SKEW CATEGORIES, SMASH PRODUCT CATEGORIES 
AND QUASI-KOSZUL CATEGORIES

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Abstract. Let \( \mathcal{A} \) be a small additive Krull-Schmidt locally radical finite category over a field \( K \) and let \( G \) be a finite group. We show that if \( \mathcal{A} \) is a free \( G \)-category (resp. \( G \)-graded category), then \( \mathcal{A} \) is quasi-Koszul if and only if the skew (resp. smash product) category \( G \ast \mathcal{A} \) (resp. \( \mathcal{A} \ast G \)) is.

1. Introduction

Koszul algebras were originally defined by Priddy in [9] and have arisen in many contexts: algebraic geometry, Lie theory, combinatorics, non-commutative geometry and topology. There exist numerous generalizing notations of Koszul algebras. For example, Berger [2] introduced the notation of \( N \)-Koszul algebras, and Cassidy and Shelton [3] introduced the notation of \( K_2 \) algebras. Very recently, Martínez-Villa and Solberg [7] introduced the notation of (weakly, quasi-)Koszul categories to obtain a naturally associated Koszul theory for any finite dimensional algebra.

Throughout this paper we will consider small additive categories \( \mathcal{A} \) over a field \( K \), free \( G \)-categories and \( G \)-graded categories where \( G \) is a finite group. For \( G \)-categories and \( G \)-graded categories, Cibils and Marcos [4] defined their skew categories and smash products categories, respectively, which provide a generalized category version of the Cohen-Montgomery Duality Theorem [5]. Inspired by the closeness of the (quasi-)Koszulity of algebras under the skew products and smash products [6, 10], we show that

Theorem. Let \( \mathcal{A} \) be a small additive Krull-Schmidt locally radical finite category over a field \( K \) and let \( G \) be a finite group.

(i) If \( \mathcal{A} \) is a free \( G \)-category, then \( \mathcal{A} \) is quasi-Koszul if and only if the skew category \( G \ast \mathcal{A} \) is.

(ii) If \( \mathcal{A} \) is \( G \)-graded, then \( \mathcal{A} \) is quasi-Koszul if and only if \( \mathcal{A} \) is.

In Section 2 we recall the definitions of \( G \)-category, of skew category, and of \( G \)-graded category and give some basic facts. In Section 3 we recall the definitions of the syzygy of functor and of quasi-Koszul category and prove the Theorem.
2. Skew categories and smash product categories

2.1. Skew categories. A $G$-category is a $K$-category $\mathcal{A}$ with firstly a set action of $G$ on the set of objects $\mathcal{A}_0$ and secondly $K$-module maps $g : \text{Hom}_\mathcal{A}(X, Y) \to \text{Hom}_\mathcal{A}(gX, gY)$ for each $g \in G$ and $(X, Y) \in \mathcal{A}_0 \times \mathcal{A}_0$ satisfying $g(\varphi \psi) = (g\varphi)(g\psi)$ in case the morphism $\varphi \psi$ is composed in $\mathcal{A}$. Moreover for $g, h \in G$ and morphism $\psi$ we have $(gh)\psi = g(h\psi)$ and $e\psi = \psi$ where $e$ is the neutral element of $G$. In other words there is a group homomorphism from $G$ to the group of autofunctors of $\mathcal{A}$. If the action of $G$ is free on $\mathcal{A}_0$, we say that $\mathcal{A}$ is a free $G$-category.

Definition 1 ([3] Definition 2.3]). Let $\mathcal{A}$ be a $G$-category. The skew category $G \ast \mathcal{A}$ is a $K$-category with $(G \ast \mathcal{A})_0 = \mathcal{A}_0$ and morphisms $\text{Hom}_{G \ast \mathcal{A}}(X, Y) = \bigoplus_{g \in G} \text{Hom}_\mathcal{A}(gX, Y)$. Composition of morphisms is provided by the composition of $\mathcal{A}$.

