ON GEODESICS OF FINSLER METRICS V IA NAVIGATION PROBLEM

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Abstract. This paper is devoted to a study of geodesics of Finsler metrics via Zermelo navigation. We give a geometric description of the geodesics of the Finsler metric produced from any Finsler metric and any homothetic field in terms of navigation representation, generalizing a result previously only known in the case of Randers metrics with constant S-curvature. As its application, we present explicitly the geodesics of the Funk metric on a strongly convex domain.

1. Introduction

A smooth curve in a Finsler manifold is called a geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [21]). Recently, R. Bryant [5] showed that a Finsler metric on \( S^2 \) of constant flag curvature \( K = 1 \) with reversible geodesics is actually a Riemannian one. M. Crampin’s result tells us that if a Randers metric \( \alpha + \beta \) has reversible geodesics, then \( \beta \) is closed [8]. C. Robles investigated geodesics of Randers metrics of constant S-curvature [17]. Randers metrics of constant flag curvature (or quadratic Riemann curvature) have constant S-curvature [3, 10]. In terms of the navigation representation, they are produced from Riemannian metrics and homothetic vector fields [13, 16, 22, 23]. In fact, C. Robles classified geodesics in Randers manifolds of constant flag curvature [17].

The aim of this paper is to give a geometric description of the geodesics of the Finsler metric \( \tilde{F} \) obtained from an arbitrary Finsler metric \( F \) and an arbitrary homothetic field \( V \) of \( F \) in terms of the navigation representation. Precisely we show the following:

Theorem 1.1. Let \( F = F(x, y) \) be a Finsler metric on a manifold \( M \) and let \( V \) be a vector field on \( M \) with \( F(x, V_x) < 1 \). Suppose that \( V \) is homothetic with dilation \( c \). Let \( \tilde{F} = \tilde{F}(x, y) \) denote the Finsler metric on \( M \) defined in (3.20). Then the geodesics of \( \tilde{F} \) are given by \( \psi_t (\gamma(a(t))) \) where \( \psi_t \) is the flow of \(-V; \gamma(t)\)
is a geodesic of $F$ and $a(t)$ is defined by

$$a(t) := \begin{cases} \frac{e^{2ct} - 1}{2c}, & \text{if } c \neq 0; \\ t, & \text{if } c = 0. \end{cases}$$

Our result generalizes a theorem previously only known in the case of Randers metrics with constant $S$-curvature [17]. As its application, we represent explicitly the geodesics of the Funk metric on a strongly convex domain (see Proposition 4.1).

Recall that a vector field $V$ on a Finsler manifold $(M, F)$ is a homothetic field of $F$ with dilation $c$ if the corresponding flow $\phi_t$ is homothetic with dilation $c$. In particular $V$ is called a Killing field if $c = 0$.

It is worth mentioning our recent result that for a non-Killing homothetic field $V$, the navigation representation has the flag curvature decreasing property [15].

For interesting results of geodesics on Finsler spheres, we refer the reader to [1, 9, 11].

2. Preliminaries

Let us recall firstly the definition of the Finsler metrics.

**Definition 2.1** ([2]). Let $M$ be a finite-dimensional manifold. A function $F : TM \to [0, +\infty)$ is a Finsler metric if it satisfies

(a) $F$ is $C^\infty$ on $TM \setminus \{0\}$;
(b) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_xM$, $x \in M$ and $\lambda > 0$;
(c) for every $y \in T_xM \setminus \{0\}$, the quadratic form

$$g_{x,y}(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t = s = 0}, \quad \forall u, v \in T_xM$$

is positive definite.

In this case, $(M, F)$ is called a Finsler manifold. Let $SM$ be the projective sphere bundle of $M$, obtained from $TM$ by identifying nonzero vectors which differ from each other by a positive multiplicative factor. Each geometrical quantity on $TM$, homogeneous of degree zero, is considered to sit on $SM$. Define

$$\omega := F_y^i dx^i.$$ 

Then $\omega$ is a differential form on $SM$. It is easy to verify that

$$\omega \wedge (d\omega)^{n-1} \neq 0, \quad n = \dim M$$

(cf. [3]), i.e., that $\omega$ defines a contact structure on $SM$. This form $\omega$ is known in the calculus of variations as the Hilbert form.

