A GENERALIZED POINCARÉ INEQUALITY FOR A CLASS OF CONSTANT COEFFICIENT DIFFERENTIAL OPERATORS

DEREK GUSTAFSON

(Communicated by Matthew J. Gursky)

Abstract. We study first order differential operators $P = \mathcal{P}(D)$ with constant coefficients. The main question is under what conditions the following full gradient $L^p$ estimate holds:

$$\|D(f - f_0)\|_{L^p} \leq C\|Pf\|_{L^p}, \text{ for some } f_0 \in \ker P.$$ 

We show that the constant rank condition is sufficient. The concept of the Moore-Penrose generalized inverse of a matrix comes into play.

1. Introduction

The aim of this paper is to investigate a class of generalized Poincaré inequalities. We begin by recalling the classical Poincaré Inequality.

**Theorem 1.1.** For each $f \in \mathcal{D}'(\mathbb{R}^n)$ such that $\nabla f \in L^p(\mathbb{R}^n)$ and each ball $B = B(x_0, r) \subset \mathbb{R}^n$, there exists a constant $f_B$ such that

$$\int_B |f - f_B|^p \leq C(n, p)r^p \int_B |\nabla f|^p.$$ 

We view $f_B$ as an element in $\mathcal{D}'(\mathbb{R}^n)$ with $\nabla f_B = 0$, that is, $f_B \in \ker \nabla$.

Our investigation of ways to generalize this to other differential operators ended up relying on full gradient estimates. This leads to our main question:

**Question 1.2.** For what homogeneous first order partial differential operators $\mathcal{P} = \sum_{i=1}^n A_i \frac{\partial}{\partial x_i}$ is it true that for every $f \in \mathcal{D}'(\mathbb{R}^n, \mathbb{U})$ such that $\mathcal{P}f \in L^p(\mathbb{R}^n, \mathbb{V})$, there exists $f_0 \in \mathcal{D}'(\mathbb{R}^n, \mathbb{U})$ such that $\mathcal{P}f_0 = 0$ and

$$\sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} (f - f_0) \right\|_p \leq C_p(\mathcal{P}) \|\mathcal{P}f\|_p.$$ 

Notice that with the change from $\nabla$ to $\mathcal{P}$ we also had to change some other details. First, there is no need for the ball that appears in the classical Poincaré inequality; our methods have been able to achieve global estimates. These are based on Calderón-Zygmund estimates while the classical Poincaré inequality is based on the fundamental theorem of calculus. Our estimates are on the first order partial
derivatives of \( f \), not \( f \) itself. The local \( L^p \) estimates of \( f - f_0 \) will follow from (1.1) by the usual Poincaré inequality. We will confine our investigations to the case where \( \mathcal{P} \) has constant coefficients.

In section 2 we review elliptic complexes and provide a previously known result, see [4] for example, that derives a generalized Poincaré inequality using elliptic complexes. In section 3 we review the notion of a generalized inverse of a matrix, and use this to prove a new generalized Poincaré inequality. In section 4 we prove a structure theorem for elliptic complexes that allows us to see the relationship between these two generalized Poincaré inequalities. This paper is based upon results contained in the author’s dissertation [6].

2. Elliptic complexes

Let \( U, V, \) and \( W \) be finite dimensional inner product spaces, whose inner products are denoted by \( \langle \, , \rangle_U \), \( \langle \, , \rangle_V \), and \( \langle \, , \rangle_W \), respectively, or just \( \langle \, , \rangle \) when the space is clear. Let \( \mathcal{P} \) and \( \mathcal{Q} \) be the first order differential operators with constant coefficients

\[
\mathcal{P} = \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}, \quad \mathcal{Q} = \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i},
\]

where the \( A_i \) are linear operators from \( U \) to \( V \) and the \( B_i \) are linear operators from \( V \) to \( W \). We will use \( \mathcal{P}(\xi) = \sum_{i=1}^{n} \xi_i A_i \) and \( \mathcal{Q}(\xi) = \sum_{i=1}^{n} \xi_i B_i \) to denote the symbols of \( \mathcal{P} \) and \( \mathcal{Q} \), respectively. We denote by \( \mathcal{D}'(\mathbb{R}^n, V) \) the space of distributions valued in \( V \). We define a short elliptic complex of order 1 over \( \mathbb{R}^n \) to be

\[
\mathcal{D}'(\mathbb{R}^n, U) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n, V) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n, W)
\]

such that the symbol complex

\[
\mathbb{R}^n \xrightarrow{\mathcal{P}(\xi)} V \xrightarrow{\mathcal{Q}(\xi)} W
\]

is exact for all \( \xi \neq 0 \in \mathbb{R}^n \).

