ON ZERO-DIMENSIONALITY
AND THE CONNECTED COMPONENT
OF LOCALLY PSEUDOCOMPACT GROUPS

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Dedicated to Wis Comfort on the occasion of his 78th birthday

ABSTRACT. A topological group is *locally pseudocompact* if it contains a non-empty open set with pseudocompact closure. In this paper, we prove that if G is a group with the property that every closed subgroup of G is locally pseudocompact, then G_0 is dense in the component of the completion of G, and G/G_0 is zero-dimensional. We also provide examples of hereditarily disconnected pseudocompact groups with strong minimality properties of arbitrarily large dimension, and thus show that G/G_0 may fail to be zero-dimensional even for totally minimal pseudocompact groups.

1. INTRODUCTION

A Tychonoff space is *zero-dimensional* if it has a base consisting of *clopen* (open-and-closed) sets. With each topological group G there are associated functorial subgroups related to the connectedness properties of G, defined as follows (cf. [11, 1.1.1]):

(a) G_0 denotes the connected component of the identity;
(b) $q(G)$ denotes the *quasi-component* of the identity, that is, the intersection of all clopen sets containing the identity;
(c) $z(G)$ denotes the intersection of all kernels of continuous homomorphisms from G into zero-dimensional groups;
(d) $o(G)$ denotes the intersection of all open subgroups of G.

It is well known that these subgroups are closed and normal (cf. [19, 7.1], [12, 2.2], and [22, 1.32(b)]). Clearly, $G_0 \subseteq q(G) \subseteq z(G) \subseteq o(G)$, and all four are equal for locally compact groups.

Theorem 1.1 ([19, 7.7, 7.8]). *Let L be a locally compact group. Then L/L_0 is zero-dimensional and $L_0 = q(L) = z(L) = o(L)$.*

The aim of the present paper is to investigate to what extent the condition of local compactness can be relaxed in Theorem 1.1. Although Theorem 1.1 might appear as a result about connectedness, it has far more to do with different degrees of *disconnectedness.* Following [18], we say that a space X is *hereditarily disconnected* if its connected components are singletons, and X is *totally disconnected* if its quasi-components are singletons. Clearly,

$$\text{zero-dimensional} \stackrel{(*)}{\Longrightarrow} \text{totally disconnected} \stackrel{(**)}{\Longrightarrow} \text{hereditarily disconnected},$$

and by Vedenissoff's classic theorem, both implications are reversible for locally compact (Hausdorff) spaces; that is, the three properties are equivalent for such spaces (cf. [36]).

It is well known that the quotient G/G_0 is hereditarily disconnected for every topological group G (cf. [19, 7.3] and [22, 1.32(c)]). Thus, if the implications $(*)$ and $(**)$ are reversible for G/G_0, then G/G_0 is zero-dimensional, and so $G_0 = q(G) = z(G)$. If in addition $z(G) = o(G)$, then Theorem 1.1 holds for G. This phenomenon warrants introducing some terminology.

Definition 1.2. A topological group G is *Vedenissoff* if the quotient G/G_0 is zero-dimensional; if in addition $z(G) = o(G)$, then we say that G is *strongly Vedenissoff.*

Our goal is to identify classes of (strongly) Vedenissoff groups and to find examples of non-Vedenissoff groups that have many compactness-like properties. The latter will demonstrate how close a group must be to being locally compact (or compact) in order to be Vedenissoff. (Not every Vedenissoff group is strongly Vedenissoff. Indeed, $\mathbb{Q}/\mathbb{Z}$ is zero-dimensional, but has no proper open subgroups, and so $z(\mathbb{Q}/\mathbb{Z}) \neq o(\mathbb{Q}/\mathbb{Z})$. However, thanks to Theorem 2.7(a) below, these two notions coincide in the class of groups that are considered in this paper.)

A Tychonoff space X is *pseudocompact* if every continuous real-valued map on X is bounded. A topological group G is *locally pseudocompact* if there is a neighborhood U of the identity such that $\mathrm{cl}_G U$ is pseudocompact. (Clearly, every metrizable locally pseudocompact group is locally compact.) We say that G is *hereditarily* [*locally*] *pseudocompact* if every closed subgroup of G is [locally] pseudocompact. (Note that the adjective *hereditarily* applies only to closed subgroups here, and not to all subgroups. Indeed, by Corollary 2.6 below, if every subgroup of a topological group is locally pseudocompact, then the group is discrete, which is of no interest for the present paper.) More than fifteen years ago, Dikranjan proved the following theorem.

Theorem 1.3 ([10, 1.2, 2.6])**.** *Let G be a hereditarily pseudocompact group. Then G/G_0 is zero-dimensional, $G_0 = q(G)$, and G_0 is dense in $(\widetilde{G})_0$; that is, G is strongly Vedenissoff.*

We obtain in this paper a theorem that simultaneously generalizes Theorems 1.1 and 1.3 and provides a positive solution to a problem posed by Comfort and Lukács (cf. [3, 4.13]).

Theorem A. *Let G be a hereditarily locally pseudocompact group. Then G/G_0 is zero-dimensional and $G_0 = q(G) = z(G) = o(G)$; that is, G is strongly Vedenissoff.*

The next example shows that the condition of hereditary local pseudocompactness in Theorem A cannot be replaced with (local) pseudocompactness.