Since $\omega$ is a contact form, there exists a unique vector field $X$ on $SM$ that satisfies

$$\omega(X) = 1, \quad X.(d\omega) = 0.$$ 

This vector field $X$ is known as the Reeb vector field [4]. It is easy to see that a $C^\infty$-curve is a (constant Finslerian speed) geodesic if its canonical lift in $SM$ is an integral curve of the Reeb vector field $X$ [4].

Every vector $y \in T_xM \setminus \{0\}$ uniquely determines a covector $p \in T_y^*M \setminus \{0\}$ by

$$p(u) := \frac{1}{2} \frac{d}{dt} (F^2(x, y + tu))|_{t = 0}, \quad u \in T_xM.$$ 

The resulting map $L^F_x : y \in T_xM \to p \in T_y^*M$ is called the Legendre transformation at $x$. The family $L^F := \{L^F_x \mid x \in M\}$ is called the Legendre transformation.
Define a non-negative scalar function \( H = H(x, p) \) by

\[
H(x, p) := \max_{y \in T_x M \setminus \{0\}} \frac{p(y)}{F(x, y)}.
\]

The function \( H \) is \( C^\infty \) on \( T^*M \setminus \{0\} \) and \( H_x := H|_{T^*_x M} \) is a Minkowski norm on \( T^*_x M \) for \( x \in M \). Such a function is called a Cartan metric \([12, 15]\) (co-Finsler metric in an alternative terminology \([18, 19]\)). The pair \((M, H)\) is called a Cartan manifold.

Every covector \( p \in T^*_x M \setminus \{0\} \) uniquely determines a vector \( y \in T_x M \setminus \{0\} \) by

\[
q(y) := \frac{1}{2} \frac{d}{dt} \left( H^2(x, p + tq) \right)_{|t=0}, \quad q \in T^*_x M.
\]

The resulting map \( L^F_x : p \in T^*_x M \to y \in T_x M \) is called the inverse Legendre transformation at \( x \). Indeed \( L^F_x \) and \( L^{F*}_x \) are inverses of each other. Moreover, they preserve the Minkowski norms

\[
H(x, p) = F(x, (L^{F*}_x)^{-1}p).
\]

Recently, one of the important approaches in discussing the Finsler metric is the (Zermelo) navigation problem. For instance, Bao-Robles-Shen have classified Randers metrics of constant flag curvature via the navigation problem in a Riemannian manifold \([3]\).

The main technique of the navigation problem is described as follows. Given a Finsler metric \( F \) and a vector field \( V \) with \( F(x, V_x) < 1 \), define a new Finsler metric \( \tilde{F} \) by

\[
F(x, \frac{y}{F(x, y)} + V_x) = 1, \quad \forall x \in M, \ y \in T_x M.
\]

A (local) flow (a local one-parameter group in an alternative terminology) on a manifold \( M \) is a map \( \phi : (-\epsilon, \epsilon) \times M \to M \), also denoted by \( \phi_t := \phi(t, \cdot) \), satisfying

- \( \phi_0 = \text{id} : M \to M; \)
- \( \phi_s \circ \phi_t = \phi_{s+t} \) for any \( s, t \in (-\epsilon, \epsilon) \) with \( s + t \in (-\epsilon, \epsilon) \).

Hence, the lift of a flow \( \phi_t \) on \( M \) is a flow \( \hat{\phi}_t \) on \( T^*M \),

\[
\hat{\phi}_t(x, p) := (\phi_t(x), (\phi^*_t)^{-1}(p)).
\]

By the relationship between vector fields and flows, \((2.2)\) induces a natural way to lift a vector field \( V \) on \( M \) to a vector field \( X^*_V \) on \( T^*M \).

