ON GEODESICS OF FINSLER METRICS
VIA NAVIGATION PROBLEM

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Abstract. This paper is devoted to a study of geodesics of Finsler metrics via Zermelo navigation. We give a geometric description of the geodesics of the Finsler metric produced from any Finsler metric and any homothetic field in terms of navigation representation, generalizing a result previously only known in the case of Randers metrics with constant S-curvature. As its application, we present explicitly the geodesics of the Funk metric on a strongly convex domain.

1. Introduction

A smooth curve in a Finsler manifold is called a geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [21]). Recently, R. Bryant [5] showed that a Finsler metric on \( S^2 \) of constant flag curvature \( K = 1 \) with reversible geodesics is actually a Riemannian one. M. Crampin’s result tells us that if a Randers metric \( \alpha + \beta \) has reversible geodesics, then \( \beta \) is closed [8]. C. Robles investigated geodesics of Randers metrics of constant \( S \)-curvature [17]. Randers metrics of constant flag curvature (or quadratic Riemann curvature) have constant \( S \)-curvature [3, 10]. In terms of the navigation representation, they are produced from Riemannian metrics and homothetic vector fields [13, 16, 22, 23]. In fact, C. Robles classified geodesics in Randers manifolds of constant flag curvature [17].

The aim of this paper is to give a geometric description of the geodesics of the Finsler metric \( \hat{F} \) obtained from an arbitrary Finsler metric \( F \) and an arbitrary homothetic field \( V \) of \( F \) in terms of the navigation representation. Precisely we show the following:

Theorem 1.1. Let \( F = F(x, y) \) be a Finsler metric on a manifold \( M \) and let \( V \) be a vector field on \( M \) with \( F(x, V_x) < 1 \). Suppose that \( V \) is homothetic with dilation \( c \). Let \( \hat{F} = \hat{F}(x, y) \) denote the Finsler metric on \( M \) defined in (3.20). Then the geodesics of \( \hat{F} \) are given by \( \psi_t(\gamma(a(t))) \) where \( \psi_t \) is the flow of \( -V \), \( \gamma(t) \)
is a geodesic of $F$ and $a(t)$ is defined by
\[ a(t) := \begin{cases} 
\frac{c^2 t^2 - 1}{2c}, & \text{if } c \neq 0; \\
t, & \text{if } c = 0.
\end{cases} \]

Our result generalizes a theorem previously only known in the case of Randers metrics with constant $S$-curvature \cite{17}. As its application, we represent explicitly the geodesics of the Funk metric on a strongly convex domain (see Proposition 4.1).

Recall that a vector field $V$ on a Finsler manifold $(M, F)$ is a homothetic field of $F$ with dilation $c$ if the corresponding flow $\phi_t$ is homothetic with dilation $c$. In particular $V$ is called a Killing field if $c = 0$.

It is worth mentioning our recent result that for a non-Killing homothetic field $V$, the navigation representation has the flag curvature decreasing property \cite{15}.

For interesting results of geodesics on Finsler spheres, we refer the reader to \cite{1, 9, 11}.

2. Preliminaries

Let us recall firstly the definition of the Finsler metrics.

**Definition 2.1** \cite{2}. Let $M$ be a finite-dimensional manifold. A function $F : TM \to [0, +\infty)$ is a Finsler metric if it satisfies

(a) $F$ is $C^\infty$ on $TM \setminus \{0\}$;
(b) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_x M$, $x \in M$ and $\lambda > 0$;
(c) for every $y \in T_x M \setminus \{0\}$, the quadratic form
\[ g_{x,y}(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv) \bigg|_{t=s=0}, \quad \forall u, v \in T_x M \]

is positive definite.

In this case, $(M, F)$ is called a Finsler manifold. Let $SM$ be the projective sphere bundle of $M$, obtained from $TM$ by identifying nonzero vectors which differ from each other by a positive multiplicative factor. Each geometrical quantity on $TM$, homogeneous of degree zero, is considered to sit on $SM$. Define
\[ \omega := F_y^j dx^j. \]

Then $\omega$ is a differential form on $SM$. It is easy to verify that
\[ \omega \wedge (d\omega)^{n-1} \neq 0, \quad n = \dim M \]

(cf. \cite{3}), i.e., that $\omega$ defines a contact structure on $SM$. This form $\omega$ is known in the calculus of variations as the Hilbert form.