Example 1.4. Comfort and van Mill showed that for every natural number n there exists an abelian pseudocompact group G_n such that G_n is totally disconnected, but $\dim G_n = n$ (cf. [5, 7.7]). In particular, the converse of the implication $(*)$ may fail for these groups G_n, and they are not Vedenissoff. This shows that pseudocompact groups need not be Vedenissoff.

Although pseudocompactness alone is too weak a property to imply that the group is Vedenissoff, it turns out that it is sufficient in the presence of some additional compactness-like properties. Recall that a (Hausdorff) topological group G is *minimal* if there is no coarser (Hausdorff) group topology (cf. [31] and [16]), and G is *totally minimal* if every (Hausdorff) quotient of G is minimal (cf. [13]). Equivalently, G is totally minimal if every continuous surjective homomorphism $G \to H$ is open.

An unpublished result of Shakhmatov states that the converse of $(*)$ holds for minimal pseudocompact groups. Specifically, Shakhmatov proved that every pseudocompact totally disconnected group admits a coarser zero-dimensional group topology, and thus minimal pseudocompact totally disconnected groups are zero-dimensional (cf. [10, 1.6]). We prove a generalization of Shakhmatov's result:

Theorem B.

   (a) *Every locally pseudocompact totally disconnected group admits a coarser zero-dimensional group topology.*

   (b) *Every minimal, locally pseudocompact, totally disconnected group is zero-dimensional and thus strongly Vedenissoff.*

Theorem C. *Let G be a totally minimal locally pseudocompact group. Then $G_0 = q(G)$ if and only if G/G_0 is zero-dimensional, in which case G is strongly Vedenissoff.*

Local pseudocompactness cannot be omitted from Theorem B. Indeed, more than twenty years ago, Arhangel′skiĭ asked whether every totally disconnected topological group admits a coarser zero-dimensional group topology. Megrelishvili answered this question in the negative by constructing a minimal totally disconnected group that is not zero-dimensional (cf. [23]). In particular, the converse of the implication $(*)$ fails for minimal groups.

Our last result is a negative one, and it is a far reaching extension of the result of Comfort and van Mill cited in Example 1.4. Recall that a group G is *perfectly (totally) minimal* if the product $G \times H$ is *(totally) minimal* for every *(totally) minimal* group H (cf. [32]).

Theorem D. *For every natural number n or $n = \omega$, there exists an abelian pseudocompact group G_n such that G_n is perfectly totally minimal, hereditarily disconnected, but $\dim G_n = n$.*

There are many known examples of pseudocompact groups for which the equality $G_0 = q(G)$ fails (cf. [9, Theorem 11], [11, 1.4.10], and [3, 4.4(d), 5.6]). By Theorem C, one has $(G_n)_0 \neq q(G_n)$ for each of the groups G_n provided by Theorem D, and thus the G_n are not totally disconnected. This shows that the converse of the implication $(**)$ may fail for totally minimal pseudocompact groups.

The paper is structured as follows. In §2, we recall some well-known facts on locally pseudocompact and locally compact groups, their G_δ-topologies, and their

connectedness properties. The proofs of Theorems B and C are also presented in §2. We devote §3 to the proof of Theorem A. Finally, in §4, we prove a general theorem concerning the embedding of groups with minimality properties as quasi-components of pseudocompact groups with the same minimality properties, which yields Theorem D.

2. Preliminaries on locally pseudocompact groups

All topological groups here are assumed to be Hausdorff, and thus Tychonoff (cf. [19, 8.4] and [22, 1.21]). Except when specifically noted, no algebraic assumptions are imposed on the groups; in particular, our groups are not necessarily abelian. A "neighborhood" of a point means an *open* set containing the point.

Although, in general, there are a number of useful uniform structures on a topological group that induce its topology, in this note, we adhere to the two-sided uniformity and the notions of precompactness and completeness that derive from it (cf. [30], [37], [28], [19, (4.11)-(4.15)], and [22, Section 1.3]). A fundamental property of this notion of completeness is that for every topological group G there is a complete topological group $\widetilde{G}$ (unique up to a topological isomorphism) that contains G as a dense topological subgroup; in other words, $\widetilde{G}$ is a *group completion* of G (cf. [28] and [22, 1.46]).

Theorem 2.1 ([22, 1.49(a), 1.51]).

    (a) *Let G be a topological group, and H a subgroup. Then $\widetilde{H} = \mathrm{cl}_{\widetilde{G}} H$.*

    (b) *If G is a locally compact group, then G is complete, that is, $\widetilde{G} = G$.*

A subset X of a topological group G is *precompact* if for every neighborhood U of the identity, there is a finite $S \subseteq X$ such that $X \subseteq (SU) \cap (US)$. (Some authors refer to precompact sets as *bounded* ones.) A topological group G is *locally precompact* if G admits a base of precompact neighborhoods at the identity. Since every pseudocompact subset of a topological group is precompact (cf. [7, 1.11]), locally pseudocompact groups are locally precompact.

Weil showed in 1937 that the completion of a locally precompact group with respect to its left or right uniformity admits the structure of a locally compact *group* containing G as a dense topological subgroup (cf. [37]). This (one-sided) *Weil-completion* coincides with the Raĭkov-completion $\widetilde{G}$ constructed in 1946 (cf. [28]). Therefore, G is locally precompact if and only if $\widetilde{G}$ is locally compact.