A vector field \( V \) on a Finsler manifold \( (M, F) \) is called homothetic with dilation \( c \) if its flow \( \phi_t \) satisfies

\[
F(\phi_t(x), \phi_t(x)) = e^{2ct} F(x, y), \quad \forall x \in M, \ y \in T_x M.
\]

Similarly, a vector field \( V \) on a Cartan manifold \( (M, H) \) is called homothetic with dilation \( c \) if its flow \( \phi_t \) satisfies

\[
H(\phi_t(x), (\phi^*_t)^{-1}(p)) = e^{-2ct} H(x, p), \quad \forall x \in M, \ p \in T^*_x M.
\]

**Lemma 2.2.** Let \( V \) be a homothetic field on a Finsler manifold \( (M, F) \) with dilation \( c \) and \( H \) its Cartan metric defined by \((2.1)\). Then \( V \) is a homothetic field of \( H \) with dilation \( c \).
Proof. By using (2.1) and (2.3) we have

\[ H(\phi_t(x), (\phi_t^*)^{-1}(p)) = \max_{\tilde{y} \in T_0 M \setminus \{0\}} \frac{\left( (\phi_t^*)^{-1}(\tilde{y}) \right) (\tilde{y})}{F(\phi_t(x), \tilde{y})} \]

\[ = \max_{\tilde{y} \in T_0 M \setminus \{0\}} \frac{p((\phi_t^*)^{-1}(\tilde{y}))}{F(\phi_t(x), \tilde{y})} \]

\[ = \max_{y \in T_x M \setminus \{0\}} \frac{p(y)}{F(\phi_t(x), y)} \]

\[ = e^{-2ct} \max_{y \in T_x M \setminus \{0\}} \frac{p(y)}{e^{2ct}F(x, y)} = e^{-2ct}H(x, p), \]

where \( y := (\phi_t^*)^{-1}(\tilde{y}) \). It follows that \( V \) is a homothetic field of \( H \) with dilation \( c \). \( \square \)

3. Geodesics of a Finsler metric via navigation problem

In this section, we give a geometric description of geodesics of Finsler metrics via the homothetic navigation problem in a Finsler manifold. First, we show the following:

Lemma 3.1. Let \( N \) be a manifold, and let \( V \) and \( W \) be vector fields on \( N \) that satisfy

\[ [V, W] = -cV \]

for some constant \( c \). Let \( \phi_t \) and \( \psi_t \) be local 1-parameter groups of \( V \) and \( W \) respectively. Then \( \psi_t \circ \phi_{a(t)} \) is a local 1-parameter group of the vector field \( V + W \), where \( a(t) \) is defined by

\[ a(t) := \begin{cases} \frac{e^ct}{c}, & \text{if } c \neq 0; \\ t, & \text{if } c = 0. \end{cases} \]

Proof. Direct calculations yield

\[ \frac{da}{dt} = e^{ct}. \]

Since \( V \) is the induced vector field from \( \phi \),

\[ \frac{d}{dt} \phi_t(x) \big|_{t=t_0} = V_{\phi_{t_0}(x)}. \]

Let \( \eta_t := \phi_{a(t)}, u := a(t) \). By using (3.2) and (3.3) we obtain

\[ \frac{d}{dt} [\eta_t(x)]_{t=s} = \frac{d\phi_u}{du} \big|_{u=a(s)} \frac{da}{dt} \big|_{t=s} = e^{cs}V_{\phi_{a(s)}(x)} = e^{cs}V_{\eta_s(x)}. \]