Since $\omega$ is a contact form, there exists a unique vector field $X$ on $SM$ that satisfies
\[ \omega(X) = 1, \quad X_\omega(d\omega) = 0. \]

This vector field $X$ is known as the Reeb vector field \cite{4}. It is easy to see that a $C^\infty$-curve is a (constant Finslerian speed) geodesic if its canonical lift in $SM$ is an integral curve of the Reeb vector field $X$ \cite{4}.

Every vector $y \in T_x M \setminus \{0\}$ uniquely determines a covector $p \in T^*_x M \setminus \{0\}$ by
\[ p(u) := \frac{1}{2} \frac{d}{dt} \left. (F^2(x, y + tu)) \right|_{t=0}, \quad u \in T_x M. \]

The resulting map $L^F_x : y \in T_x M \to p \in T^*_x M$ is called the Legendre transformation at $x$. The family $L^F := \{ L^F_x \mid x \in M \}$ is called the Legendre transformation.
Define a non-negative scalar function \( H = H(x, p) \) by

\[
(2.1) \quad H(x, p) := \max_{y \in T_x M \setminus \{0\}} \frac{p(y)}{F(x, y)}.
\]

The function \( H \) is \( C^\infty \) on \( T^* M \setminus \{0\} \) and \( H_x := H|_{T_x^* M} \) is a Minkowski norm on \( T_x^* M \) for \( x \in M \). Such a function is called a Cartan metric \([12, 15]\) (co-Finsler metric in an alternative terminology \([18, 19]\)). The pair \((M, H)\) is called a Cartan manifold.

Every covector \( p \in T_x^* M \setminus \{0\} \) uniquely determines a vector \( y \in T_x M \) by

\[
q(y) := \frac{1}{2} \frac{d}{dt} (H^2(x, p + tq)) |_{t=0}, \quad q \in T_x^* M.
\]

The resulting map \( L_x^F : p \in T_x^* M \to y \in T_x M \) is called the inverse Legendre transformation at \( x \). Indeed \( L^F_x \) and \( L_x^{F*} \) are inverses of each other. Moreover, they preserve the Minkowski norms

\[
H(x, p) = F(x, (L_x^F)^{-1} p).
\]

Recently, one of the important approaches in discussing the Finsler metric is the (Zermelo) navigation problem. For instance, Bao-Robles-Shen have classified Randers metrics of constant flag curvature via the navigation problem in a Riemannian manifold \([3]\).

The main technique of the navigation problem is described as follows. Given a Finsler metric \( F \) and a vector field \( V \) with \( F(x, V_x) < 1 \), define a new Finsler metric \( \tilde{F} \) by

\[
F(x, \frac{y}{F(x, y)} + V_x) = 1, \quad \forall x \in M, \; y \in T_x M.
\]

A (local) flow (a local one-parameter group in an alternative terminology) on a manifold \( M \) is a map \( \phi : (-\epsilon, \epsilon) \times M \to M \), also denoted by \( \phi_t := \phi(t, \cdot) \), satisfying

- \( \phi_0 = \text{id} : M \to M; \)
- \( \phi_s \circ \phi_t = \phi_{s+t} \) for any \( s, t \in (-\epsilon, \epsilon) \) with \( s + t \in (-\epsilon, \epsilon) \).

Hence, the lift of a flow \( \phi_t \) on \( M \) is a flow \( \hat{\phi}_t \) on \( T^* M \),

\[
(2.2) \quad \hat{\phi}_t(x, p) := (\phi_t(x), (\phi_t^*)_t^{-1}(p)).
\]

By the relationship between vector fields and flows, \((2.2)\) induces a natural way to lift a vector field \( V \) on \( M \) to a vector field \( X^*_V \) on \( T^* M \).

A vector field \( V \) on a Finsler manifold \((M, F)\) is called homothetic with dilation \( c \) if its flow \( \hat{\phi}_t \) satisfies

\[
(2.3) \quad F(\phi_t(x), \phi_t(y)) = e^{2ct} F(x, y), \quad \forall x \in M, \; y \in T_x M.
\]

Similarly, a vector field \( V \) on a Cartan manifold \((M, H)\) is called homothetic with dilation \( c \) if its flow \( \hat{\phi}_t \) satisfies

\[
(2.4) \quad H(\phi_t(x), (\phi_t^*)_t^{-1}(p)) = e^{-2ct} H(x, p), \quad \forall x \in M, \; p \in T_x^* M.
\]