Theorem 2.2 below, which summarizes the main results of [6] and [7], provides a characterization of (locally) pseudocompact groups. Recall that a G_δ-*subset* of a space $(X, \mathcal{T})$ is a set of the form $\bigcap_{n < \omega} U_n$ with each $U_n \in \mathcal{T}$. The G_δ-*topology* on X is the topology generated by the G_δ-subsets of $(X, \mathcal{T})$. A subset of X is G_δ-*open* (respectively, G_δ-*closed*, G_δ-*dense*) if it is open (respectively, closed, dense) in the G_δ-topology on X.

Theorem 2.2 ([6] and [7]). *A topological group G is [locally] pseudocompact if and only if G is [locally] precompact and G_δ-dense in $\widetilde{G}$, in which case $\widetilde{G} = \beta G$ [$\beta \widetilde{G} = \beta G$]. In particular, every precompact locally pseudocompact group is pseudocompact.*

Since the G_δ-topology of groups plays an important role in the present work, we introduce some notation and then record a few useful facts. We let $\Lambda(G)$ denote the

set of closed G_δ-subgroups of the topological group G, that is, closed subgroups of G that are also G_δ-subsets of G, and we set $\Lambda_c(G) := \{K \in \Lambda(G) \mid K \text{ is compact}\}$ and $\Lambda_c^*(G) := \{K \in \Lambda_c(G) \mid K \triangleleft G\}$.

Theorem 2.3. *Let G be a topological group. Then:*

(a) ([22, 2.5]) *the G_δ-topology is a group topology on G;*

(b) ([19, 8.7]) *if G is locally compact, then $\Lambda_c(G)$ is a base at the identity for the G_δ-topology on G;*

(c) ([19, 8.7]) *if G is locally compact and σ-compact, then $\Lambda_c^*(G)$ is a base at the identity for the G_δ-topology on G.*

We have already mentioned that the adjective *hereditarily* used in the term *hereditarily locally pseudocompact* applies only to the closed subgroups of a given group. The next theorem shows that it would be uninteresting to interpret *hereditarily* as applying to all subgroups.

Theorem 2.4. *Let G be a locally pseudocompact group. If every countable subgroup of G is locally pseudocompact, then G is discrete.*

Before we proceed to the proof of Theorem 2.4, we formulate a well-known observation that will be frequently used later on too.

Lemma 2.5. *Let H be a topological group, D a dense subgroup, and O an open subgroup of H. Then $\mathrm{cl}_H(D \cap O) = O$.*

Proof. Since D is dense and O is open in H, one has $\mathrm{cl}_H O = \mathrm{cl}_H(D \cap O)$. On the other hand, O is closed in H, because every open subgroup of a topological group is also closed (cf. [19, 5.5]). This completes the proof. □

Proof of Theorem 2.4. The proof consists of two steps.

Step 1. The completion $L := \widetilde{G}$ is locally compact, and so by Theorem 2.3(b), $\Lambda_c(L)$ is a base at the identity of the G_δ-topology on L. Pick $K \in \Lambda_c(L)$. Since G is locally pseudocompact, by Theorem 2.2, G is G_δ-dense in L. By Theorem 2.3(a), the G_δ-topology is a group topology on L, and thus, by Lemma 2.5, $P := K \cap G$ is G_δ-dense in K. Let S be a countable subgroup of P. By our assumption, S is locally pseudocompact, and by Theorem 2.2, S is pseudocompact. Thus, S is finite, because there are no countably infinite homogeneous pseudocompact spaces (cf. [17, 1.3]), and so P is finite. In particular, K is finite. Therefore, L has a countable pseudocharacter, and hence L is metrizable (cf. [18, 3.3.4]). Consequently, G is metrizable.

Step 2. Let S be a countable subgroup of G. By our assumption, S is locally pseudocompact, and thus it is locally compact, because G is metrizable. Consequently, by the Baire Category Theorem, S is discrete. Therefore, every countable subgroup of the metrizable group G is discrete. Hence, G is discrete, as desired. □

Corollary 2.6. *If every subgroup of a topological group G is locally pseudocompact, then G is discrete.* □

Next, we summarize the relationship between connectedness properties of locally pseudocompact groups and their completions.

Theorem 2.7. *Let G be a locally pseudocompact group. Then:*

(a) ([8, 1.4], [3, 4.5]) $q(G) = o(G) = (\widetilde{G})_0 \cap G;$

(b) ([33], [3, 4.8]) *G is zero-dimensional if and only if $\widetilde{G}$ is zero-dimensional;*
(c) ([8, 1.7], [3, 4.11(b)]) *G/G_0 is zero-dimensional if and only if G_0 is dense in $(\widetilde{G})_0$, in which case $G_0 = q(G)$.*

We turn to proving Theorems B and C, which follow from Theorem 2.7. Recall that a group topology is *linear* if it admits a base at the identity consisting of subgroups. Since every open subgroup is also closed, every linear group topology is zero-dimensional. We prove a slightly stronger version of Theorem B.

Theorem B$'$.

(a) *Every locally pseudocompact totally disconnected group admits a coarser linear group topology.*
(b) *Every minimal, locally pseudocompact, totally disconnected group has a linear topology, and thus it is strongly Vedenissoff.*

Proof. (a) The family of open subgroups in a topological group is closed under conjugation and formation of finite intersections. Since G is totally disconnected, $q(G) = \{e\}$, and since G is locally pseudocompact, by Theorem 2.7(a), $q(G) = o(G)$. Thus, $o(G) = \{e\}$. Therefore, the family of open subgroups in G forms a base at the identity for a Hausdorff group topology on G, and it is obviously coarser than the topology of G. Clearly, this topology is linear.