From an elliptic complex, we form the adjoint complex

\[
\mathcal{D}'(\mathbb{R}^n, W) \xrightarrow{\mathcal{Q}^*} \mathcal{D}'(\mathbb{R}^n, V) \xrightarrow{\mathcal{P}^*} \mathcal{D}'(\mathbb{R}^n, U).
\]

Here \( \mathcal{P}^* \) is the formal adjoint defined by

\[
\int_{\mathbb{R}^n} \langle \mathcal{P}^* f, g \rangle_U = \int_{\mathbb{R}^n} \langle f, \mathcal{P} g \rangle_V
\]

for \( f \in C_0^\infty(\mathbb{R}^n, V) \) and \( g \in C_0^\infty(\mathbb{R}^n, U) \). So, we have

\[
\mathcal{P}^* = -\sum_{i=1}^{n} A_i^* \frac{\partial}{\partial x_i},
\]

and similarly for \( \mathcal{Q}^* \). Here, we have identified \( U^*, V^*, \) and \( W^* \) with \( U, V \) and \( W \), respectively, by use of their inner products. Note that the adjoint complex is elliptic if and only if the original complex is.

From this, we define an associated second order Laplace-Beltrami operator by

\[
\Delta = \Delta_V = -\mathcal{P} \mathcal{P}^* - \mathcal{Q} \mathcal{Q} : \mathcal{D}'(\mathbb{R}^n, V) \to \mathcal{D}'(\mathbb{R}^n, V),
\]
with symbol denoted by $\triangle(\xi) : V \to V$. Linear algebra shows that for every $v \in V$, $\langle -\triangle(\xi)v, v \rangle = |P^*(\xi)v|^2 + |Q(\xi)v|^2 \geq 0$. That equality only occurs when $\xi = 0$ follows from the definition of an elliptic complex. Thus, the linear operator $\triangle(\xi) : V \to V$ is invertible for $\xi \neq 0$. We also have that as a function in $\xi$, $\triangle(\xi)$ is homogeneous of degree 2. So, letting $c = \max |\xi| = 1 \|\triangle^{-1}(\xi) : V \to V\|$, we get the estimate

$$\|\triangle^{-1}(\xi)\| \leq c |\xi|^{-2}.$$ 

So, solving the Poisson equation $\triangle \varphi = F$ with $F \in C^\infty_0(\mathbb{R}^n, V)$, we find the second derivatives of $\varphi$ by noting that $\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(\xi) = \xi_i \xi_j \triangle^{-1}(\xi) \tilde{F}(\xi)$.

Since $\xi_i \xi_j \triangle^{-1}(\xi) : V \to V$ is bounded, this gives rise to a Calderón-Zygmund type singular integral operator, $R_{ij} F = \frac{\partial^2}{\partial x_i \partial x_j} \varphi$ which is bounded on $L^p$ for $1 < p < \infty$. We will refer to these as the second order Riesz type transforms, due to the similarities with the classical Riesz transforms. A detailed discussion of Calderón-Zygmund singular integral operators and, in particular, the classical Riesz transforms, can be found in [8].

We refer the reader to [2], [4], [9], and [10] for further reading on elliptic complexes.

We now present a previously known generalized Poincaré inequality; see [4] for example.

