PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 8, August 2011, Pages 3015–3024
S 0002-9939(2011)10726-3
Article electronically published on January 13, 2011

ON GEODESICS OF FINSLER METRICS VIA NAVIGATION PROBLEM

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(Communicated by Jianguo Cao)

Abstract. This paper is devoted to a study of geodesics of Finsler metrics via Zermelo navigation. We give a geometric description of the geodesics of the Finsler metric produced from *any* Finsler metric and *any* homothetic field in terms of navigation representation, generalizing a result previously only known in the case of Randers metrics with constant S-curvature. As its application, we present explicitly the geodesics of the Funk metric on a strongly convex domain.

1. Introduction

A smooth curve in a Finsler manifold is called a *geodesic* if it is locally the shortest path connecting any two nearby points on this curve (cf. [21]). Recently, R. Bryant [5] showed that a Finsler metric on S^2 of constant flag curvature $K = 1$ with reversible geodesics is actually a Riemannian one. M. Crampin's result tells us that if a Randers metric $\alpha + \beta$ has reversible geodesics, then β is closed [8]. C. Robles investigated geodesics of Randers metrics of constant S-curvature [17]. Randers metrics of constant flag curvature (or quadratic Riemann curvature) have constant S-curvature [3, 10]. In terms of the navigation representation, they are produced from Riemannian metrics and homothetic vector fields [13, 16, 22, 23]. In fact, C. Robles classified geodesics in Randers manifolds of constant flag curvature [17].

The aim of this paper is to give a geometric description of the geodesics of the Finsler metric $\tilde{F}$ obtained from an *arbitrary* Finsler metric F and an *arbitrary* homothetic field V of F in terms of the navigation representation. Precisely we show the following:

Theorem 1.1. *Let $F = F(x, y)$ be a Finsler metric on a manifold M and let V be a vector field on M with $F(x, V_x) < 1$. Suppose that V is homothetic with dilation c. Let $\tilde{F} = \tilde{F}(x, y)$ denote the Finsler metric on M defined in* (3.20). *Then the geodesics of $\tilde{F}$ are given by $\psi_t(\gamma(a(t)))$ where ψ_t is the flow of $-V$, $\gamma(t)$*

Received by the editors June 30, 2010 and, in revised form, August 3, 2010.
2010 *Mathematics Subject Classification.* Primary 58B20.
This work is supported by the National Natural Science Foundation of China 11071005.
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is a geodesic of F and $a(t)$ is defined by

$$a(t) := \begin{cases} \frac{e^{2ct}-1}{2c}, & \text{if } c \neq 0; \\ t, & \text{if } c = 0. \end{cases}$$

Our result generalizes a theorem previously only known in the case of Randers metrics with constant S-curvature [17]. As its application, we represent explicitly the geodesics of the Funk metric on a strongly convex domain (see Proposition 4.1).

Recall that a vector field V on a Finsler manifold (M, F) is said to be a *homothetic field of F with dilation c* if the corresponding flow ϕ_t is homothetic with dilation c. In particular V is called a *Killing field* if $c = 0$.

It is worth mentioning our recent result that for a non-Killing homothetic field V, the navigation representation has the flag curvature decreasing property [15].

For interesting results of geodesics on Finsler spheres, we refer the reader to [1, 9, 11].

2. Preliminaries

Let us recall firstly the definition of the Finsler metrics.

Definition 2.1 ([2])**.** Let M be a finite-dimensional manifold. A function $F : TM \to [0, +\infty)$ is a *Finsler metric* if it satisfies

(a) F is C^∞ on $TM\backslash\{0\}$;

(b) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_xM$, $x \in M$ and $\lambda > 0$;

(c) for every $y \in T_xM\backslash\{0\}$, the quadratic form

$$g_{x,y}(u, v) \equiv \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \quad \forall u, v \in T_xM$$

is positive definite.

In this case, (M, F) is called a *Finsler manifold*. Let SM be the projective sphere bundle of M, obtained from TM by identifying nonzero vectors which differ from each other by a positive multiplicative factor. Each geometrical quantity on TM, homogeneous of degree zero, is considered to sit on SM. Define

$$\omega := F_{y^i} dx^i.$$

Then ω is a differential form on SM. It is easy to verify that

$$\omega \wedge (d\omega)^{n-1} \neq 0, \qquad n = \dim M$$

(cf. [6]), i.e., that ω defines a contact structure on SM. This form ω is known in the calculus of variations as the *Hilbert form*.