(b) follows from (a) and the definition of minimality. $\qquad\square$

Theorem C$'$.
Let G be a totally minimal locally pseudocompact group. Then $G_0 = q(G)$ if and only if G/G_0 is zero-dimensional, in which case G is strongly Vedenissoff.

Proof. Suppose that $G_0 = q(G)$. Then the quotient $G/G_0 = G/q(G)$ is minimal, locally pseudocompact, and totally disconnected. Thus, by Theorem B(b), G/G_0 is zero-dimensional. By Theorem 2.7(a), $q(G) = o(G)$, and in particular, $z(G) = o(G)$. Therefore, G is strongly Vedenissoff, as required. The converse follows by Theorem 2.7(c). $\qquad\square$

In the definition of a linear topology, we asked for a base at the identity consisting of open subgroups. Some authors define the same term by requiring the existence of a base at the identity consisting of open *normal* subgroups. This stronger property warrants defining a new functorial subgroup of a topological group G, namely, the intersection $o^*(G)$ of all open *normal* subgroups of a group G. If a locally compact group L admits a base at the identity consisting of neighborhoods that are invariant under conjugation (that is, L is so-called *balanced* or admits *small invariant neighborhoods*), which is the case for compact or abelian groups, then $o(L) = o^*(L)$. There are, however, many locally compact groups that do not have this property, a fact that explains our choice of terminology.

Examples 2.8.

(a) The semidirect product $L := \{0,1\}^{\mathbb{Z}} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on the compact group $K := \{0,1\}^{\mathbb{Z}}$ by shifts, is locally compact and zero-dimensional, and thus $o(L)$ is trivial. However, K is the smallest open normal subgroup of L, and therefore $o^*(L) = K$.
(b) For $p \in \mathbb{P}$, let $\mathbb{Q}_p$ denote the (locally compact) field of p-adic numbers. The discrete multiplicative group $\mathbb{Q}^{\times}$ of non-zero rationals acts on $\mathbb{Q}_p$ by multiplication. The semidirect product $L := (\mathbb{Q}_p, +) \rtimes \mathbb{Q}^{\times}$ is locally compact

and zero-dimensional (and so, again, $o(L)$ is trivial), but $\mathbb{Q}_p$ is the smallest open normal subgroup of L, and therefore $o^*(L) = \mathbb{Q}_p$.

(c) In general, let G be a locally compact group and let D be a subgroup of $\mathrm{Aut}(G)$ such that G contains no proper D-invariant open subgroup, and put $L := G \rtimes D$, where D is equipped with the discrete topology. Then, by Theorem 1.1, the locally compact group L has the property that $o(L) = L_0 = G_0$ and $o^*(L) = G$.

3. Proof of Theorem A

Theorem A. *Let G be a hereditarily locally pseudocompact group. Then G/G_0 is zero-dimensional and $G_0 = q(G) = z(G) = o(G)$; that is, G is strongly Vedenissoff.*

In this section, we present the proof of Theorem A. By Theorem 2.7(a),

$$q(G) = z(G) = o(G)$$

for every locally pseudocompact group G. We have already noted that if G/G_0 is zero-dimensional, then $G_0 = q(G) = z(G)$. Thus, it suffices to show that G/G_0 is zero-dimensional for every hereditarily locally pseudocompact group G. Since every (Hausdorff) quotient of a hereditarily locally pseudocompact group is again hereditarily locally pseudocompact, and the quotient G/G_0 is hereditarily disconnected (cf. [19, 7.3] and [22, 1.32(c)]), it suffices to prove the following statement.

Theorem 3.1. *Let G be a hereditarily locally pseudocompact, hereditarily disconnected group. Then $\widetilde{G}$ is hereditarily disconnected, and G is zero-dimensional.*

In the setting of Theorem 3.1, if $\widetilde{G}$ is hereditarily disconnected, then by Theorem 1.1, $\widetilde{G}$ is zero-dimensional (because it is locally compact), and so G is zero-dimensional too. Thus, it suffices to show that $\widetilde{G}$ is hereditarily disconnected whenever G is so. We prove the contrapositive of this statement, namely, that if $(\widetilde{G})_0$ is non-trivial, then G_0 is non-trivial too. The proof is broken down into several steps: First, it is shown in Proposition 3.2 that Theorem 3.1 holds in the case where the completion $\widetilde{G}$ of G is a direct product of a zero-dimensional compact group and the real line $\mathbb{R}$. Then, in Proposition 3.4, it is proven that if $\widetilde{G}$ contains a non-trivial compact connected subgroup, then G_0 is non-trivial. Finally, it is shown that if the component of $\widetilde{G}$ is non-trivial, but contains no compact connected subgroup, then G_0 contains $\mathbb{R}$ as a closed subgroup.

Proposition 3.2. *Let N be a zero-dimensional compact group, and G a dense hereditarily locally pseudocompact subgroup of $N \times \mathbb{R}$. Then one has $\{e\} \times \mathbb{R} \subseteq G$.*

In order to prove Proposition 3.2, we recall a notion and a result that is well-known to profinite group theorists. A topological group P is *topologically finitely generated* if it contains a dense finitely generated group, that is, if there exists a finite subset F of P such that $P = \mathrm{cl}_P \langle F \rangle$.