**Lemma 2.2.** Let \( V \) be a homothetic field on a Finsler manifold \((M, F)\) with dilation \( c \) and \( H \) its Cartan metric defined by \((2.1)\). Then \( V \) is a homothetic field of \( H \) with dilation \( c \).
Proof. By using (2.1) and (2.3) we have
\[
H(\phi_t(x), (\phi_t^*)^{-1}(p)) = \max_{\tilde{y} \in T_{\phi_t}(x) \setminus \{0\}} \left[ (\phi_t^*)^{-1}(p) \right](\tilde{y}) \frac{F(\phi_t(x), \tilde{y})}{p((\phi_t^*)^{-1}(\tilde{y}))}
\]
\[
= \max_{\tilde{y} \in T_{\phi_t}(x) \setminus \{0\}} \left[ (\phi_t^*)^{-1}(p) \right] F(\phi_t(x), \tilde{y}) \frac{p(y)}{p(y)}
\]
\[
= \max_{y \in T_{x} \setminus \{0\}} \frac{p(y)}{e^{2ct}F(x, y)} = e^{-2ct} \max_{y \in T_{x} \setminus \{0\}} \frac{p(y)}{F(x, y)} = e^{-2ct} H(x, p),
\]
where \( y := (\phi_t^*)^{-1}(\tilde{y}) \). It follows that \( V \) is a homothetic field of \( H \) with dilation \( c \). □

3. GEODESICS OF A FINSLER METRIC VIA NAVIGATION PROBLEM

In this section, we give a geometric description of geodesics of Finsler metrics via the homothetic navigation problem in a Finsler manifold. First, we show the following:

**Lemma 3.1.** Let \( N \) be a manifold, and let \( V \) and \( W \) be vector fields on \( N \) that satisfy
\[
[V, W] = -c V
\]
for some constant \( c \). Let \( \phi_t \) and \( \psi_t \) be local 1-parameter groups of \( V \) and \( W \) respectively. Then \( \psi_t \circ \phi_{a(t)} \) is a local 1-parameter group of the vector field \( V + W \), where \( a(t) \) is defined by
\[
a(t) = \begin{cases} 
\frac{e^{ct} - 1}{c}, & \text{if } c \neq 0; \\
t, & \text{if } c = 0.
\end{cases}
\]

**Proof.** Direct calculations yield
\[
\frac{da}{dt} = e^{ct}.
\]
Since \( V \) is the induced vector field from \( \phi \),
\[
\frac{d}{dt} \phi_t(x)|_{t=t_0} = V_{\phi_{a(t)}}(x).
\]
Let \( \eta_t := \phi_{a(t)}, u := a(t) \). By using (3.2) and (3.3) we obtain
\[
\frac{d}{dt} [\eta_t(x)]_{t=s} = \frac{d}{du} |_{u=a(s)} \frac{da}{dt} |_{t=s} = e^{cs} V_{\phi_{a(s)}}(x) = e^{cs} V_{\eta_t(x)}.
\]
From (3.1), one obtains \( [W, V] = c V \). It follows that
\[
c \psi_{t*} V = \psi_{t*} (c V)
\]
\[
= \psi_{t*} [W, V]
\]
\[
= [\psi_{t*} W, \psi_{t*} V] = [W, \psi_{t*} V].
\]
This gives
\[ c\psi_t^* V_p(f) = [W, \psi_t^* V]_p f \]
\[ = ([LW(\psi_t^* V)]_p f \]
\[ = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\psi_t^* V - (\psi_t + \Delta t)_* V]_p f = -\frac{d}{dt} [\psi_t^* V]_p f \]
for a point \( p \in N \) and a function \( f \in C^\infty(N) \). We set
\[ y(t) := \psi_t^* V_p(f). \]
Substituting (3.7) into (3.6) yields
\[ dy \]
\[ \frac{dy}{dt} = -cy. \]
Solving (3.8), we get
\[ y = C_1 e^{-ct}. \]
Plugging (3.7) into (3.9) yields
\[ (\psi_t)_* V = e^{-ct} V. \]
By using (3.11), for any function \( f \in C^\infty(N) \) we have
\[ \frac{d}{dt} [\psi_t \circ \eta_t(x)]_{t=s} f = \lim_{\Delta s \to 0} \frac{1}{\Delta s} [f \circ \psi_{s+\Delta s} \circ \eta_{s+\Delta s}(x) - f \circ \psi_s \circ \eta_s(x)] \]
\[ = \lim_{\Delta s \to 0} \frac{1}{\Delta s} [f \circ \psi_{s+\Delta s} \circ \eta_{s+\Delta s}(x) - f \circ \psi_s \circ \eta_s(x)] \]
\[ + \lim_{\Delta s \to 0} \frac{1}{\Delta s} [f \circ \psi_s \circ \eta_{s+\Delta s}(x) - f \circ \psi_s \circ \eta_s(x)] \]
\[ = \lim_{\Delta s \to 0} \frac{f \circ \psi_{s+\Delta s} - f \circ \psi_s}{\Delta s} (\eta_{s+\Delta s}(x)) \]
\[ + (f \circ \psi_s)^* \left( \lim_{\Delta s \to 0} \frac{\eta_{s+\Delta s}(x) - \eta_s(x)}{\Delta s} \right) \]
\[ = W_{\psi_s \circ \eta_s(x)} f + \psi_s^* e^{cs} V|_{\eta_s(x)} f. \]
It follows that
\[ \frac{d}{dt} [\psi_t \circ \eta_t(x)]_{t=s} = W_{\psi_s \circ \eta_s(x)} + \psi_s^* e^{cs} V|_{\eta_s(x)}. \]
By using (3.11) we have
\[ \psi_s^* (e^{cs} V|_{\eta_s(x)}) = e^{cs} \psi_s^* (V|_{\eta_s(x)}) = e^{cs} e^{-cs} V_{\psi_s \circ \eta_s(x)} = V_{\psi_s \circ \eta_s(x)}. \]
Plugging this into (3.12) yields
\[ \frac{d}{dt} [\psi_t \circ \eta_t(x)]_{t=s} = W_{\psi_s \circ \eta_s(x)} + V_{\psi_s \circ \eta_s(x)} = (W + V)_{\psi_s \circ \eta_s(x)}. \]
It follows that \( \psi_t \circ \eta_t \) is a local 1-parameter group of the vector field \( V + W \).
Lemma 3.2. For a homothetic field $V$ on a Cartan manifold $(M, H)$ with dilation $c$, we have