From (3.1), one obtains \([W, V] = cV\). It follows that

\[ c\psi_{ts}V = \psi_{ts}(cV) \]

\[ = \psi_{ts}[W, V] \]

\[ = [\psi_{ts}W, \psi_{ts}V] = [W, \psi_{ts}V]. \]
This gives
\[ c\psi_t \cdot V_p(f) = [W, \psi_t \cdot V]_p f \]
\[ = ([CW, \psi_t \cdot V])_p f \]
\[ = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \psi_t \cdot V - (\psi_t + \Delta t)_* V \right]_p f = -\frac{d}{dt} [\psi_t \cdot V]_p f \]
for a point \( p \in N \) and a function \( f \in C^\infty(N) \). We set
\[ y(t) := \psi_t \cdot V_p(f). \]
Substituting (3.7) into (3.6) yields
\[ \frac{dy}{dt} = -cy. \]
Solving (3.8), we get
\[ y = C_1 e^{-ct}. \]
Plugging (3.7) into (3.9) yields
\[ (\psi_t)_* V = e^{-ct} V. \]
By using (3.4), for any function \( f \in C^\infty(N) \) we have
\[ \frac{d}{dt} [\psi_t \circ \eta_t(x)]_{t=s} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[ f \circ \psi_{s+\Delta s} \circ \eta_{s+\Delta s}(x) - f \circ \psi_s \circ \eta_s(x) \right] \]
\[ = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[ f \circ \psi_{s+\Delta s} \circ \eta_{s+\Delta s}(x) - f \circ \psi_s \circ \eta_{s+\Delta s}(x) \right] \]
\[ + \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[ f \circ \psi_s \circ \eta_{s+\Delta s}(x) - f \circ \psi_s \circ \eta_s(x) \right] \]
\[ = \lim_{\Delta s \to 0} \frac{f \circ \psi_{s+\Delta s} - f \circ \psi_s}{\Delta s} \left( \eta_{s+\Delta s}(x) \right) \]
\[ + (f \circ \psi_s)_* \left( \lim_{\Delta s \to 0} \frac{\eta_{s+\Delta s}(x) - \eta_s(x)}{\Delta s} \right) \]
\[ = W_{\psi_s} \circ \eta_s(x) f + \psi_s (e^{cs} V |_{\eta_s(x)}) f. \]
It follows that
\[ \frac{d}{dt} [\psi_t \circ \eta_t(x)]_{t=s} = W_{\psi_s} \circ \eta_s(x) + \psi_s (e^{cs} V |_{\eta_s(x)}). \]
By using (3.11) we have
\[ \psi_s (e^{cs} V |_{\eta_s(x)}) = e^{cs} \psi_s (V |_{\eta_s(x)}) = e^{cs} e^{-cs} V_{\psi_s \circ \eta_s(x)} = V_{\psi_s \circ \eta_s(x)}. \]
Plugging this into (3.12) yields
\[ \frac{d}{dt} [\psi_t \circ \eta_t(x)]_{t=s} = W_{\psi_s \circ \eta_s(x)} + V_{\psi_s \circ \eta_s(x)} = (W + V)_{\psi_s \circ \eta_s(x)}. \]
It follows that \( \psi_t \circ \eta_t \) is a local 1-parameter group of the vector field \( V + W \). \( \square \)
Lemma 3.2. For a homothetic field $V$ on a Cartan manifold $(M, H)$ with dilation $c$, we have

\[(3.13) \quad [X^\flat, X^*_V] = 2cX^\flat,\]

where $X^\flat = (L^F)_*X$ and $X^*_V$ is the lift of $V$ to $T^*M$.

Proof. In natural coordinates, we have

\[(3.14) \quad X^*_V = v^i \frac{\partial}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial p_i},\]

where $V = v^i \frac{\partial}{\partial x^i}$ [15 (5.3)]. Note that $V$ is homothetic with respect to $H$ with dilation $c$. Differentiating (2.4) with respect to $t$ at $t = 0$ yields

\[(3.15) \quad X^*_V(H) = -2cH.\]

By using (3.14), we have

\[(3.16) \quad X^*_V(H) = v^i \frac{\partial H}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial H}{\partial p_i} = -2cH.\]

Differentiating (3.16) with respect to $p_k$ gives

\[(3.17) \quad v^i \frac{\partial^2 H}{\partial x^i \partial p_k} - \sum_i p_j \frac{\partial v^j}{\partial x^i} \frac{\partial^2 H}{\partial p_i \partial p_k} = -2c \frac{\partial H}{\partial p_k}.\]