Theorem 3.3 ([29, 2.5.1], [15, 2.1(a)]). *If P is a topologically finitely generated compact zero-dimensional group, then P is metrizable.*

Proof of Proposition 3.2. As G is locally pseudocompact, by Theorem 2.2, G is G_δ-dense in $N \times \mathbb{R}$. The set $A_r := N \times \{r\}$ is a G_δ-set in $N \times \mathbb{R}$ for every $r \in \mathbb{R}$, and so $A_r \cap G \neq \emptyset$. Thus, for every $r \in \mathbb{R}$, there is $g \in N$ such that $(g, r) \in G$.

Let $g_1, g_2 \in N$ be such that $x_1 := (g_1, 1) \in G$ and $x_2 := (g_2, \sqrt{2}) \in G$, and put $P := \mathrm{cl}_N \langle g_1, g_2 \rangle$ and $H := \mathrm{cl}_G \langle x_1, x_2 \rangle$. Since P is a closed subgroup of N, it is a compact topologically finitely generated zero-dimensional group, and by Theorem 3.3, P is metrizable. Thus, the product $P \times \mathbb{R}$ is metrizable, and so H is metrizable, being a subgroup of $P \times \mathbb{R}$. On the other hand, H is locally pseudocompact, being a closed subgroup of the hereditarily locally pseudocompact group G. Therefore, H is locally compact. Hence, by Theorem 2.1, H is closed not only in G, but also in $N \times \mathbb{R}$.

Let $\pi_2 \colon N \times \mathbb{R} \to \mathbb{R}$ denote the second projection, and put $\pi := \pi_{2|H}$. Since N is compact and H is closed in $N \times \mathbb{R}$, π_2 is a closed map (cf. [18, 3.1.16]), and thus $\pi(H)$ is closed in $\mathbb{R}$ and π is a closed map too. This implies that π is surjective, because $\pi(H)$ contains the dense subgroup $\langle 1, \sqrt{2} \rangle$ of $\mathbb{R}$. Consequently, π is a quotient map, and $\mathbb{R}$ is topologically isomorphic to a quotient of H. Therefore, H is not zero-dimensional, and by Theorem 1.1, H_0 is non-trivial.

Since $H_0 \subseteq \{e\} \times \mathbb{R} = (N \times \mathbb{R})_0$, one has $H_0 = \{e\} \times \mathbb{R}$, because $\mathbb{R}$ has no non-trivial proper connected subgroups. Hence, $\{e\} \times \mathbb{R} = H_0 \subseteq H \subseteq G$, as desired. $\square$

Proposition 3.4. *Let G be a hereditarily locally pseudocompact group such that the completion $\widetilde{G}$ is σ-compact. If $\widetilde{G}$ contains a non-trivial compact connected subgroup, then G_0 is non-trivial.*

Proof. Let C be a non-trivial connected compact subgroup of $\widetilde{G}$. By Theorem 2.3(c), $\Lambda_c^*(\widetilde{G})$ is a base at the identity for the G_δ-topology on $\widetilde{G}$. So, we may pick $K \in \Lambda_c^*(\widetilde{G})$. Since K is a normal subgroup of $\widetilde{G}$, the set KC is a subgroup of $\widetilde{G}$, and KC is compact, because both K and C are compact. Furthermore, KC is G_δ-open in $\widetilde{G}$, as it contains the G_δ-set K. By Theorem 2.2, G is G_δ-dense in $\widetilde{G}$. By Theorem 2.3(a), the G_δ-topology is a group topology on $\widetilde{G}$. Consequently, by Lemma 2.5, $P := KC \cap G$ is G_δ-dense in KC; in particular, P is dense in KC, and thus, by Theorem 2.1, $\widetilde{P} = KC$.

We show that P is hereditarily pseudocompact. Let S be a closed subgroup of P. Since KC is compact, it is closed in $\widetilde{G}$, and so P is a closed subgroup of G. Thus, S is a closed subgroup of G, and by our assumption, S is locally pseudocompact. As S is contained in the compact group KC, it is precompact. Consequently, by Theorem 2.2, S is pseudocompact. This shows that P is hereditarily pseudocompact. Therefore, by Theorem 1.3, P_0 is dense in $(\widetilde{P})_0 = (KC)_0$, and hence

$$\{e\} \neq C \subseteq (KC)_0 \subseteq \mathrm{cl}_{KC} P_0 \subseteq \mathrm{cl}_{\widetilde{G}} G_0.$$

In particular, G_0 cannot be trivial, as desired. $\square$

One last ingredient of the proof of Theorem 3.1 is a result that is often referred to as Iwasawa's Theorem (although it also relies on the work of Yamabe).

Theorem 3.5 ([38, Theorem 5′], [21, Theorem 13])**.** *Let L be a connected locally compact group. Then there is a compact connected subgroup C and closed subgroups $H_1, \ldots, H_r$ such that each H_i is topologically isomorphic to the additive group $\mathbb{R}$, and L is homeomorphic to $H_1 \times \cdots \times H_r \times C$.*

Proof of Theorem 3.1. We prove the contrapositive of the theorem. Let G be a hereditarily locally pseudocompact group such that $(\widetilde{G})_0 \neq \{e\}$. We show that $G_0 \neq \{e\}$.

Step 1. As G is locally precompact, its completion $L := \widetilde{G}$ is locally compact. Let U be a neighborhood of the identity in L such that $\mathrm{cl}_L\, U$ is compact. Put $L' := \langle U \rangle$, the subgroup generated by U, and $G' := L' \cap G$. We claim that by replacing G with G' if necessary, we may assume that $\widetilde{G}$ is σ-compact from the outset.