Since ω is a contact form, there exists a unique vector field X on SM that satisfies

$$\omega(X) = 1, \qquad X \lrcorner (d\omega) = 0.$$

This vector field X is known as the *Reeb vector field* [4]. It is easy to see that a C^∞-curve is a (constant Finslerian speed) geodesic if its canonical lift in SM is an integral curve of the Reeb vector field X [4].

Every vector $y \in T_xM\backslash\{0\}$ uniquely determines a covector $p \in T_x^*M\backslash\{0\}$ by

$$p(u) := \frac{1}{2}\frac{d}{dt}\left(F^2(x, y + tu)\right)|_{t=0}, \quad u \in T_xM.$$

The resulting map $L_x^F : y \in T_xM \to p \in T_x^*M$ is called the *Legendre transformation* at x. The family $L^F := \left\{L_x^F \mid x \in M\right\}$ is called the *Legendre transformation*.

Define a non-negative scalar function $H = H(x, p)$ by

$$H(x, p) := \max_{y \in T_x M \setminus \{0\}} \frac{p(y)}{F(x, y)}. \tag{2.1}$$

The function H is C^∞ on $T^*M \setminus \{0\}$ and $H_x := H|_{T_x^* M}$ is a Minkowski norm on $T_x^* M$ for $x \in M$. Such a function is called a *Cartan metric* [12, 15] (co-Finsler metric in an alternative terminology [18, 19]). The pair (M, H) is called a *Cartan manifold*.

Every covector $p \in T_x^* M \setminus \{0\}$ uniquely determines a vector $y \in T_x M \setminus \{0\}$ by

$$q(y) := \frac{1}{2} \frac{d}{dt} \left(H^2(x, p + tq) \right) |_{t=0}, \quad q \in T_x^* M.$$

The resulting map $L_x^{F*} : p \in T_x^* M \to y \in T_x M$ is called the *inverse Legendre transformation* at x. Indeed L_x^F and L_x^{F*} are inverses of each other. Moreover, they preserve the Minkowski norms

$$H(x, p) = F\left(x, (L_x^F)^{-1} p\right).$$

Recently, one of the important approaches in discussing the Finsler metric is the (Zermelo) navigation problem. For instance, Bao-Robles-Shen have classified Randers metrics of constant flag curvature via the navigation problem in a Riemannian manifold [3].

The main technique of the navigation problem is described as follows. Given a Finsler metric F and a vector field V with $F(x, V_x) < 1$, define a new Finsler metric $\tilde{F}$ by

$$F(x, \frac{y}{\tilde{F}(x, y)} + V_x) = 1, \quad \forall x \in M, \ y \in T_x M.$$

A *(local) flow* (a *local one-parameter group* in an alternative terminology) on a manifold M is a map $\phi : (-\epsilon, \epsilon) \times M \to M$, also denoted by $\phi_t := \phi(t, \cdot)$, satisfying

- $\phi_0 = \mathrm{id} : M \to M$;
- $\phi_s \cdot \phi_t = \phi_{s+t}$ for any $s, t \in (-\epsilon, \epsilon)$ with $s + t \in (-\epsilon, \epsilon)$.

Hence, the lift of a flow ϕ_t on M is a flow $\hat{\phi}_t$ on T^*M,

$$\hat{\phi}_t(x, p) := (\phi_t(x), (\phi_t^*)^{-1}(p)). \tag{2.2}$$

By the relationship between vector fields and flows, (2.2) induces a natural way to lift a vector field V on M to a vector field X_V^* on T^*M.

A vector field V on a Finsler manifold (M, F) is called *homothetic with dilation* c if its flow ϕ_t satisfies

$$F(\phi_t(x), \phi_{t*}(y)) = e^{2ct} F(x, y), \quad \forall x \in M, \ y \in T_x M. \tag{2.3}$$

Similarly, a vector field V on a Cartan manifold (M, H) is called *homothetic with dilation* c if its flow ϕ_t satisfies

$$H(\phi_t(x), (\phi_t^*)^{-1}(p)) = e^{-2ct} H(x, p), \quad \forall x \in M, \ p \in T_x^* M. \tag{2.4}$$

Lemma 2.2. *Let V be a homothetic field on a Finsler manifold (M, F) with dilation c and H its Cartan metric defined by* (2.1)*. Then V is a homothetic field of H with dilation c.*