**Theorem 2.1.** Let $1 < p < \infty$, and let

$$D'(\mathbb{R}^n, \mathbb{X}) \xrightarrow{R} D'(\mathbb{R}^n, \mathbb{U}) \xrightarrow{P} D'(\mathbb{R}^n, V) \xrightarrow{Q} D'(\mathbb{R}^n, W)$$

be an elliptic complex of order 1, and let $f \in D'(\mathbb{R}^n, U)$ such that $Pf \in L^p(\mathbb{R}^n, V)$. Then there exists $f_0 \in D'(\mathbb{R}^n, U) \cap \ker P$ with

$$\sum_j \left\| \frac{\partial}{\partial x_j} (f - f_0) \right\|_p \leq C \|Pf\|_p.$$

**Proof.** Here we shall need not only the Laplace-Beltrami operator for functions valued in $V$, but also the Laplace-Beltrami operator for functions valued in $U$, $-\triangle_U = RR^* + P^*P$. There exists $\varphi \in D'(\mathbb{R}^n, U)$ such that $\triangle_U \varphi = f$. Note that because of the exactness of the elliptic complex, we have the identity

$$\triangle_U \varphi = -P \varphi + QP \varphi - Q^* Q \varphi = -P^* \varphi - PR^* \varphi = \varphi \triangle_U \varphi = Pf.$$

Let $f_0 = f + P^* \varphi$. Now it simply remains to verify that $f_0$ satisfies the conclusions of the theorem. First,

$$Pf_0 = Pf + P^* \varphi = Pf + P^*P \varphi + P^* \varphi + P^* \varphi = Pf - P \triangle_U \varphi = 0.$$
Also,
\[
\sum_j \left\| \frac{\partial}{\partial x_j}(f - f_0) \right\|_p = \sum_j \left\| - \frac{\partial}{\partial x_j} P^* P \varphi \right\|_p \leq \sum_{i,j} \left\| A^* \frac{\partial^2}{\partial x_i \partial x_j} P \varphi \right\|_p \\
\leq \sum_{i,j} \left\| A^* R_{ij} P f \right\|_p \leq \sum_{i,j} \left\| A^* \right\| \left\| C_{i,j} \right\| \left\| P f \right\|_p \\
\leq C \left\| P f \right\|_p. \quad \Box
\]

3. Generalized inverses

Before we are able to present the second theorem, we need to look at the theory of generalized inverses.

**Proposition 3.1.** For \( A \in \operatorname{Hom}(U, V) \), there exists a unique \( A^\dagger \in \operatorname{Hom}(V, U) \), called the Moore-Penrose generalized inverse, with the following properties:

1. \( A A^\dagger A = A : U \to V \),
2. \( A^\dagger A A^\dagger = A^\dagger : V \to U \),
3. \( (AA^\dagger)^* = AA^\dagger : V \to V \),
4. \( (A^\dagger A)^* = A^\dagger A : U \to U \).

The linear map \( A^\dagger \) has properties similar to inverse matrices that make it valuable as a tool.

**Proposition 3.2.** For \( \lambda \neq 0 \), \( (\lambda A)^\dagger = \lambda^{-1} A^\dagger \).

**Proposition 3.3.** For a continuous matrix-valued function \( P = P(\xi) \), the function \( P^\dagger = P^\dagger(\xi) \) is continuous at \( \xi \) if and only if there is a neighborhood of \( \xi \) on which \( P \) has constant rank.

**Proposition 3.4.** \( AA^\dagger \) is the orthogonal projection onto the image of \( A \). \( A^\dagger A \) is the orthogonal projection onto the orthogonal complement of the kernel of \( A \).

For a more detailed discussion of generalized inverses and the proofs of these results, consult [1] and the references cited there. Generalized inverses are the additional tools we need for the following theorem.

**Theorem 3.5.** Let \( P : \mathcal{D}'(\mathbb{R}^n, U) \to \mathcal{D}'(\mathbb{R}^n, V) \) be a differential operator of order 1 with constant coefficients and symbol \( P(\xi) \) which is of constant rank for \( \xi \neq 0 \), and let \( f \in \mathcal{D}'(\mathbb{R}^n, U) \) such that \( Pf \in L^p(\mathbb{R}^n, V) \), \( 1 < p < \infty \). Then there exists \( f_0 \in \mathcal{D}'(\mathbb{R}^n, U) \) such that \( Pf_0 = 0 \) and
\[
\sum_j \left\| \frac{\partial}{\partial x_j} (f - f_0) \right\|_p \leq C \left\| Pf \right\|_p.
\]

Note that this is the same constant rank condition investigated in [3] in relation to quasiconvexity of variational integrals.