Differentiating (3.16) with respect to $x^k$ yields

\[(3.18) \quad \frac{\partial v^i}{\partial x^k} \frac{\partial H}{\partial x^i} + v^i \frac{\partial^2 H}{\partial x^i \partial x^k} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial^2 H}{\partial p_i \partial x^k} = -2c \frac{\partial H}{\partial x^k}.\]

By Lemma 4.5 in [15], $X^\flat$ is the Hamiltonian vector field for $H$. Hence it has the local expression

\[(3.19) \quad X^\flat = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}.\]

It follows from (3.14) and (3.19) that

\[
[X^\flat, X^*_V] = \left[ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}, v^j \frac{\partial}{\partial x^j}, p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right] = \left[ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}, v^j \frac{\partial}{\partial x^j}, p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right] \sum_i \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial p_i} \left[ \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_j} \right] = \left[ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}, p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right].
\]

Recall that

\[ [fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]. \]
for any vector fields $X$, $Y$ and functions $f$, $g$. It follows that

$$[X^b, X_V] = \frac{\partial H}{\partial p_i} \frac{\partial^2 H}{\partial x^i \partial x^j} - v^j \frac{\partial^2 H}{\partial x^i \partial x^j} + v^j \frac{\partial^2 H}{\partial x^i \partial p_l} \frac{\partial}{\partial p_i}$$

$\square$

Proof of Theorem 1.1. Let $V$ be a homothetic field on a Finsler manifold $(M, F)$ with dilation $c$ and let $X$ be the Reeb vector field. Applying the Legendre transformation, we obtain a Cartan metric $H(x, p)$. From Lemma 2.2, $V$ is a homothetic field of $(M, H)$ with dilation $c$. By Lemma 3.2 and (3.14), we have $[X^b, X_{-V}^b] = -2cX^b$. Let $\phi_t$ and $\hat{\psi}_t$ be local 1-parameter groups of $X^b$ and $X_{-V}^b$ respectively. By using (3.14) we get $X^b + X_{-V}^b = X^b - X_{-V}^b$. Taking this together with Lemma 3.1 we obtain that $\psi_t \circ \phi_{a(t)}$ is a local 1-parameter group of $X^b - X_{-V}^b$, where

$$a(t) = \begin{cases} \frac{e^{ct} - 1}{2c}, & \text{if } c \neq 0; \\ t, & \text{if } c = 0. \end{cases}$$

Recall that a navigation problem makes use of a Finsler metric $F$ and a vector field $V$ with $F(x, V_x) < 1$ and produces a new Finsler metric $\tilde{F}$ by solving the equation

$$F(x, y + \tilde{F}(x, y)V) = \tilde{F}(x, y).$$

Let $\tilde{X}$ be the Reeb vector field of $\tilde{F}$ and $\tilde{X}^b = (L_{\tilde{F}})_* \tilde{X}$. By Lemma 6.2 in [15], we have

$$\tilde{X}^b = X^b - X_{-V}^b.$$ 

It follows that $\hat{\psi}_t \circ \phi_{a(t)}$ is a local 1-parameter group of $\tilde{X}^b$.

Since $L_{\tilde{F}}(x, y) = (x, L^F_x(y))$, we see that any geodesic of $F$ is precisely the projection of an integral curve of $X^b$. Similarly, a geodesic of $\tilde{F}$ is precisely the projection of an integral curve of $\tilde{X}^b$. Note that

$$\hat{\psi}_t(x, p) = (\psi_t(x), (\psi_t^* p)^{-1}(p)),$$

where $\psi_t$ is the flow produced by $-V$.