Proof. By using (2.1) and (2.3) we have

$$\begin{aligned} H(\phi_t(x),\, (\phi_t^*)^{-1}(p)) &= \max_{\tilde{y}\in T_{\phi_t(x)}M\setminus\{0\}} \frac{\left[(\phi_t^*)^{-1}(p)\right](\tilde{y})}{F(\phi_t(x),\, \tilde{y})} \\ &= \max_{\tilde{y}\in T_{\phi_t(x)}M\setminus\{0\}} \frac{p\left((\phi_{t*})^{-1}(\tilde{y})\right)}{F(\phi_t(x),\, \tilde{y})} \\ &= \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{F(\phi_t(x),\, \phi_{t*}(y))} \\ &= \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{e^{2ct}F(x,\, y)} \\ &= e^{-2ct} \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{F(x,\, y)} = e^{-2ct}H(x,\, p), \end{aligned}$$

where $y := (\phi_{t*})^{-1}(\tilde{y})$. It follows that V is a homothetic field of H with dilation c. □

3. Geodesics of a Finsler metric via navigation problem

In this section, we give a geometric description of geodesics of Finsler metrics via the homothetic navigation problem in a Finsler manifold. First, we show the following:

Lemma 3.1. *Let N be a manifold, and let V and W be vector fields on N that satisfy*

$$[V,\, W] = -cV \tag{3.1}$$

for some constant c. Let ϕ_t and ψ_t be local 1-parameter groups of V and W respectively. Then $\psi_t \circ \phi_{a(t)}$ is a local 1-parameter group of the vector field $V+W$, where $a(t)$ is defined by

$$a(t) := \begin{cases} \frac{e^{ct}-1}{c}, & \text{if } c \neq 0; \\ t, & \text{if } c = 0. \end{cases}$$

Proof. Direct calculations yield

$$\frac{da}{dt} = e^{ct}. \tag{3.2}$$

Since V is the induced vector field from ϕ,

$$\frac{d}{dt}\phi_t(x)|_{t=t_0} = V_{\phi_{t_0}(x)}. \tag{3.3}$$

Let $\eta_t := \phi_{a(t)},\ u := a(t)$. By using (3.2) and (3.3) we obtain

$$\frac{d}{dt}\left[\eta_t(x)\right]_{t=s} = \frac{d\phi_u}{du}|_{u=a(s)}\frac{da}{dt}|_{t=s} = e^{cs}V|_{\phi_{a(s)}(x)} = e^{cs}V|_{\eta_s(x)}. \tag{3.4}$$

From (3.1), one obtains $[W,\, V] = cV$. It follows that

$$\begin{aligned} c\psi_{t*}V &= \psi_{t*}(cV) \\ &= \psi_{t*}[W,\, V] \\ &= [\psi_{t*}W,\, \psi_{t*}V] = [W,\, \psi_{t*}V]. \end{aligned} \tag{3.5}$$

This gives

$$\begin{aligned} c\psi_{t*}V_p(f) &= [W,\, \psi_{t*}V]_p f \\ &= [(\mathcal{L}_W(\psi_{t*}V)]_p\, f \\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\,[\psi_{t*}V-(\psi_{t+\Delta t})_*V]_p\, f = -\frac{d}{dt}\,[\psi_{t*}V]_p\, f \end{aligned} \tag{3.6}$$

for a point $p\in N$ and a function $f\in\mathcal{C}^\infty(N)$. We set

$$y(t) := \psi_{t*}V_p(f). \tag{3.7}$$

Substituting (3.7) into (3.6) yields

$$\frac{dy}{dt} = -cy. \tag{3.8}$$

Solving (3.8), we get

$$y = C_1 e^{-ct}. \tag{3.9}$$

Plugging (3.7) into (3.9) yields

$$\psi_{t*}V_p(f) = C_1(p,\, f)e^{-ct}. \tag{3.10}$$

Taking $t=0$, then $C_1(p,\, f) = V_p f$. Plugging this into (3.10) yields $(\psi_t)_*V_p(f) = e^{-ct}V_p f$ for any point $p\in N$ and any function $f\in\mathcal{C}^\infty(N)$. It follows that