Since L' contains U, it is an open subgroup of L, and thus it is also closed. Consequently, G' is open and closed in G, and G' is also hereditarily locally pseudocompact. By Lemma 2.5, one has $L' = \mathrm{cl}_L\, G'$, and so, by Theorem 2.1, $L' = \widetilde{G'}$. As L' is generated by U, it is compactly generated, and in particular, it is σ compact (cf. [19, 5.12, 5.13]). Since L' is an open subgroup of L, by Theorem 1.1, $L_0 = o(L) \subseteq L'$, and therefore $L_0 = L_0'$ (because $L' \subseteq L$ implies $L_0' \subseteq L_0$). Similarly, one has $G_0 = G_0'$. Hence, it suffices to show that $G_0' \neq \{e\}$, and by replacing G with G' if necessary, we may assume that $\widetilde{G}$ is σ-compact.

Step 2. Since $\widetilde{G}$ is σ-compact, if $\widetilde{G}$ contains a non-trivial compact connected subgroup, then by Proposition 3.4, G_0 is non-trivial, and we are done. Thus, from now on, we assume that $\widetilde{G}$ contains no non-trivial compact connected subgroups.

If N is a compact subgroup of $\widetilde{G}$, then $N_0 = \{e\}$, and by Theorem 1.1, N is zero-dimensional. Thus, every compact subgroup of $\widetilde{G}$ is zero-dimensional. In particular, $(\widetilde{G})_0$ contains no non-trivial compact connected subgroups. Therefore, by Theorem 3.5, our assumption $(\widetilde{G})_0 \neq \{e\}$ yields that there is a closed subgroup H of $\widetilde{G}$ such that $H \cong \mathbb{R}$ (that is, H is topologically isomorphic to the additive group $\mathbb{R}$).

Step 3. By Theorem 2.3(c), $\Lambda_c^*(\widetilde{G})$ is a base at the identity for the G_δ-topology on $\widetilde{G}$, and so we may pick $N \in \Lambda_c^*(\widetilde{G})$. Since N is a normal subgroup, $(\widetilde{G})_0$ acts continuously on N by conjugation, and the orbit of x is a connected subspace of N. By Step 2, N is zero-dimensional. Thus, the orbit of each $x \in N$ is a singleton. Therefore, $g^{-1}xg = x$ for every $g \in (\widetilde{G})_0$ and $x \in N$. In particular, the elements of H and N commute (elementwise). We note that this argument, concerning the commuting of connected and zero-dimensional normal subgroups, is due to K. H. Hofmann (cf. [20]).

Let $f\colon \mathbb{R} \to H$ be a topological isomorphism. The continuous surjection $h\colon N \times \mathbb{R} \to NH$ given by $h(x,r) = xf(r)$ is a homomorphism, because f is a homomorphism, and N and H commute (elementwise). We show that h is a topological isomorphism. Since $H \cong \mathbb{R}$, the only compact subgroup of H is the trivial one, and thus $N \cap H = \{e\}$. Therefore, h is injective. Since N is compact and normal, and H is closed in $\widetilde{G}$, the subgroup NH is closed in $\widetilde{G}$, and so NH is locally compact. The domain $N \times \mathbb{R}$ of h is also locally compact and σ-compact. Consequently, by the Open Mapping Theorem, h is open (cf. [19, 5.29]). Hence, h is a topological isomorphism.

The subgroup NH is G_δ-open in $\widetilde{G}$, because it contains the G_δ-set N. By Theorem 2.2, G is G_δ-dense in $\widetilde{G}$. By Theorem 2.3(a), the G_δ-topology is a group topology on $\widetilde{G}$. So, by Lemma 2.5, $P := NH \cap G$ is G_δ-dense in NH; in particular, P is dense in NH, and by Theorem 2.1, $\widetilde{P} = NH$. Since P is a closed subgroup of G, it is hereditarily locally pseudocompact. Thus, $P' := h^{-1}(P)$ satisfies the conditions of Proposition 3.2. Consequently, $H = h(\{e\} \times \mathbb{R}) \subseteq h(P') \subseteq G$. Hence, G_0 is non-trivial, as desired. $\qquad\square$

4. Proof of Theorem D

Theorem D. *For every natural number n or $n = \omega$, there exists an abelian pseudocompact group G_n such that G_n is perfectly totally minimal, hereditarily disconnected, but $\dim G_n = n$.*

In this section, we prove Theorem D by establishing a general construction that allows one to "realize" minimal abelian groups as quasi-components of minimal pseudocompact groups. A weaker version of Theorem D, which provides totally minimal pseudocompact groups, was announced in [11, 1.4.2]. The novelty of Theorem D, in addition to its complete proof, is that we obtain *perfectly* totally minimal pseudocompact groups.

Theorem D$'$. *Let A be a precompact abelian group that is contained in a connected compact abelian group C. Then there exists a pseudocompact abelian group G such that $A \cong q(G)$ and $C \cong (\widetilde{G})_0$, and in particular, $\dim G = \dim C$. Furthermore, if $C = \widetilde{A}$ and*

- (a) *A is minimal, then G may be chosen to be minimal;*
- (b) *A is totally minimal, then G may be chosen to be totally minimal;*
- (c) *A is perfectly minimal, then G may be chosen to be perfectly minimal;*
- (d) *A is perfectly totally minimal, then G may be chosen to be perfectly totally minimal.*

Theorem D$'$ follows a line of "embedding" results, which state that certain (locally) precompact groups embed into (locally) pseudocompact groups as a particular (e.g., functorial) closed subgroup (cf. [4, 2.1], [34], [5, 7.6], [35], [8, 3.6], and [3, 5.6]). The novelty is that minimality properties of the group A are inherited by the group G that is constructed. By the celebrated Prodanov-Stoyanov Theorem, every minimal abelian group is precompact (cf. [26] and [27]), and so the condition that the group A is precompact is not restrictive at all.