$$[X^b, X^b_V] = 2cX^b,$$

where $X^b = (L^F)_* X$ and $X^b_V$ is the lift of $V$ to $T^* M$.

Proof. In natural coordinates, we have

$$X^b_V = v^i \frac{\partial}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial p_i},$$

where $V = v^i \frac{\partial}{\partial x^i}$ [15 (5.3)]. Note that $V$ is homothetic with respect to $H$ with dilation $c$. Differentiating (2.4) with respect to $t$ at $t = 0$ yields

$$X^b_V(H) = -2cH.$$

By using (3.14), we have

$$X^b_V(H) = v^i \frac{\partial H}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial H}{\partial p_i}.$$

It follows that

$$v^i \frac{\partial H}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial H}{\partial p_i} = -2cH.$$

Differentiating (3.16) with respect to $p_k$ gives

$$v^i \frac{\partial^2 H}{\partial x^i \partial p_k} = \sum_i \frac{\partial v^k}{\partial x^i} \frac{\partial H}{\partial p_i} - \sum_i \frac{\partial v^j}{\partial x^i} \frac{\partial^2 H}{\partial p_i \partial p_k} = -2c \frac{\partial H}{\partial p_k}.$$

Differentiating (3.17) with respect to $x^k$ yields

$$\frac{\partial v^i}{\partial x^k} \frac{\partial H}{\partial x^i} + v^i \frac{\partial^2 H}{\partial x^i \partial x^k} - p_j \frac{\partial^2 v^j}{\partial x^i \partial x^k} \frac{\partial H}{\partial p_i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial^2 H}{\partial p_i \partial p_k} = -2c \frac{\partial H}{\partial x^k}.$$

By Lemma 4.5 in [15], $X^b$ is the Hamiltonian vector field for $H$. Hence it has the local expression

$$X^b = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}.$$

It follows from (3.14) and (3.19) that

$$[X^b, X^b_V] = \left[ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}, v^j \frac{\partial}{\partial x^j} - p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right]$$

$$= \left[ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}, v^j \frac{\partial}{\partial x^j} \right] - \left[ \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}, v^j \frac{\partial}{\partial x^j} \right]$$

$$- \left[ \frac{\partial H}{\partial p_i} \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j}, p_k \frac{\partial}{\partial x^j} \right] + \left[ \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}, p_k \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial p_j} \right].$$