$$(\psi_t)_*V = e^{-ct}V. \tag{3.11}$$

By using (3.4), for any function $f\in\mathcal{C}^\infty(N)$ we have

$$\begin{aligned} \frac{d}{dt}\,[\psi_t\circ\eta_t(x)]_{t=s}\, f &= \lim_{\Delta s\to 0}\frac{1}{\Delta s}\,[f\circ\psi_{s+\Delta s}\circ\eta_{s+\Delta s}(x) - f\circ\psi_s\circ\eta_s(x)] \\ &= \lim_{\Delta s\to 0}\frac{1}{\Delta s}\,[f\circ\psi_{s+\Delta s}\circ\eta_{s+\Delta s}(x) - f\circ\psi_s\circ\eta_{s+\Delta s}(x)] \\ &\quad + \lim_{\Delta s\to 0}\frac{1}{\Delta s}\,[f\circ\psi_s\circ\eta_{s+\Delta s}(x) - f\circ\psi_s\circ\eta_s(x)] \\ &= \lim_{\Delta s\to 0}\frac{f\circ\psi_{s+\Delta s} - f\circ\psi_s}{\Delta s}(\eta_{s+\Delta s}(x)) \\ &\quad + (f\circ\psi_s)_*\left(\lim_{\Delta s\to 0}\frac{\eta_{s+\Delta s}(x)-\eta_s(x)}{\Delta s}\right) \\ &= W_{\psi_s\circ\eta_s(x)}f + \psi_{s*}(e^{cs}V|_{\eta_s(x)})f. \end{aligned}$$

It follows that

$$\frac{d}{dt}\,[\psi_t\circ\eta_t(x)]_{t=s} = W_{\psi_s\circ\eta_s(x)} + \psi_{s*}(e^{cs}V|_{\eta_s(x)}). \tag{3.12}$$

By using (3.11) we have

$$\psi_{s*}(e^{cs}V|_{\eta_s(x)}) = e^{cs}\psi_{s*}(V|_{\eta_s(x)}) = e^{cs}e^{-cs}V_{\psi_s\circ\eta_s(x)} = V_{\psi_s\circ\eta_s(x)}.$$

Plugging this into (3.12) yields

$$\frac{d}{dt}\,[\psi_t\circ\eta_t(x)]_{t=s} = W_{\psi_s\circ\eta_s(x)} + V_{\psi_s\circ\eta_s(x)} = (W+V)_{\psi_s\circ\eta_s(x)}.$$

It follows that $\psi_t\circ\eta_t$ is a local 1-parameter group of the vector field $V+W$. $\square$

Lemma 3.2. *For a homothetic field V on a Cartan manifold (M, H) with dilation c, we have*

$$[X^\flat, X_V^*] = 2cX^\flat, \tag{3.13}$$

where $X^\flat = (L^F)_* X$ and X_V^* is the lift of V to T^*M.

Proof. In natural coordinates, we have

$$X_V^* = v^i \frac{\partial}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial p_i}, \tag{3.14}$$

where $V = v^i \frac{\partial}{\partial x^i}$ [15, (5.3)]. Note that V is homothetic with respect to H with dilation c. Differentiating (2.4) with respect to t at $t = 0$ yields

$$X_V^*(H) = -2cH. \tag{3.15}$$

By using (3.14), we have

$$X_V^*(H) = v^i \frac{\partial H}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial H}{\partial p_i}.$$

It follows that

$$v^i \frac{\partial H}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial H}{\partial p_i} = -2cH. \tag{3.16}$$

Differentiating (3.16) with respect to p_k gives

$$v^i \frac{\partial^2 H}{\partial x^i \partial p_k} - \sum_i \frac{\partial v^k}{\partial x^i} \frac{\partial H}{\partial p_i} - \sum_i p_j \frac{\partial v^j}{\partial x^i} \frac{\partial^2 H}{\partial p_i \partial p_k} = -2c \frac{\partial H}{\partial p_k}. \tag{3.17}$$

Differentiating (3.16) with respect to x^k yields

$$\frac{\partial v^i}{\partial x^k} \frac{\partial H}{\partial x^i} + v^i \frac{\partial^2 H}{\partial x^i \partial x^k} - p_j \frac{\partial^2 v^j}{\partial x^i \partial x^k} \frac{\partial H}{\partial p_i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial^2 H}{\partial p_i x^k \partial =} - 2c \frac{\partial H}{\partial x^k}. \tag{3.18}$$

By Lemma 4.5 in [15], $X^\flat$ is the Hamiltonian vector field for H. Hence it has the local expression

$$X^\flat = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}. \tag{3.19}$$

It follows from (3.14) and (3.19) that

$$\begin{aligned}
[X^\flat, X_V^*] &= \left[\frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}, v^j \frac{\partial}{\partial x^j} - p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right] \\
&= \left[\frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}, v^j \frac{\partial}{\partial x^j} \right] - \left[\frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}, v^j \frac{\partial}{\partial x^j} \right] \\
&\quad - \left[\frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}, p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right] + \left[\frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}, p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right].
\end{aligned}$$