An additive $K$-category $\mathcal{A}$ is said to be Krull-Schmidt if any object in $\mathcal{A}$ is a finite direct sum of objects with a local endomorphism algebra. Recall that the radical of a category $\mathcal{A}$, $\text{rad}\mathcal{A}$, is a sub-bifunctor of $\text{Hom}_\mathcal{A}(–, –)$ given by $\text{rad}\mathcal{A}(X, Y) = \{ \psi \in \text{Hom}_\mathcal{A}(X, Y) | \varphi \psi \in \text{rad}(\text{End}_\mathcal{A}(X)) \text{ for all } \varphi \in \text{Hom}_\mathcal{A}(Y, X) \}$. Observe that we also have (see [8] Lemmas 4.1 and 4.2)

$\text{rad}_\mathcal{A}(X, Y) = \bigoplus_{i=1}^m \psi_i$ with $\psi_i \in \text{rad}_\mathcal{A}(X, X_i)$, $\psi_i' \in \text{rad}_\mathcal{A}(X_i, Y)$ for $i = 1, \cdots, m$ and $\psi = \sum_{i=1}^m \psi_i \psi_i'$. Inductively define $\text{rad}_\mathcal{A}^n := \text{rad}_\mathcal{A} \circ \text{rad}_\mathcal{A}^{n-1}$. A $K$-category $\mathcal{A}$ is called locally radical finite if $\text{rad}_\mathcal{A}(X, Y)/\text{rad}_\mathcal{A}^{n+1}(X, Y)$ is finite dimensional over $K$ for all $X, Y \in \mathcal{A}$ and $i \geq 0$.

Lemma. If $X, Y \in \mathcal{A}$, then $\text{rad}_{G \ast \mathcal{A}}(X, Y) \simeq KG \otimes_K \text{rad}_\mathcal{A}(X, Y)$ and $\text{rad}_{G \ast \mathcal{A}}^n \simeq KG \otimes_K \cdots \otimes_K KG \otimes_K \text{rad}_\mathcal{A}^n$, $n \geq 1$.

Proof. Clearly $\text{rad}_\mathcal{A}(gX, Y) \simeq \text{rad}_\mathcal{A}(X, Y)$ for all $g \in G$. Assume that $\psi = \sum_{g \in G} \psi_g$ and $\varphi = \sum_{h \in G} \varphi_h$ where $\psi_g \in \text{Hom}_\mathcal{A}(gX, Y)$ and $\varphi_h \in \text{Hom}_\mathcal{A}(hY, X)$. Then $\varphi \cdot \psi \in \text{rad}(\text{End}_{G \ast \mathcal{A}}(X))$ if and only if $\varphi_h(h\psi_g) \in \text{rad}(\text{End}_{G \ast \mathcal{A}}(X))$ for all $g, h \in G$ if and only if $h\psi_g \in \text{rad}_{G \ast \mathcal{A}}(gX, Y)$ for all $g \in G$. Thus we obtain that

$\text{rad}_{G \ast \mathcal{A}}(X, Y) = \{ \psi \in \text{Hom}_{G \ast \mathcal{A}}(X, Y) | \varphi \psi \in \text{rad}(\text{End}_{G \ast \mathcal{A}}(X)) \}$

for all $\varphi \in \text{Hom}_{G \ast \mathcal{A}}(X, Y)$

$= \{ \psi = \sum_{g \in G} \psi_g, \psi_g \in \text{Hom}_\mathcal{A}(gX, Y) | \sum_{g, h \in G} \varphi_h \psi_g \in \text{rad}(\text{End}_{G \ast \mathcal{A}}(X)) \}$

for all $\varphi_h \in \text{Hom}_\mathcal{A}(hY, X), h \in G$

$= \{ \psi = \sum_{g \in G} \psi_g, \psi_g \in \text{Hom}_\mathcal{A}(gX, Y) | \varphi_h(h\psi_g) \in \text{rad}(\text{End}_{G \ast \mathcal{A}}(X)) \}$

for all $\varphi_h \in \text{Hom}_\mathcal{A}(hY, X), h \in G$

$= \{ \psi = \sum_{g \in G} \psi_g | \psi_g \in \text{rad}_\mathcal{A}(gX, Y) \}

\simeq KG \otimes_K \text{rad}_\mathcal{A}(gX, Y)$

$= KG \otimes_K \text{rad}_\mathcal{A}(X, Y).$
By induction, we have
\[
\operatorname{rad}_{G, sf}^{n}(X, Y) = \operatorname{rad}_{G, sf} \cdot \operatorname{rad}_{G, sf}^{n-1}(X, Y)
\]
\[
\simeq \operatorname{rad}_{G, sf} \cdot (\bigoplus_{K} \cdots \otimes_{K} K \otimes_{K} \operatorname{rad}_{G, sf}^{n-1}(X, Y))
\]
\[
\simeq \bigoplus_{K} \cdots \otimes_{K} K \otimes_{K} \operatorname{rad}_{G, sf}^{n}(X, Y).
\]
\[\square\]

