LARGE CHARACTER DEGREES OF SOLVABLE 3'-GROUPS

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Abstract. We prove that if G is a finite solvable group and 3 \n \n \n \n |G:F(G)|, then the index of the Fitting subgroup of G is at most the square of the largest irreducible character degree of G.

1. Introduction

Let G be a finite group and denote by \( b(G) = \max\{\psi(1) \mid \psi \in \text{Irr}(G)\} \) the largest degree of an irreducible character of G. In [5] Gluck proves that in all finite groups the index of the Fitting subgroup \( F(G) \) in G is bounded by a polynomial function of \( b(G) \). For a solvable group, Gluck further shows that \( |G:F(G)| \leq b(G)^{13/2} \) and conjectures that \( |G:F(G)| \leq b(G)^2 \). This has been verified by Espuelas [1] for G of odd order. Espuelas’ result has been extended in [4] to G a solvable group with abelian Sylow 2-subgroups by Dolfi and Jabara. The best general bound \( |G:F(G)| \leq b(G)^3 \) is given by Moretó and Wolf in [6]. In this note we prove Gluck’s conjecture for all solvable groups with order not divisible by 3.

2. Gluck’s conjecture for solvable 3'-groups

Theorem 2.1. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V. By [9, Theorem 2.2], G will have a uniquely determined normal subgroup E which is a direct product of extraspecial \( p \)-groups for various \( p \) and \( e = \sqrt{|E/Z(E)|} \). Assume \( e = 5, 7 \) or \( e \geq 10 \) and \( e \neq 16 \); then G will have at least 5 regular orbits on V.

Proof. This follows from [9, Theorem 3.1] and [10, Theorem 3.1].

Theorem 2.2. Suppose that G is a finite solvable group and V is a faithful, irreducible and quasi-primitive \( \mathbb{F}G \)-module and char(\( \mathbb{F} \)) = r. Assume 3 \n \n \n \n |G|; then G has at least 3 regular orbits on \( V \oplus V \).

Proof. By [9, Theorem 2.2], G will have a uniquely determined normal subgroup E which is a direct product of extraspecial \( p \)-groups for various \( p \) and \( e = \sqrt{|E/Z(E)|} \).

Since 3 \n \n \n \n |G|, 3 \n \n \n \n |e and G will have at least 5 regular orbits on V unless \( e = 1, 2, 4, 8, 16 \) by Theorem 2.1. Since 3 \n \n \n \n |G|, G will have at least \( r \) regular orbits on \( V \oplus V \) by [3, Theorem 3.4]. Assume \( e = 2, 4, 8, 16 \); then \( r \geq 3 \) and G will have at least 3 regular orbits on \( V \oplus V \).

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Thus we may assume $e = 1$ and $r = 2$. As in [7], if $V$ is a finite vector space of dimension $n$ over $\text{GF}(q)$, where $q$ is a prime power, we denote by $\Gamma(q^n) = \Gamma(V)$ the semilinear group of $V$, i.e.,

$$\Gamma(V) = \{ x \mapsto ax^\sigma \mid x \in \text{GF}(q^n), a \in \text{GF}(q^n)^\times, \sigma \in \text{Gal}(\text{GF}(q^n)/\text{GF}(q))\}.$$ 

Since $e = 1$ we have $G \leq \Gamma(V) \leq \Gamma(2^d) \cong G_1$ by [7, Corollary 2.3(b)]. For any $0 \neq v \in V$, $|\mathbf{C}_G(v)| = d$. We can hence assume that $\mathbf{C}_G(v)$ is the Galois group of $V = \text{GF}(2^d)$. So the elements of $V$ that do not belong to a regular orbit of $\mathbf{C}_G(v)$ are in the union of the subfields $\text{GF}(2^{d/m})$, $m$ varying among the prime divisors of $d$. Since the number of distinct prime divisors of $d$ is at most $\log_2(d)$, it is enough to prove that $f(d) = (2^d - 1) - \log_2(d) \cdot (2^{d/2} - 1) - 2d$ is positive. It is not hard to check that $f(d) > 0$ for all $d \geq 4$. Thus we are left with the cases when $d = 1, 2, 3$:

1. Let $d = 1$; then $G \leq \Gamma(2^1)$ and $G$ is trivial. The result is clear.
2. Let $d = 2$; then $G \leq \Gamma(2^2) \cong S_3$. Since $3 \nmid |G|$, $G \not\cong Z_2$ and the result is clear.
3. Let $d = 3$; then $G \leq \Gamma(2^3)$. Since $3 \nmid |G|$, $G \cong Z_7$ and the result is clear.