We first show how Theorem D follows from Theorem D$'$ and then proceed to proving the latter. To that end, we recall a characterization due to Stoyanov for groups that are not only perfectly totally minimal, but their powers have the same property too (cf. [32]). For an abelian topological group G, let $wtd(G)$ denote the subgroup of elements x in G for which there exists a positive integer m such that for every sequence $\{k_n\}_{n=1}^{\infty}$ of integers, one has $m^n k_n x \longrightarrow 0$ in G. In other words,

$$wtd(G) := \{x \in G \mid \exists m > 0, \forall \{k_n\}_{n=1}^{\infty} \in \mathbb{N}^{\omega}, m^n k_n x \longrightarrow 0\}.$$

Theorem 4.1 ([32], [14, 6.1.18]). *Let P be a precompact abelian group. Then P^{λ} is perfectly totally minimal for every cardinal λ if and only if $wtd(\widetilde{P}) \subseteq P$.*

Proof of Theorem D. Put $P := \mathbb{Q}/\mathbb{Z}$. Then $\widetilde{P} = \mathbb{R}/\mathbb{Z}$ and $wtd(\widetilde{P}) = \mathbb{Q}/\mathbb{Z} = P$, and by Theorem 4.1, $A_n := P^n$ is perfectly totally minimal for every natural number n or $n = \omega$, and A is contained in the connected compact group $C_n := (\mathbb{R}/\mathbb{Z})^n$. By Theorem D$'$(d), there exists a perfectly totally minimal pseudocompact group G_n such that $A_n \cong q(G_n)$ and $\dim G_n = \dim C_n = n$. Since

$$(G_n)_0 \subseteq q(G_n)_0 \cong (A_n)_0 = \{0\},$$

the group G_n is hereditarily disconnected, as desired. $\qquad\square$

We proceed now to proving Theorem D$'$. The proof has two ingredients: a zero-dimensional pseudocompact group H with good minimality properties, and

a discontinuous homomorphism $h\colon \widetilde{H} \to C$ with kernel H. The desired group G will be the sum of the graph of h and the group A formed in the product $H \times C$.

Lemma 4.2. *For every infinite cardinal λ, there exists a pseudocompact zero-dimensional group H such that:*
 (i) *H is perfectly totally minimal;*
 (ii) *$r_0(\widetilde{H}/H) \geq 2^\lambda$.*

Proof. Let $\mathbb{P}$ denote the set of prime integers, and for $p \in \mathbb{P}$, let $\mathbb{Z}_p$ denote the group of p-adic integers. Put $N := \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{\omega_1}$. We think of elements of N as tuples $(x_{p,\alpha})$, where $p \in \mathbb{P}$ and $\alpha < \omega_1$. We define three subgroups of N:
 (1) $E_1 := \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p^{\omega_1}$ consists of elements x such that $(\exists \alpha)(x_{p,\alpha} \neq 0)$ only for finitely many primes p (or equivalently, $E_1 = wtd(N)$);
 (2) $E_2 := \prod_{p \in \mathbb{P}} S_p$, where S_p is the Σ-product of ω_1-many copies of $\mathbb{Z}_p$ (or equivalently, E_2 consists of elements x such that all but countably many coordinates $x_{p,\alpha}$ of x are zero);
 (3) $E := E_1 + E_2$.
We claim that $H := E^\lambda$ has the desired properties.

The group E is G_δ-dense in N, because it contains E_2, which is clearly G_δ-dense. Thus, H is G_δ-dense in the compact group N^λ, and in particular, by Theorem 2.1, $\widetilde{H} = N^\lambda$. Therefore, by Theorem 2.2, H is pseudocompact. The group H is zero-dimensional, being a subspace of the zero-dimensional group N^λ.

Since E is G_δ-dense in the compact group N, in particular, it is dense, and by Theorem 2.1, $N = \widetilde{E}$. Thus, $wtd(\widetilde{E}) = wtd(N) = E_1 \subseteq E$, and therefore by Theorem 4.1, $H = E^\lambda$ is perfectly totally minimal.

In order to prove that $r_0(\widetilde{H}/H) \geq 2^\lambda$, it suffices to show that $r_0(N/E) \geq 1$, as $\widetilde{H}/H \cong (N/E)^\lambda$. Letting Δ denote the "diagonal" subgroup of N, that is, the subgroup generated by d such that $d_{p,\alpha} = 1$ for every p and α, we prove that $E \cap \Delta = \{0\}$. In fact, we show a bit more, namely, that every element in E has at least one zero coordinate. Let $x = y + z \in E$, where $y \in E_1$ and $z \in E_2$. By the definition of E_1, there exists $q \in \mathbb{P}$ such that $y_{q,\alpha} = 0$ for every $\alpha < \omega_1$. Since $z \in E_2$, all but countably many coordinates of z are zero. In particular, there exists $\gamma < \omega_1$ such that $z_{q,\gamma} = 0$. Therefore, $x_{q,\gamma} = y_{q,\gamma} + z_{q,\gamma} = 0$. Hence, $E \cap \Delta = \{0\}$ and $r_0(N/E) \geq 1$, as desired. $\square$

We consider the next lemma to be part of the folklore of pseudocompact abelian groups (cf. [2, 3.6, 3.10]), and we provide its proof only for the sake of completeness.