2.2. Smash product categories. A \(G\)-graded category \(\mathcal{B}\) is a \(K\)-category together with a decomposition of morphisms \(\operatorname{Hom}_{\mathcal{B}}(X, Y) = \bigoplus_{g \in G} \operatorname{Hom}_{\mathcal{B}}(X, Y)_{g}\) such that \(\operatorname{Hom}_{\mathcal{B}}(Y, Z)_{g} \cdot \operatorname{Hom}_{\mathcal{B}}(X, Y)_{h} \subset \operatorname{Hom}_{\mathcal{B}}(X, Z)_{gh}\). In particular, \(\operatorname{Hom}_{\mathcal{B}}(X, X)\) is a \(G\)-graded algebra for each \(X \in \mathcal{B}\).

**Definition 2** ([K, Definition 3.1]). Let \(\mathcal{B}\) be a \(G\)-graded category. The **smash product category** \(\mathcal{B} \# G\) is a category with \((\mathcal{B} \# G)_{0} = \mathcal{B}_{0} \times G\) and morphisms \(\operatorname{Hom}_{\mathcal{B} \# G}((X, g), (Y, h)) = \operatorname{Hom}_{\mathcal{B}}(X, Y)_{h^{-1}g}\). Composition of morphisms is provided by the graded composition of \(\mathcal{B}\).

Define a \(G\)-action on \(\mathcal{B} \# G\) by \(g(X, h) = (X, gh)\) and \(gf = f \in \operatorname{Hom}_{\mathcal{B} \# G}((X, gh), (Y, gl))\) if \(f \in \operatorname{Hom}_{\mathcal{B} \# G}((X, h), (Y, l))\). Then the category \(\mathcal{B} \# G\) is a free \(G\)-category and the categories \(G \ast (\mathcal{B} \# G)\) and \(\mathcal{B}\) are equivalent [K, Corollary 3.3]. By the Lemma, we have the following:

**Corollary.** If \(X, Y \in \mathcal{B}\), then \(KG \otimes_{K} \operatorname{rad}_{\mathcal{B} \# G}((X, g), (Y, h)) = \operatorname{rad}_{\mathcal{B}}(X, Y)\) for all \(g, h \in G\).

3. Quasi-Koszul categories

3.1. Syzygy of functor. Let \(\mathcal{A}\) be an additive Krull-Schmidt locally radical finite \(K\)-category. We denote by \(\operatorname{Mod}\mathcal{A}\) the category of contravariant additive functors from \(\mathcal{A}\) to the category \(\operatorname{Mod}K\) of \(K\)-vector spaces with natural transforms as morphisms.

A family of objects \(\{F(Y_{i})|Y_{i} \in \mathcal{A}, i \in I\}\) is said to be a **family of generators** for \(F \in \operatorname{Mod}\mathcal{A}\) if for very \(Y \in \mathcal{A}\),

\[
F(Y) = \sum_{i \in I} F(\phi_{i})(F(Y_{i}))
\]

where all but a finite number of \(\phi_{i} : Y \rightarrow Y_{i}\) are zero, and \(F \in \operatorname{Mod}\mathcal{A}\) is **finitely generated** if it has a finite set of generator.

Denote by \(\operatorname{mod}\mathcal{A}\) the full subcategory of \(\operatorname{Mod}\mathcal{A}\) consisting of all finitely generated contravariant additive functors of \(\operatorname{Mod}\mathcal{A}\). Then the category \(\operatorname{mod}\mathcal{A}\) has, by [K, Corollary 4.13], minimal projective resolution. This gives rise to one-to-one correspondence between the indecomposable objects in \(\mathcal{A}\) and the simple objects in \(\operatorname{Mod}\mathcal{A}\), where an indecomposable object \(X\) in \(\mathcal{A}\) gives rise to the simple object \(S_{X} := \operatorname{Hom}_{\mathcal{A}}(-, X)/\operatorname{rad}_{\mathcal{A}}(-, X)\) in \(\operatorname{Mod}\mathcal{A}\). Furthermore Yoneda’s Lemma implies that an \(\mathcal{A}\)-module is projective in \(\operatorname{mod}\mathcal{A}\) if and only if it is isomorphic to \(\operatorname{Hom}_{\mathcal{A}}(-, X)\) for some \(X \in \mathcal{A}\).