$\square$

**Theorem 2.3.** Suppose that $G$ is a finite solvable group and $V$ is a faithful and completely reducible $G$-module (possibly of mixed characteristic). Assume $3 \nmid |G|$; then $G$ has at least 3 regular orbits on $V \oplus V$.

**Proof.** We work by induction on $|GV|$.

Assume first that $V = U \oplus W$ with $U$ and $W$ proper $G$-submodules. Then by inductive hypothesis $G/\mathbf{C}_G(U)$ has 3 regular orbits on $U \oplus U$ and $G/\mathbf{C}_G(W)$ has 3 regular orbits on $W \oplus W$. Since $\mathbf{C}_G(U) \cap \mathbf{C}_G(W) = 1$, it follows that $G$ has 3 regular orbits on $U \oplus U \oplus W \oplus W \cong V \oplus V$.

Therefore, we can assume that $V$ is irreducible.

Assume $V$ is quasi-primitive; then the result follows from Theorem 2.2.

Now we assume that $V$ is not quasi-primitive, then there exists $N$ normal in $G$ such that $V_N = V_1 \oplus \cdots \oplus V_m$ for $m > 1$ homogeneous components $V_i$ of $V_N$. If $N$ is maximal with this property, then $S = G/N$ primitivey permutes the $V_i$. Also $V = V_1^G$, induced from $\mathbf{N}_G(V_1)$. If $H = \mathbf{N}_G(V_1)/\mathbf{C}_G(V_1)$, then $H$ acts faithfully and irreducibly on $V_1$ and $G$ is isomorphic to a subgroup of $H \cdot S$.

By induction $H$ will have at least 3 regular orbits on $V_1 \oplus V_1$. $S$ is a solvable primitive permutation group on $\Omega = \{ V_1, \ldots, V_m \}$. By [8, Proposition 3.2(2)], $G$ will have at least 5 regular orbits on $V \oplus V$ unless $m \leq 4$. Since $3 \nmid |S|$, the only case left is when $|\Omega| = 2$ and $S \cong S_2$. In this case $G$ will have at least 3 regular orbits on $V \oplus V$. $\square$

**Corollary 2.4.** Suppose that $G$ is a finite solvable group and $V$ is a faithful and completely reducible $G$-module (possibly of mixed characteristic). Assume $3 \nmid |G|$; then there exists $v \in V$ such that $|\mathbf{C}_G(v)| \leq \sqrt{|G|}$.

**Proof.** By Theorem 2.3, there is an element $(v, u) \in V \oplus V$ such that $\mathbf{C}_G((v, u)) = \mathbf{C}_G(v) \cap \mathbf{C}_G(u) = 1$. Thus, we have

$$|\mathbf{C}_G(v)| \cdot |\mathbf{C}_G(u)| = \frac{|\mathbf{C}_G(v)|}{|\mathbf{C}_G(v) \cap \mathbf{C}_G(u)|} = |\mathbf{C}_G(v)| |\mathbf{C}_G(u)| \leq |G|.$$

It follows that either $|\mathbf{C}_G(v)| \leq \sqrt{|G|}$ or $|\mathbf{C}_G(u)| \leq \sqrt{|G|}$. $\square$
Theorem 2.5. Let $G$ be a finite solvable group and $3 
mid |G : F(G)|$. Then $|G : F(G)| \leq b(G)^2$.

Proof. Let $U = F(G)/\Phi(G)$ and $\overline{G} = G/F(G)$. $U$ is a faithful and completely reducible $\overline{G}$-module by Gaschütz's theorem [7, Theorem 1.12]. Let $V = \text{Irr}(F(G)/\Phi(G))$. $V$ is a faithful and completely reducible $\overline{G}$-module by [7, Proposition 12.1]. By Corollary 2.4, there exists $\lambda \in V$ such that $\overline{I} = I_{\overline{G}}(\lambda) = \{g \in \overline{G} | \lambda g = \lambda\}$ satisfies $|\overline{I}| \leq |\overline{G}|^{1/2}$. Consider $\lambda$ as a character of $F(G)$ with a kernel containing $\Phi(G)$. Let $I$ be the preimage of $\overline{I}$ in $G$. Now $I = I_G(\lambda) = \{g \in G | \lambda g = \lambda\}$. Take $\mu \in \text{Irr}(I|\lambda)$. Now $ψ = I^G \in \text{Irr}(G)$. Thus we have $|G : F(G)| = |G : I||I : F(G)| \leq |G : I|^2 \leq ψ(1)^2 \leq b(G)^2$. □

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References


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