Recall that

$$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$$
Let \( \tilde{c} \) respectively. By using (3.14) we get for any vector fields \( F \)
\[ X^\flat = \sum_i \frac{\partial V^i}{\partial x^i} \] 
with functions \( f, g \). It follows that
\[
[X^\flat, X_\psi^\flat] = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} \frac{\partial^2 H}{\partial x^j \partial x^k} - v^j \frac{\partial^2 H}{\partial x^j \partial x^k} \frac{\partial}{\partial p_i} + v^j \frac{\partial^2 H}{\partial x^j \partial x^k} \frac{\partial}{\partial p_i} 
- p_k \frac{\partial H}{\partial p_i} \frac{\partial^2 v^k}{\partial x^j \partial x^k} \frac{\partial}{\partial p_j} + p_k \frac{\partial v^k}{\partial x^j} \frac{\partial^2 H}{\partial x^j \partial x^k} \frac{\partial}{\partial p_i} 
+ \frac{\partial H}{\partial v^i} \frac{\partial}{\partial x^j} \frac{\partial v^j}{\partial p_j} - p_k \frac{\partial v^k}{\partial x^j} \frac{\partial^2 H}{\partial x^j \partial x^k} \frac{\partial}{\partial p_i} 
= - \left( v^j \frac{\partial^2 H}{\partial x^j \partial p_k} - \sum_i \frac{\partial v^i}{\partial x^i} \frac{\partial H}{\partial p_i} \right) \frac{\partial}{\partial x^k} 
+ \left( \frac{\partial v^i}{\partial x^i} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial p_j} + v^j \frac{\partial^2 v^j}{\partial x^j \partial x^k} \frac{\partial}{\partial p_k} \right) \frac{\partial}{\partial p_i} 
= 2c \frac{\partial H}{\partial p_k} \frac{\partial}{\partial x^k} - 2c \frac{\partial H}{\partial x^k} \frac{\partial}{\partial p_k} = 2c X^\flat. 
\]

Here we have used (3.17), (3.18) and (3.19).

**Proof of Theorem 1.1.** Let \( V \) be a homothetic field on a Finsler manifold \( (M, F) \) with dilation \( c \) and let \( X \) be the Reeb vector field. Applying the Legendre transformation, we obtain a Cartan metric \( H(x, p) \). From Lemma 2.2, \( V \) is a homothetic field of \( (M, H) \) with dilation \( c \). By Lemma 3.2 and (3.14), we have \( [X^\flat, X^\flat_\psi] = -2cX^\flat \). Let \( \phi_t \) and \( \tilde{\phi}_t \) be local 1-parameter groups of \( X^\flat \) and \( X^\flat_\psi \) respectively. By using (3.14) we get \( X^\flat + X^\flat_\psi = X^\flat - X^\flat_\psi \). Taking this together with Lemma 3.1 we obtain that \( \psi_t \circ \phi_{a(t)} \) is a local 1-parameter group of \( X^\flat - X^\flat_\psi \), where
\[
a(t) = \begin{cases} \frac{e^{ct} - 1}{2c}, & \text{if } c \neq 0; \\
t, & \text{if } c = 0. \end{cases}
\]

Recall that a navigation problem makes use of a Finsler metric \( F \) and a vector field \( V \) with \( F(x, V_x) < 1 \) and produces a new Finsler metric \( \tilde{F} \) by solving the equation
\[
(3.20) \quad F(x, y + \tilde{F}(x, y)V) = \tilde{F}(x, y).
\]

Let \( \tilde{X} \) be the Reeb vector field of \( \tilde{F} \) and \( \tilde{X}^\flat = (L_{\tilde{F}})_* \tilde{X} \). By Lemma 6.2 in [15], we have
\[
\tilde{X}^\flat = X^\flat - X^\flat_\psi.
\]

It follows that \( \psi_t \circ \phi_{a(t)} \) is a local 1-parameter group of \( \tilde{X}^\flat \).

Since \( L_{\tilde{F}}(x, y) = (x, L^F_x(y)) \), we see that any geodesic of \( F \) is precisely the projection of an integral curve of \( X^\flat \). Similarly, a geodesic of \( \tilde{F} \) is precisely the projection of an integral curve of \( \tilde{X}^\flat \). Note that
\[
\hat{\psi}_t(x, p) = (\psi_t(x), (\psi_t^*)^{-1}(p)),
\]
where \( \psi_t \) is the flow produced by \( -V \).