**Definition 3.** The **syzygy functor** \(\Omega(F)\) of \(F \in \operatorname{Mod}\mathcal{A}\) is defined by the exact sequence

\[
0 \rightarrow \Omega(F) \rightarrow \operatorname{Hom}_{\mathcal{A}}(-, X) \rightarrow F \rightarrow 0
\]
for some $X \in \mathcal{A}$. Inductively, we can define the $n$-th syzygy functor $\Omega^n(\mathcal{F})$ of $\mathcal{F}$.

**Remark.** By Freyd’s Theorem (see e.g. [3] p. 16, Theorem 3.1) $\text{Mod}\mathcal{A}$ is an abelian category with coproduct and a faithful set of small projectives. Thus $\Omega(\mathcal{F}) : \mathcal{A} \to \text{Mod}K, M \mapsto \Omega(\mathcal{F})(M)$ is a contravariant functor for any $\mathcal{F} \in \text{mod}\mathcal{A}$.

### 3.2. Quasi-Koszul categories

Recall that an ideal in the category $\mathcal{A}$ is a sub-bifunctor of the Hom-functor. For two ideals $\mathcal{I}$ and $\mathcal{J}$ we define in a natural way inclusion, intersection and product. In particular the product of two ideals $\mathcal{I}$ and $\mathcal{J}$ is given by

$$\mathcal{I}\mathcal{J}(X, Z) = \{ \psi \in \text{Hom}_\mathcal{A}(X, Z) | \psi \}$$

$$= \sum_{i=1}^{n} \varphi_i \phi_i, \varphi_i \in \text{Hom}_\mathcal{A}(X, Y_i), \varphi \in \text{Hom}_\mathcal{A}(Y_i, Z), Y_i \in \mathcal{A} \}$$

for $X, Z \in \mathcal{A}$.

For an $n$-fold product of an ideal $\mathcal{I}$ with itself we write $\mathcal{I}^n$. Let $\mathcal{I}$ be an ideal in the category. For a contravariant functor $\mathcal{F} : \mathcal{A} \to \text{Mod}K$ define $\mathcal{IF} : \mathcal{A} \to \text{Mod}K$ as the subfunctor of $\mathcal{F}$ given by

$$\mathcal{IF}(M) = \sum_{X \in \mathcal{A}} \mathcal{F}(\mathcal{I}(M, X))$$

for all objects $M \in \mathcal{A}$. Clearly $\mathcal{IF} \subseteq \mathcal{F}$ for any ideal $\mathcal{I}$ and any contravariant functor $\mathcal{F}$ in $\text{Mod}\mathcal{A}$.

**Definition 4** (II §5). A contravariant functor $\mathcal{F}$ in $\text{Mod}\mathcal{A}$ is quasi-Koszul if it has a finitely generated projective resolution

$$\cdots \to \text{Hom}_\mathcal{A}(-, X_i) \to \cdots \to \text{Hom}_\mathcal{A}(-, X_0) \to \mathcal{F} \to 0$$

and $\text{rad}_\mathcal{A}^i(\mathcal{F}) = \text{rad}_\mathcal{A}^2(-, X_{i-1}) \cap \Omega^i(\mathcal{F})$ for all $i \geq 1$.

The category $\mathcal{A}$ is quasi-Koszul if every simple functor in $\text{Mod}\mathcal{A}$ is quasi-Koszul.

### 3.3. Proof of Theorem

Since the categories $G \ast (\mathcal{A} \# G)$ and $\mathcal{A}$ are equivalent, it is sufficient to prove (i). Assume that $\mathcal{A}$ is quasi-Koszul and $\mathcal{F}$ is a simple contravariant functor in $\text{Mod}\mathcal{A}$ having a finitely generated projective resolution

$$\cdots \to \text{Hom}_\mathcal{A}(-, X_i) \to \cdots \to \text{Hom}_\mathcal{A}(-, X_0) \to \mathcal{F} \to 0$$

and that $\text{rad}_\mathcal{A}^i(\mathcal{F}) = \text{rad}_\mathcal{A}^2(-, X_{i-1}) \cap \Omega^i(\mathcal{F})$ for all $i \geq 1$.