Recall that

$$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$$

for any vector fields X, Y and functions f, g. It follows that

$$\begin{aligned}
[X^\flat,\, X_V^*] &= \frac{\partial H}{\partial p_i}\frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial x^j} - v^j\frac{\partial^2 H}{\partial p_i \partial x_j}\frac{\partial}{\partial x^i} + v^j\frac{\partial^2 H}{\partial x^i \partial x^j}\frac{\partial}{\partial p_i} \\
&\quad - p_k\frac{\partial H}{\partial p_i}\frac{\partial^2 v^k}{\partial x^i \partial x^j}\frac{\partial}{\partial p_j} + p_k\frac{\partial v^k}{\partial x^j}\frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial}{\partial x^i} \\
&\quad + \frac{\partial H}{\partial x_i}\frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial p_j} - p_k\frac{\partial v^k}{\partial x^j}\frac{\partial^2 H}{\partial x^i \partial p_j}\frac{\partial}{\partial p_i} \\
&= -\left(v^j\frac{\partial^2 H}{\partial x^j \partial p_k} - \sum_i \frac{\partial v^k}{\partial x^i}\frac{\partial H}{\partial p_i} - \sum_j p_i\frac{\partial v^i}{\partial x^j}\frac{\partial^2 H}{\partial p_j \partial p_k}\right)\frac{\partial}{\partial x^k} \\
&\quad + \left(\frac{\partial v^i}{\partial x^k}\frac{\partial H}{\partial x^i} + v^j\frac{\partial^2 H}{\partial x^j \partial x^k} - p_j\frac{\partial^2 v^j}{\partial x^i \partial x^k}\frac{\partial H}{\partial p_i} - p_i\frac{\partial v^i}{\partial x^j}\frac{\partial^2 H}{\partial p_j \partial x^k}\right)\frac{\partial}{\partial p_k} \\
&= 2c\frac{\partial H}{\partial p_k}\frac{\partial}{\partial x^k} - 2c\frac{\partial H}{\partial x^k}\frac{\partial}{\partial p_k} = 2cX^\flat.
\end{aligned}$$

Here we have used (3.17), (3.18) and (3.19). □

Proof of Theorem 1.1. Let V be a homothetic field on a Finsler manifold $(M,\, F)$ with dilation c and let X be the Reeb vector field. Applying the Legendre transformation, we obtain a Cartan metric $H(x,\, p)$. From Lemma 2.2, V is a homothetic field of $(M,\, H)$ with dilation c. By Lemma 3.2 and (3.14), we have $[X^\flat,\, X_{-V}^*] = -2cX^\flat$. Let ϕ_t and $\hat{\psi}_t$ be local 1-parameter groups of $X^\flat$ and X_{-V}^* respectively. By using (3.14) we get $X^\flat + X_{-V}^* = X^\flat - X_V^*$. Taking this together with Lemma 3.1, we obtain that $\hat{\psi}_t \circ \phi_{a(t)}$ is a local 1-parameter group of $X^\flat - X_V^*$, where

$$a(t) = \begin{cases} \frac{e^{2ct}-1}{2c}, & \text{if } c \neq 0; \\ t, & \text{if } c = 0. \end{cases}$$

Recall that a navigation problem makes use of a Finsler metric F and a vector field V with $F(x,\, V_x) < 1$ and produces a new Finsler metric $\tilde{F}$ by solving the equation

$$F(x,\, y + \tilde{F}(x,\, y)V) = \tilde{F}(x,\, y). \tag{3.20}$$

Let $\tilde{X}$ be the Reeb vector field of $\tilde{F}$ and $\tilde{X}^\flat = (L^{\tilde{F}})_*\tilde{X}$. By Lemma 6.2 in [15], we have

$$\tilde{X}^\flat = X^\flat - X_V^*.$$

It follows that $\hat{\psi}_t \circ \phi_{a(t)}$ is a local 1-parameter group of $\tilde{X}^\flat$.

Since $L^F(x,\, y) = \big(x,\, L_x^F(y)\big)$, we see that any geodesic of F is precisely the projection of an integral curve of $X^\flat$. Similarly, a geodesic of $\tilde{F}$ is precisely the projection of an integral curve of $\tilde{X}^\flat$. Note that

$$\hat{\psi}_t(x,\, p) = \big(\psi_t(x),\, (\psi_t^*)^{-1}(p)\big)\,,$$

where ψ_t is the flow produced by $-V$.