Let \( \pi : T^*M \setminus \{0\} \to M \) be the natural projection. It follows that
\[
\pi \circ \hat{\psi}_t(x, p) = \pi (\psi_t(x), (\psi_t^*)^{-1}(p)) = \psi_t(x) = \psi_t \circ \pi(x, p)
\]

for any \( x \in M \) and \( p \in T_xM \setminus \{0\} \). Hence we have
\[
\pi \circ \hat{\psi}_t = \psi_t \circ \pi.
\]
(3.21)

It follows that
\[
\pi \circ \hat{\psi}_t \circ \phi_{a(t)} = \psi_t \circ \pi \circ \phi_{a(t)} = \psi_t (\gamma(a(t))),
\]
where \( \gamma(t) := \pi(\phi_t(x)) \). Thus we have proved Theorem 1.1.

**Remark 3.1.** The reader should note that the navigation problem adopted here differs from those of C. Robles and Z. Shen [17, 20], where the navigation problem is defined by
\[
F(x, y \tilde{F}) = 1;
\]
i.e., the \( \tilde{F} \) we define with \((F, V)\) is precisely the \( \tilde{F} \) that Shen defines with \((F, -V)\).

### 4. Geodesics of Funk metrics on convex domains

In this section we are going to represent explicitly the geodesics of the Funk metric on a strongly convex domain.

Given a Minkowski norm \( \varphi : \mathbb{E} \to \mathbb{R} \) on a vector space \( \mathbb{E} \), one can construct \( \Omega := \{ v \in \mathbb{E} | \varphi(v) < 1 \} \), \( T_\Omega \mathbb{E} \simeq \mathbb{E} \). A domain \( \Omega \) in \( \mathbb{E} \) defined by a Minkowski norm \( \varphi \) is called a strongly convex domain [19]. Thus \((\Omega, F(x, y))\) is a Minkowski manifold, where \( F(x, y) := \varphi(y) \). For each \( x \in \Omega \), identify \( T_x\Omega \) with \( \mathbb{E} \). Thus \( V_x := x \) is a radical vector field on \( \Omega \) satisfying \( F(x, V_x) = \varphi(x) < 1 \). Moreover \( V \) is a homothetic field of \( F \) with dilation \( c = \frac{1}{2} \) [15 Example 1]. Define a 1-parameter transformation \( \psi_t \) on \( \Omega \) by
\[
\psi_t(x) = e^{-t}x.
\]
Note that \( T_{\psi_t(x)}\Omega \simeq \mathbb{E} \) for any \( t \). It is easy to see that
\[
\psi_t(y) = e^{-t}y, \quad \text{for } \forall y \in T_x\Omega.
\]
Thus we have
\[
F (\psi_t(x), \psi_t(y)) = \varphi(\psi_t(y)) = e^{-t} \varphi(y) = e^{-t}F(x, y)
\]
for any \((x, y) \in T\Omega\). It follows that \( \varphi_t \) is homothetic. A direct calculation yields
\[
\frac{d\psi_t(x)}{dt} \bigg|_{t=0} = -x = -V_x.
\]
Thus \( \psi_t \) is the flow of the vector field \(-V\). By using the Minkowski metric \( F \) and the homothetic field \( V \), we produce a new Finsler metric \( \tilde{F} \) in terms of the navigation problem. \( \tilde{F} \) is called the **Funk metric** on a strongly convex domain \( \Omega \) (cf. [7 Example 3.4.3]).

Since \( F \) is Minkowskian, it is locally projectively flat. This implies that the unit speed geodesic through \( x \) with tangent vector \( y(\neq 0) \) is \( \gamma(t) := x + \frac{t}{\varphi(y)} y \). Note that \( V \) is a homothetic field with dilation \( c = \frac{1}{2} \). We have
\[
a(t) = e^t - 1
\]
(see Theorem 1.1). It follows that

$$\psi_t(\gamma(a(t))) = e^{-t} \left[ x + \frac{e^t - 1}{\varphi(y)} y \right].$$

By Theorem 1.1, we have the following.

**Proposition 4.1.** Let $\varphi : E \to \mathbb{R}$ be a Minkowski norm and $\Omega$ its strongly convex domain. Assume that $\tilde{F}$ is the Funk metric on $\Omega$ defined by

$$\varphi \left( \frac{y}{F(x, y)} + x \right) = 1.$$ 

Then the geodesics of $\tilde{F}$ are given by

$$e^{-t} \left[ x + \frac{e^t - 1}{\varphi(y)} y \right].$$

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