**Proof.** From the symbol \( P(\xi) : U \to V \), we have its generalized inverse \( P^\dagger(\xi) : V \to U \). We use this to define pseudodifferential operators \( R_j \), which we will refer to as the first order Riesz type transforms. For \( h \in C_c^\infty(\mathbb{R}^n, V) \), we define \( R_j h(x) = (2\pi)^{-n/2} \int e^{i\xi j} \xi_j P^\dagger(i\xi) h(\xi) d\xi \). Note that since \( P(\xi) \) is homogeneous of degree 1, we get that for \( \lambda \neq 0 \),
\[
(\lambda \xi_j) P^\dagger(i\lambda \xi) = \lambda \xi_j (\lambda P(i\xi))^\dagger = \xi_j P(i\xi).
\]
So, $\xi_j \mathcal{P}^\dagger(i\xi)$ is homogeneous of degree 0. Since $\mathcal{P}(\xi)$ is a polynomial, $\mathcal{P}^\dagger(i\xi)$ is infinitely differentiable on $|\xi| = 1$ by Theorem 4.3 of [5], which gives a formula for the derivative of $\mathcal{P}^\dagger(\xi)$ in terms of $\mathcal{P}(\xi)$, $\mathcal{P}(\xi)$, and the derivative of $\mathcal{P}(\xi)$. Thus, $R_j$ extends continuously to a Calderón-Zygmund singular integral operator from $L^p(\mathbb{R}^n, \mathcal{V})$ to $L^p(\mathbb{R}^n, \mathcal{U})$. Recalling the definition of the operator $\mathcal{P}^\dagger = \sum_j A_j \partial \partial x_j$, we note that

$$\sum_j A_j R_j h = (2\pi)^{-n/2} \int e^{ix \cdot \xi} \mathcal{P}(i\xi) \mathcal{P}^\dagger(i\xi) \hat{h}(\xi) d\xi.$$ 

So, if $h = \mathcal{P}g$, then

$$\sum_j A_j R_j h = (2\pi)^{-n/2} \int e^{ix \cdot \xi} \mathcal{P}(i\xi) \mathcal{P}^\dagger(i\xi) \mathcal{P}(i\xi) \hat{g}(\xi) d\xi = (2\pi)^{-n/2} \int e^{ix \cdot \xi} \mathcal{P}(i\xi) \hat{g}(\xi) d\xi = \mathcal{P}g = h.$$ 

Also, since this is defined by a Calderón-Zygmund singular integral operator, the identity

$$\sum_j A_j R_j = Id$$

extends to the $L^p$ closure of the set $\{\mathcal{P}g : g \in C_0^\infty\}$, which is sufficient for the subsequent computations.

The reader may wish to notice that

$$\left( \frac{\partial}{\partial x_j} R_k h \right)^\wedge = (i\xi_j) \left( i\xi_k \mathcal{P}^\dagger(i\xi) \hat{h}(\xi) \right) = (i\xi_k) \left( i\xi_j \mathcal{P}^\dagger(i\xi) \hat{h}(\xi) \right) = \left( \frac{\partial}{\partial x_k} R_j h \right)^\wedge,$$

which means that

$$\frac{\partial}{\partial x_j} R_k h = \frac{\partial}{\partial x_k} R_j h.$$ 

Thus,

$$\frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_k} f - R_k \mathcal{P} f \right) = \frac{\partial}{\partial x_k} \left( \frac{\partial}{\partial x_j} f - R_j \mathcal{P} f \right).$$

This can be viewed as saying that for any basis vector $v_l$ of $\mathcal{V}$,

$$\text{curl} \left[ \begin{array}{c} \left\langle \frac{\partial}{\partial x_1} f - R_1 \mathcal{P} f, v_l \right\rangle_{\mathcal{V}} \\ \vdots \\ \left\langle \frac{\partial}{\partial x_n} f - R_n \mathcal{P} f, v_l \right\rangle_{\mathcal{V}} \end{array} \right] = 0.$$ 