Let $\pi : T^* \setminus \{0\} \rightarrow M$ be the natural projection. It follows that

$$\pi \circ \hat{\psi}_t(x, p) = \pi(\psi_t(x), (\psi_t^* p)^{-1}(p)) = \psi_t(x) = \psi_t \circ \pi(x, p)$$
for any $x \in M$ and $p \in T_xM \setminus \{0\}$. Hence we have
\begin{equation}
\pi \circ \hat{\psi}_t = \psi_t \circ \pi.
\end{equation}

It follows that
\[ \pi \circ \hat{\psi}_t \circ \phi_a(t) = \psi_t \circ \pi \circ \phi_a(t) = \psi_t (\gamma_a(t)), \]
where $\gamma(t) := \pi(\phi_t(x))$. Thus we have proved Theorem 1.1.

**Remark 3.1.** The reader should note that the navigation problem adopted here differs from those of C. Robles and Z. Shen [17, 20], where the navigation problem is defined by
\[ F(x, y) - V = 1; \]
i.e., the $\hat{F}$ we define with $(F, V)$ is precisely the $\hat{F}$ that Shen defines with $(F, -V)$.

### 4. Geodesics of Funk metrics on convex domains

In this section we are going to represent explicitly the geodesics of the Funk metric on a strongly convex domain.

Given a Minkowski norm $\varphi : \mathbb{E} \to \mathbb{R}$ on a vector space $\mathbb{E}$, one can construct $\Omega := \{ v \in \mathbb{E} | \varphi(v) < 1 \}$, $T_\Omega \Omega \simeq \mathbb{E}$. A domain $\Omega$ in $\mathbb{E}$ defined by a Minkowski norm $\varphi$ is called a strongly convex domain [19]. Thus $(\Omega, F(x, y))$ is a Minkowski manifold, where $F(x, y) := \varphi(y)$. For each $x \in \Omega$, identify $T_x \Omega$ with $\mathbb{E}$. Thus $V_x := x$ is a radical vector field on $\Omega$ satisfying $F(x, V_x) = \varphi(x) < 1$. Moreover $V$ is a homothetic field of $F$ with dilation $c = \frac{1}{2}$ [15]. Define a 1-parameter transformation $\psi_t$ on $\Omega$ by
\[ \psi_t(x) = e^{-t}x. \]
Note that $T_{\psi_t(x)} \Omega \simeq \mathbb{E}$ for any $t$. It is easy to see that
\[ \psi_t(y) = e^{-t}y, \quad \text{for } \forall \, y \in T_x \Omega. \]
Thus we have
\[ F(\psi_t(x), \psi_t(y)) = \varphi(\psi_t(y)) = \varphi(e^{-t}y) = e^{-t} \varphi(y) = e^{-t}F(x, y) \]
for any $(x, y) \in T\Omega$. It follows that $\varphi_t$ is homothetic. A direct calculation yields
\[ \frac{d\psi_t(x)}{dt} \bigg|_{t=0} = -x = -V_x. \]
Thus $\psi_t$ is the flow of the vector field $-V$. By using the Minkowski metric $F$ and the homothetic field $V$, we produce a new Finsler metric $\hat{F}$ in terms of the navigation problem. $\hat{F}$ is called the Funk metric on a strongly convex domain $\Omega$ (cf. [17 Example 3.4.3]).

Since $F$ is Minkowskian, it is locally projectively flat. This implies that the unit speed geodesic through $x$ with tangent vector $y(\neq 0)$ is $\gamma(t) := x + \frac{t}{\varphi(y)} y$. Note that $V$ is a homothetic field with dilation $c = \frac{1}{2}$. We have
\[ a(t) = e^t - 1 \]
(see Theorem \[1\]). It follows that

\[
\psi_t(\gamma(a(t))) = e^{-t} \left[ x + \frac{e^t - 1}{\varphi(y)} y \right].
\]

By Theorem 1.1, we have the following.

**Proposition 4.1.** Let \( \varphi : E \to \mathbb{R} \) be a Minkowski norm and \( \Omega \) its strongly convex domain. Assume that \( F \) is the Funk metric on \( \Omega \) defined by

\[
\varphi\left(\frac{y}{F(x, y)} + x\right) = 1.
\]

Then the geodesics of \( F \) are given by

\[
e^{-t} \left[ x + \frac{e^t - 1}{\varphi(y)} y \right].
\]

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