Lemma 4.3. *Let K_1 and K_2 be compact topological groups, and let $h\colon K_1 \to K_2$ be a surjective homomorphism such that $\ker h$ is G_δ-dense in K_1. Then the graph Γ_h of h is a G_δ-dense subgroup of the product $K_1 \times K_2$, and in particular, Γ_h is pseudocompact.*

Proof. Let B be a non-empty G_δ-subset of $K_1 \times K_2$. Without loss of generality, we may assume that B is of the form $B_1 \times B_2$, where B_i is a G_δ-set in K_i. Pick $x_2 \in B_2$. Since h is surjective, there is $x_1 \in K_1$ such that $h(x_1) = x_2$. The translate $B_1 x_1^{-1}$ is a non-empty G_δ-set in K_1, and thus we may pick $x_0 \in B_1 x_1^{-1} \cap \ker h$, because $\ker h$ is G_δ-dense in K_1. Since $h(x_0 x_1) = h(x_1) = x_2$, one obtains that $(x_0 x_1, x_2) \in \Gamma_h \cap (B_1 \times B_2)$. This shows that Γ_h meets every G_δ-set in $K_1 \times K_2$. Therefore, by Theorem 2.1, $K_1 \times K_2$ is the completion of Γ_h. Hence, by Theorem 2.2, Γ_h is pseudocompact. $\square$

A last, auxiliary, tool in the proof of Theorem D$'$ is the following observation.

Remark 4.4. Let $\mathcal{P}$ denote one of the following properties: minimal, totally minimal, perfectly minimal, perfectly totally minimal. If G contains a dense subgroup with property $\mathcal{P}$, then G also has property $\mathcal{P}$ (cf. [31, Theorem 2], [25], [1, Propositions 1 and 2], [13], [14, 2.5.1, 4.3.3], and [22, 3.21, 3.23]).

Proof of Theorem D$'$. Put $\lambda = w(C)$, and let H be the group provided by Lemma 4.2. Since $r_0(\widetilde{H}/H) \geq 2^\lambda$, the quotient $\widetilde{H}/H$ contains a free abelian group F of rank 2^λ. As $|C| \leq 2^\lambda$, one may pick a surjective homomorphism $h_1 \colon F \to C$. The group C is divisible, because it is compact and connected (cf. [19, 24.25]). Thus, h_1 can be extended to a surjective homomorphism $h_2 \colon \widetilde{H}/H \to C$.

Let $h \colon \widetilde{H} \to C$ denote the composition of h_2 with the canonical projection $\widetilde{H} \to \widetilde{H}/H$. By Theorem 2.2, H is G_δ-dense in $\widetilde{H}$, because H is pseudocompact. Thus, $\ker h$ is G_δ-dense in $\widetilde{H}$, because $H \subseteq \ker h$. Clearly, h is surjective. Therefore, by Lemma 4.3, the graph Γ_h of h is G_δ-dense in the product $\widetilde{H} \times C$.

Put $G := \Gamma_h + (\{0\} \times A)$. Since Γ_h is G_δ-dense in $H \times C$ and contained in G, the group G is G_δ-dense too. Thus, by Theorem 2.1, $\widetilde{G} = \widetilde{H} \times C$, and by Theorem 2.2, G is pseudocompact. As H is zero-dimensional, $(\widetilde{G})_0 = \{0\} \times C$, and by Theorem 2.7(a), $q(G) = (\widetilde{G})_0 \cap G = \{0\} \times A$.

We now check that $\dim G = \dim C$. Since G is pseudocompact, by Theorem 2.2, $\widetilde{G} = \beta G$, and so $\dim G = \dim \beta G = \dim \widetilde{G}$ (cf. [18, 7.1.17]). As H is zero-dimensional and pseudocompact, by Theorem 2.7(b), $\dim \widetilde{H} = 0$. Thus, by Yamanoshita's Theorem, $\dim \widetilde{G} = \dim \widetilde{H} + \dim C = \dim C$ (cf. [39], [24, Corollary 2], and [14, 3.3.12]). Therefore, $\dim G = \dim C$.

We turn to the minimality properties of G. Suppose that $C = \widetilde{A}$. The group G always contains the product $H \times A$, but in this case, $H \times A$ is dense in $\widetilde{G} = \widetilde{H} \times C$, and thus it is dense in G. Therefore, by Remark 4.4, G inherits all minimality properties of $H \times A$. Since H is perfectly totally minimal, the product $H \times A$ inherits all minimality properties of A. This shows (a)–(d). $\qquad\square$

One wonders whether the condition $C = \widetilde{A}$ is necessary for parts (a)–(d) of Theorem D$'$. If the resulting group G is to be totally minimal, then the answer is positive. Dikranjan showed that if G is a minimal pseudocompact abelian group, then $q(G)$ is dense in $(\widetilde{G})_0$ if and only if $G/q(G)$ is minimal (cf. [10, 1.7]), in which case $(\widetilde{G})_0$ is the completion of $q(G)$. This settles the question for (b) and (d). The following remark settles the question for (a) and (c).

Remark 4.5. We note (without a proof) that the techniques of Theorem D$'$ can also be used to construct, for every positive integer n or $n = \omega$, a perfectly minimal pseudocompact n-dimensional group G such that $G/q(G)$ is not minimal, and hence $q(G)$ is not dense in $(\widetilde{G})_0$.

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