Thus, assuming that we can find a distribution $f_l$ such that

$$(3.1) \quad \frac{\partial}{\partial x_j} f_l = \left\langle \frac{\partial}{\partial x_1} f - R_1 \mathcal{P} f, v_l \right\rangle_{\mathcal{V}},$$

we can define $f_0$ by

$$f_0 = \sum_l f_l v_l,$$

which means that $\frac{\partial}{\partial x_j} f_0 = \frac{\partial}{\partial x_j} f - R_j \mathcal{P} f$, for $j = 1, \ldots, n$. Then, we have

$$\mathcal{P} f_0 = \mathcal{P} f - \sum_j A_j R_j \mathcal{P} f = \mathcal{P} f - \mathcal{P} f = 0.$$
Also,
\[
\sum_j \left\| \frac{\partial}{\partial x_j} (f - f_0) \right\|_p = \sum_j \| R_j Pf \|_p \leq C \| Pf \|_p,
\]
where the constant depends on the norms of the Riesz type transforms. Thus, it only remains to prove Lemma 3.6, a distributional Poincaré lemma, which is used in (3.1). \(\square\)

**Lemma 3.6 (Distributional Poincaré Lemma).** Let \(\omega \in \mathcal{D}' \left( \mathbb{R}^n, \Lambda^l(\mathbb{R}^n) \right)\) with \(1 \leq l \leq n\), where \(\Lambda^l(\mathbb{R}^n)\) denotes the space of \(l\)-covectors on \(\mathbb{R}^n\), such that \(d\omega = 0\). Then there exists \(\nu \in \mathcal{D}' \left( \mathbb{R}^n, \Lambda^{l-1}(\mathbb{R}^n) \right)\) such that \(\omega = d\nu\).

Note that this was used in the sense that we can view a vector field \(F \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)\) as a differential form in \(\mathcal{D}'(\mathbb{R}^n, \Lambda^1(\mathbb{R}^n))\) and that the conditions \(\nabla f = F\) become \(dF = 0\) and \(df = F\) from this point of view.

**Proof.** We will use the notation \(dx_I\) to denote \(dx_{i_1} \wedge \cdots \wedge dx_{i_l}\), where \(\{x_{i_1}, \ldots, x_{i_l}\}\) is any collection of \(l\) basis elements and \(i_1 < i_2 < \cdots < i_l\). Let \(\omega = \sum I \omega_I dx_I\), where \(\omega_I \in \mathcal{D}'(\mathbb{R}^n)\). It is well known that \(\omega_I\) can be written as \(\omega_I = \Delta \eta_I = (dd^* + d^* d) \eta_I\) for \(\eta_I \in \mathcal{D}'(\mathbb{R}^n)\). See, for example, Corollary 3.6.2 in [7]. Thus, \(\omega = \Delta \eta = (dd^* + d^* d) \eta\), where \(\eta = \sum \eta_I dx_I\). Notice that
\[
\Delta d\eta = d\Delta \eta = d\omega = 0.
\]

Thus, by Weyl’s Lemma, we get that \(d\eta\) is infinitely differentiable, which gives us that \(d^* d\eta \in C^\infty(\mathbb{R}^n, \Lambda^l(\mathbb{R}^n))\). Also, we have that
\[
d \left( dd^* \eta \right) = d\omega - d d^* d\eta = 0.
\]

So, since the De Rham cohomology of \(\mathbb{R}^n\) is trivial except in dimension 0, we get that \(d^* d\eta = d\kappa\) for some \(\kappa \in C^\infty(\mathbb{R}^n, \Lambda^{l-1}(\mathbb{R}^n))\). Letting \(\nu = \kappa + d^* \eta\) we get that
\[
d\nu = d\kappa + d d^* \eta = d^* d\eta + d d^* \eta = \omega,
\]
as desired. \(\square\)

### 4. Sufficiency of Generalized Inverses

At this point we have proved Theorem 2.1 and Theorem 3.3 in an attempt to answer our question about when a generalized Poincaré inequality is true. What is unclear is if these two results are related in any way.

Since this next result is true for a broader class of elliptic complexes than what we have previously defined, we will take a moment for definitions so that we may state our result in this broader sense. For the differential operators
\[
\mathcal{P} = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha
\]
and
\[
\mathcal{Q} = \sum_{|\alpha| \leq m} B_\alpha(x) D^\alpha
\]
of order \(m\) with variable coefficients, then
\[
\mathcal{D}'(\mathbb{R}^n, \mathcal{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n, \mathcal{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n, \mathcal{W})
\]
is an elliptic complex of order \(m\) if \(\mathcal{Q}\mathcal{P} = 0\) and the symbol complex
\[
\mathcal{U} \xrightarrow{\mathcal{P}_m(x, \xi)} \mathcal{V} \xrightarrow{\mathcal{Q}_m(x, \xi)} \mathcal{W}
\]
A GENERALIZED POINCARÉ INEQUALITY 2727

is exact for every \( x \) and every \( \xi \neq 0 \). Here, \( P_m \) denotes the principal symbol of \( P \), that is, \( \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha \), and similarly for \( Q_m \).