Suppose that $\mathcal{F} = \text{Hom}_\mathcal{A}(-, X)/\text{rad}_\mathcal{A}(-, X)$ for some indecomposable object $X \in \mathcal{A}$ and define

$$\mathcal{F}^* := \text{Hom}_{G \ast \mathcal{A}}(-, X)/\text{rad}_{G \ast \mathcal{A}}(-, X).$$

Then $\mathcal{F}^*$ is a simple contravariant functor in $\text{Mod}G \ast \mathcal{A}$, and every simple functor in $\text{Mod}G \ast \mathcal{A}$ is of this form, which has a finitely generated projective resolution

$$\cdots \to \text{Hom}_{G \ast \mathcal{A}}(-, X_i) \to \cdots \to \text{Hom}_{G \ast \mathcal{A}}(-, X_0) \to \mathcal{F}^* \to 0.$$

Thus it is enough to show that

$$\text{(i) } \text{ rad}_\mathcal{A}^i(\mathcal{F}) = \text{ rad}_\mathcal{A}^2(-, X_{i-1}) \cap \Omega^i(\mathcal{F})$$

if and only if $\text{ rad}_{G \ast \mathcal{A}}^i(\mathcal{F}^*) = \text{ rad}_{G \ast \mathcal{A}}^2(-, X_{i-1}) \cap \Omega^i(\mathcal{F}^*)$. 


For any object $M \in \mathcal{A}$ and $i \geq 1$, we have
\[
\text{rad}_{G_{i,\mathcal{A}}}(\Omega_i^i(F^*)(M)) = \sum_{Y \in \mathcal{A}} \Omega_i^i(F^*)(\text{rad}_{G_{i,\mathcal{A}}}(M, Y)) = \sum_{Y \in \mathcal{A}} \Omega_i^i(F^*)(KG \otimes_K \text{rad}_{\mathcal{A}}(M, Y)) = \sum_{Y \in \mathcal{A}} KS \otimes_K KS \otimes_K \Omega_i^i(F)(\text{rad}_{\mathcal{A}}(M, Y)) = KG \otimes_K KG \otimes_K \text{rad}_{\mathcal{A}}(\Omega_i^i(F)(M));
\]
that is, for all $i \geq 1$,
\[
(\text{I}) \quad \text{rad}_{G_{i,\mathcal{A}}}(\Omega_i^i(F^*)) = KG \otimes_K KG \otimes_K \text{rad}_{\mathcal{A}}\Omega_i^i(F).
\]
By the Lemma, we have
\[
\text{rad}_{G_{i,\mathcal{A}}}(\Omega_i^i(F^*)) = KG \otimes_K KG \otimes_K \text{rad}_{\mathcal{A}}(\Omega_i^i(F^*)) \quad \text{for all } i \geq 1.
\]
Since $\Omega_i^i(F^*) \subseteq \text{rad}_{G_{i,\mathcal{A}}} \text{Hom}_{G_{i,\mathcal{A}}}(\cdot, X_{i-1})$ for all $i \geq 1$,
\[
\Omega_i^i(F^*)(M) \subseteq \text{rad}_{G_{i,\mathcal{A}}} \text{Hom}_{G_{i,\mathcal{A}}}(\cdot, X_{i-1})
\]
\[
= \sum_{Y \in \mathcal{A}} \text{Hom}_{G_{i,\mathcal{A}}}(\cdot, X_{i-1})\text{rad}_{G_{i,\mathcal{A}}}(M, Y)
\]
\[
= KG \otimes_K KG \otimes_K \text{Hom}_{\mathcal{A}}(\cdot, X_{i-1})\text{rad}_{\mathcal{A}}(M, Y)
\]
\[
= KG \otimes_K KG \otimes_K \text{rad}_{\mathcal{A}}\text{Hom}_{\mathcal{A}}(\cdot, X_{i-1})(M).
\]
On the other hand, note that $\Omega_i^i(F) \subseteq \text{rad}_{\mathcal{A}}\text{Hom}_{\mathcal{A}}(\cdot, X_{i-1})$. This yields that $\Omega_i^i(F^*) = KG \otimes_K KG \otimes_K \Omega_i^i(F)$. Hence
\[
(\text{II}) \quad \text{rad}_{G_{i,\mathcal{A}}}(\Omega_i^i(F^*)) = KG \otimes_K KG \otimes_K \text{rad}_{\mathcal{A}}^{\mathcal{A}}(\cdot, X_{i-1}) \cap \Omega_i^i(F).
\]
Now $(\ast)$ follows by combining $(\text{I})$ and $(\text{II})$. This completes the proof. \hfill $\square$

References


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