, x]$ for some element $t \in K$. Then $|C_G(y)| > |C_G(x)|$, and so the $G$-class of $y$ is smaller than that of $x$.

Proof. This is exactly Lemma 1 of [6]. \hfill \Box

Theorem 11. Suppose that $P$ is a nonabelian normal $p$-subgroup of a group $G$ such that $P$ has only two $G$-conjugacy class sizes. Then $P/(Z(G) \cap P)$, and in particular $P/Z(P)$, has exponent $p$.

Proof. We assume that the theorem is untrue and fix some element $x \in P$ such that $x^p \notin Z(G)$. Write $Z_1 = Z(P)$. Since $P$ is nonabelian, we define $Z_2/Z_1 = Z(P/Z_1)$. Notice that $Z_1 < Z_2$ and that $Z_2 \trianglelefteq G$. Let $T_2 = C_P(Z_2) < C_P(Z_1) = P$ and observe also that $T_2 \trianglelefteq G$. The proof has been divided into several steps.

Step 1. If $z \in P - T_2$, then $z^p \in Z(G)$. Consequently, $x \in T_2$.

By hypothesis $z \notin C_P(Z_2)$, so there exists some $y \in Z_2$ such that $1 \neq [y, z] \in Z_1$. Now, if we consider $y^p$, by the hypotheses of the theorem we have two possibilities: either $y^p \in Z(G)$ or $C_G(y^p) = C_G(z)$. We show that the second case is not possible. Let $y^p$ be the order of $[y, z]$. As $[y, z]$ is central in $P$, we have

$$1 = [y, z]^{p^n} = [y^{p^{n-1}}, z],$$

so $y^{p^{n-1}} \in C_G(z^p) = C_G(z)$. Which yields to $1 = [y^{p^{n-1}}, z] = [y, z]^{p^{n-1}}$, and this is the required contradiction. Then $z^p \in Z(G)$ and in particular, $x$ must lie in $T_2$.

Step 2. There exists an abelian subgroup $T \trianglelefteq G$ with $Z_1 \trianglelefteq T \subseteq P$ such that if $z \in P - T$, then $z^p \in Z(G)$. As a consequence, $x \in T$.

If $T_2$ is abelian, we can take $T = T_2$ and the step is proved. So we assume that $T_2$ is not abelian, and we may define $Z_3$ by means of $Z_3/Z(T_2) = Z(T_2/Z(T_2)) \neq 1$ and also define $T_3 = C_{T_2}(Z_3) < T_2$. Notice that $Z_1 \subseteq T_3$. Furthermore, as $Z_3/Z(T_2)$ is characteristic in $T_2/Z(T_2)$, and $Z(T_2)$ is characteristic in $T_2$, we deduce that $Z_3$ is characteristic in $T_2$, so $Z_3$ and accordingly $T_3$ are normal subgroups in $G$.

Now, we show that if $z \in T_2 - T_3$, then $z^p \in Z(G)$. As $z \notin C_P(Z_3)$, there exists some $y \in Z_3$ such that $1 \neq [y, z] \in Z(T_2)$. Arguing as in Step 1, if $C_G(z^p) = C_G(z)$ we get a contradiction, so $z^p \in Z(G)$. Therefore, for any $z \in P - T_3$, we have $z \in P - T_2$ or $z \in T_2 - T_3$, and both cases yield to $z^p \in Z(G)$.

Thus, if $T_3$ is abelian, we take $T = T_3$ and the step is proved. Otherwise, we can argue as we have done with $T_2$ and construct from $Z_3$ the subgroups $Z_4$ and $T_4 = C_P(Z_4)$, which satisfy that $z^p \in Z(G)$ for any $z \in P - T_4$. This method provides a properly descendant series of subgroups $T_i \trianglelefteq G$, with $Z_1 \subseteq T_i \subseteq P$, and satisfying the property that $z^p \in Z(G)$ for any $z \in P - T_i$. As $Z_1$ is abelian, we can get an abelian $T_i$ for some $i$, and thus $T = T_i$ is the desired subgroup.

Step 3. If $z \in T$ and $z^p \notin Z(G)$, then $C_P(z) = T$. In particular $C_P(x) = T$.

As $T$ is abelian, we have $T \subseteq C_P(z)$. Suppose that $y \in C_P(z) - T$. By Step 2, we know that $y^p \in Z(G)$, so $(zy)^p = z^py^p \notin Z(G)$. Again by Step 2, we have $zy \in T$, so $y \in T$, a contradiction.

Step 4. If $\overline{G}$ denotes $G/(Z(G) \cap P)$, then $\overline{C_P(x)} = C_P(\overline{t})$. 

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Let \( y \in P \) such that \( \overline{y} \in \overline{C_P(x)} \). If \( y^p \notin Z(G) \), then by Step 2, \( y \in T \), and by Step 3, we have \( C_P(x) = T \), so \( y \in C_P(x) \). If \( y^p \in Z(G) \), then \( (xy)^p = x^p y^p \notin Z(G) \). By Step 2, we have \( xy \in T \), which implies again \( y \in T \). This proves that \( \overline{C_P(x)} \leq \overline{C_P(x)} \), and the other containment is obvious.

**Step 5. Conclusion.**

Let \( g \in P - T \) and consider \( y = [x,g] \). Since \( T \) is abelian, we can apply Lemma 10 and get that the \( G \)-class size of \( y \in P \) is smaller than that of \( g \). This forces \( y \) to be central in \( G \), and as a consequence \( \overline{y} \in \overline{C_P(x)} = \overline{C_P(x)} \) by Step 4. Therefore, \( g \in C_P(x) = T \), which is a contradiction. This shows that the element \( x \) cannot exist, so \( P \) has exponent \( p \), and the proof finishes.

**References**


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