**Theorem 4.1.** A sequence

\[
\mathcal{D}'(\mathbb{R}^n, U) \xrightarrow{P} \mathcal{D}'(\mathbb{R}^n, V) \xrightarrow{Q} \mathcal{D}'(\mathbb{R}^n, W)
\]

with continuous coefficients is an elliptic complex if and only if all of the following hold:

(i) \( QP = 0 \).

(ii) The sequence

\[
\mathcal{U} \xrightarrow{P_m(y, \xi)} \mathcal{V} \xrightarrow{Q_m(y, \xi)} \mathcal{W}
\]

is exact for some \( y \) and some \( \xi \neq 0 \).

(iii) For each multi-index \( \gamma \) of length \( 2m \),

\[
\sum_{\alpha+\beta=\gamma, |\alpha|=|\beta|=m} B_\beta(x)A_\alpha(x) = 0
\]

as operators from \( \mathcal{U} \) to \( \mathcal{V} \).

(iv) The matrix \( P_m(x, \xi) \) has constant rank for all \( x \) and all \( \xi \neq 0 \).

(v) The matrix \( Q_m(x, \xi) \) has constant rank for all \( x \) and all \( \xi \neq 0 \).

**Proof.** We will begin by showing that an elliptic complex has the stated properties. Note that (i) and (ii) follow trivially from the definition. Since \( Q_m(x, \xi) P_m(x, \xi) = 0 \) as functions of \( \xi \), we get (iii) by equating coefficients of \( \xi^\gamma \). Since the \( A_\alpha \) and \( B_\beta \) are continuous, we get that rank \( P_m \) and rank \( Q_m \) are lower semicontinuous. By the Rank-Nullity Theorem, the fact that the symbol complex is exact, and the lower semicontinuity of rank \( P_m \), we get that rank \( Q_m \) is upper semicontinuous. Therefore rank \( Q_m \) is continuous, and, since it is valued in a discrete set, we get (v). Then, (iv) follows by the Rank-Nullity Theorem.

Now, we will assume that properties (i) through (iv) hold and show that the complex is elliptic. Property (iii) gives us that the composition \( Q_m P_m \) is identically 0, which means that image \( P_m \subseteq \ker Q_m \). Now, the Rank-Nullity Theorem, and properties (ii), (iv), and (v) give us that rank \( P_m = \text{Null } Q_m(x, \xi) \), proving ellipticity.

This result shows that Theorem 2.1 follows from Theorem 3.5. The following example shows that Theorem 3.5 is a strictly stronger result.

**Example 4.2.** Consider the differential operator \( P : \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^3) \) given by

\[
P = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y}
\end{bmatrix}.
\]

Clearly, the symbol of \( P \) has constant rank away from \( \xi = 0 \).

We will show that there is no first order constant coefficient differential operator \( Q = B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial}{\partial y} \) such that

\[
\mathcal{D}'(\mathbb{R}^2, \mathbb{R}^2) \xrightarrow{P} \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^3) \xrightarrow{Q} \mathcal{D}'(\mathbb{R}^2, \mathbb{W})
\]

is an elliptic complex. Since the cokernel of \( A_i \) has dimension 1 for each \( i \), the largest that \( \mathcal{W} \) need be is \( \mathbb{R}^2 \), corresponding to the possibility that the images of
the $B_i$ only share 0. Now, solving the equation $Q(\xi)P(\xi) = 0$ with $W = \mathbb{R}^2$, we see that $Q$ must be the zero operator, but the image of $P$ is not all of $\mathbb{R}^3$. Thus, Theorem 3.5 applies to $P$, but Theorem 2.1 does not.

**Acknowledgement**

The author would like to thank the referee for several valuable comments which helped create a clearer and more self-contained document.

**References**


Department of Mathematics, Syracuse University, Syracuse, New York 13210

E-mail address: degustaf@syr.edu