Let $\pi : T^*M \setminus \{0\} \to M$ be the natural projection. It follows that

$$\pi \circ \hat{\psi}_t(x,\, p) = \pi\left(\psi_t(x),\, (\psi_t^*)^{-1}(p)\right) = \psi_t(x) = \psi_t \circ \pi(x,\, p)$$

for any $x \in M$ and $p \in T_xM \setminus \{0\}$. Hence we have

$$\pi \circ \hat{\psi}_t = \psi_t \circ \pi. \tag{3.21}$$

It follows that

$$\pi \circ \hat{\psi}_t \circ \phi_{a(t)} = \psi_t \circ \pi \circ \phi_{a(t)} = \psi_t \left(\gamma(a(t))\right),$$

where $\gamma(t) := \pi(\phi_t(x))$. Thus we have proved Theorem 1.1.

Remark 3.1. The reader should note that the navigation problem adopted here differs from those of C. Robles and Z. Shen [17, 20], where the navigation problem is defined by

$$F(x, \frac{y}{\tilde{F}(x, y)} - V) = 1;$$

i.e., the $\tilde{F}$ we define with (F, V) is precisely the $\tilde{F}$ that Shen defines with $(F, -V)$.

4. Geodesics of Funk metrics on convex domains

In this section we are going to represent explicitly the geodesics of the Funk metric on a strongly convex domain.

Given a Minkowski norm $\varphi : \mathbb{E} \to \mathbb{R}$ on a vector space $\mathbb{E}$, one can construct $\Omega := \{v \in \mathbb{E} \,|\, \varphi(v) < 1\}$, $T_x\Omega \simeq \mathbb{E}$. A domain Ω in $\mathbb{E}$ defined by a Minkowski norm φ is called a *strongly convex domain* [19]. Thus $(\Omega, F(x, y))$ is a Minkowski manifold, where $F(x, y) := \varphi(y)$. For each $x \in \Omega$, identify $T_x\Omega$ with $\mathbb{E}$. Thus $V_x := x$ is a radical vector field on Ω satisfying $F(x, V_x) = \varphi(x) < 1$. Moreover V is a homothetic field of F with dilation $c = \frac{1}{2}$ [15, Example 1]. Define a 1-parameter transformation ψ_t on Ω by

$$\psi_t(x) = e^{-t}x.$$

Note that $T_{\psi_t(x)}\Omega \simeq \mathbb{E}$ for any t. It is easy to see that

$$\psi_{t*}(y) = e^{-t}y, \qquad \text{for} \quad \forall\, y \in T_x\Omega.$$

Thus we have

$$\begin{aligned} F\left(\psi_t(x), \psi_{t*}(y)\right) &= \varphi\left(\psi_{t*}(y)\right) \\ &= \varphi\left(e^{-t}y\right) \\ &= e^{-t}\varphi\left(y\right) = e^{-t}F(x, y) \end{aligned}$$

for any $(x, y) \in T\Omega$. It follows that φ_t is homothetic. A direct calculation yields

$$\frac{d\psi_t(x)}{dt} \,|_{t=0} = -x = -V_x.$$

Thus ψ_t is the flow of the vector field $-V$. By using the Minkowski metric F and the homothetic field V, we produce a new Finsler metric $\tilde{F}$ in terms of the navigation problem. $\tilde{F}$ is called the *Funk metric* on a strongly convex domain Ω (cf. [7, Example 3.4.3]).

Since F is Minkowskian, it is locally projectively flat. This implies that the unit speed geodesic through x with tangent vector $y(\neq 0)$ is $\gamma(t) := x + \frac{t}{\varphi(y)}y$. Note that V is a homothetic field with dilation $c = \frac{1}{2}$. We have

$$a(t) = e^t - 1$$

(see Theorem 1.1). It follows that

$$\psi_t\left(\gamma(a(t))\right) = e^{-t}\left[x + \frac{e^t - 1}{\varphi(y)}y\right].$$

By Theorem 1.1, we have the following.

Proposition 4.1. *Let $\varphi : \mathbb{E} \to \mathbb{R}$ be a Minkowski norm and Ω its strongly convex domain. Assume that $\tilde{F}$ is the Funk metric on Ω defined by*

$$\varphi\left(\frac{y}{\tilde{F}(x,\, y)} + x\right) = 1.$$

Then the geodesics of $\tilde{F}$ are given by

$$e^{-t}\left[x + \frac{e^t - 1}{\varphi(y)}y\right].$$

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