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LARGE CHARACTER DEGREES OF SOLVABLE $3'$-GROUPS

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Abstract. We prove that if G is a finite solvable group and $3 \nmid |G : \mathbf{F}(G)|$, then the index of the Fitting subgroup of G is at most the square of the largest irreducible character degree of G.

1. Introduction

Let G be a finite group and denote by $b(G) = \max\{\psi(1) \mid \psi \in \mathrm{Irr}(G)\}$ the largest degree of an irreducible character of G. In [5] Gluck proves that in all finite groups the index of the Fitting subgroup $\mathbf{F}(G)$ in G is bounded by a polynomial function of $b(G)$. For a solvable group, Gluck further shows that $|G : \mathbf{F}(G)| \leq b(G)^{13/2}$ and conjectures that $|G : \mathbf{F}(G)| \leq b(G)^2$. This has been verified by Espuelas [1] for G of odd order. Espuelas' result has been extended in [4] to G a solvable group with abelian Sylow 2-subgroups by Dolfi and Jabara. The best general bound $|G : \mathbf{F}(G)| \leq b(G)^3$ is given by Moretó and Wolf in [6]. In this note we prove Gluck's conjecture for all solvable groups with order not divisible by 3.

2. Gluck's conjecture for solvable $3'$-groups

Theorem 2.1. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V. By* [9, Theorem 2.2]*, G will have a uniquely determined normal subgroup E which is a direct product of extraspecial p-groups for various p and $e = \sqrt{|E/\mathbf{Z}(E)|}$. Assume $e = 5, 7$ or $e \geq 10$ and $e \neq 16$; then G will have at least 5 regular orbits on V.*

Proof. This follows from [9, Theorem 3.1] and [10, Theorem 3.1]. □

Theorem 2.2. *Suppose that G is a finite solvable group and V is a faithful, irreducible and quasi-primitive $\mathbb{F}G$-module and* $\mathrm{char}(\mathbb{F}) = r$. *Assume $3 \nmid |G|$; then G has at least 3 regular orbits on $V \oplus V$.*

Proof. By [9, Theorem 2.2], G will have a uniquely determined normal subgroup E which is a direct product of extraspecial p-groups for various p and $e = \sqrt{|E/\mathbf{Z}(E)|}$.

Since $3 \nmid |G|$, $3 \nmid e$ and G will have at least 5 regular orbits on V unless $e = 1, 2, 4, 8, 16$ by Theorem 2.1. Since $3 \nmid |G|$, G will have at least r regular orbits on $V \oplus V$ by [3, Theorem 3.4]. Assume $e = 2, 4, 8, 16$; then $r \geq 3$ and G will have at least 3 regular orbits on $V \oplus V$.

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Thus we may assume $e = 1$ and $r = 2$. As in [7], if V is a finite vector space of dimension n over $\mathrm{GF}(q)$, where q is a prime power, we denote by $\Gamma(q^n) = \Gamma(V)$ the semilinear group of V, i.e.,

$$\Gamma(V) = \{x \mapsto ax^\sigma \mid x \in \mathrm{GF}(q^n), a \in \mathrm{GF}(q^n)^\times, \sigma \in \mathrm{Gal}(\mathrm{GF}(q^n)/\mathrm{GF}(q))\}.$$

Since $e = 1$ we have $G \leq \Gamma(V) \leq \Gamma(2^d) \cong G_1$ by [7, Corollary 2.3(b)]. For any $0 \neq v \in V$, $|\mathbf{C}_{G_1}(v)| = d$. We can hence assume that $\mathbf{C}_{G_1}(v)$ is the Galois group of $V = \mathrm{GF}(2^d)$. So the elements of V that do not belong to a regular orbit of $\mathbf{C}_{G_1}(v)$ are in the union of the subfields $\mathrm{GF}(2^{d/m})$, m varying among the prime divisors of d. Since the number of distinct prime divisors of d is at most $\log_2(d)$, it is enough to prove that $f(d) = (2^d - 1) - \log_2(d) \cdot (2^{d/2} - 1) - 2d$ is positive. It is not hard to check that $f(d) > 0$ for all $d \geq 4$. Thus we are left with the cases when $d = 1, 2, 3$:

(1) Let $d = 1$; then $G \leq \Gamma(2^1)$ and G is trivial. The result is clear.
(2) Let $d = 2$; then $G \leq \Gamma(2^2) \cong S_3$. Since $3 \nmid |G|$, $G \lesssim Z_2$ and the result is clear.
(3) Let $d = 3$; then $G \leq \Gamma(2^3)$. Since $3 \nmid |G|$, $G \lesssim Z_7$ and the result is clear.

□

Theorem 2.3. *Suppose that G is a finite solvable group and V is a faithful and completely reducible G-module (possibly of mixed characteristic). Assume $3 \nmid |G|$; then G has at least 3 regular orbits on $V \oplus V$.*

Proof. We work by induction on $|GV|$.

Assume first that $V = U \oplus W$ with U and W proper G-submodules. Then by inductive hypothesis $G/\mathbf{C}_G(U)$ has 3 regular orbits on $U \oplus U$ and $G/\mathbf{C}_G(W)$ has 3 regular orbits on $W \oplus W$. Since $\mathbf{C}_G(U) \cap \mathbf{C}_G(W) = 1$, it follows that G has 3 regular orbits on $U \oplus U \oplus W \oplus W \cong V \oplus V$.

Therefore, we can assume that V is irreducible.

Assume V is quasi-primitive; then the result follows from Theorem 2.2.

Now we assume that V is not quasi-primitive, then there exists N normal in G such that $V_N = V_1 \oplus \cdots \oplus V_m$ for $m > 1$ homogeneous components V_i of V_N. If N is maximal with this property, then $S = G/N$ primitively permutes the V_i. Also $V = V_1^G$, induced from $\mathbf{N}_G(V_1)$. If $H = \mathbf{N}_G(V_1)/\mathbf{C}_G(V_1)$, then H acts faithfully and irreducibly on V_1 and G is isomorphic to a subgroup of $H \wr S$.

By induction H will have at least 3 regular orbits on $V_1 \oplus V_1$. S is a solvable primitive permutation group on $\Omega = \{V_1, \ldots, V_m\}$. By [8, Proposition 3.2(2)], G will have at least 5 regular orbits on $V \oplus V$ unless $m \leq 4$. Since $3 \nmid |S|$, the only case left is when $|\Omega| = 2$ and $S \cong S_2$. In this case G will have at least 3 regular orbits on $V \oplus V$. □

Corollary 2.4. *Suppose that G is a finite solvable group and V is a faithful and completely reducible G-module (possibly of mixed characteristic). Assume $3 \nmid |G|$; then there exists $v \in V$ such that $|\mathbf{C}_G(v)| \leq \sqrt{|G|}$.*

Proof. By Theorem 2.3, there is an element $(v, u) \in V \oplus V$ such that $\mathbf{C}_G((v, u)) = \mathbf{C}_G(v) \cap \mathbf{C}_G(u) = 1$. Thus, we have

$$|\mathbf{C}_G(v)| \cdot |\mathbf{C}_G(u)| = \frac{|\mathbf{C}_G(v)| \cdot |\mathbf{C}_G(u)|}{|\mathbf{C}_G(v) \cap \mathbf{C}_G(u)|} = |\mathbf{C}_G(v)\mathbf{C}_G(u)| \leq |G|.$$

It follows that either $|\mathbf{C}_G(v)| \leq \sqrt{|G|}$ or $|\mathbf{C}_G(u)| \leq \sqrt{|G|}$. □

Theorem 2.5. *Let G be a finite solvable group and $3 \nmid |G : \mathbf{F}(G)|$. Then $|G : \mathbf{F}(G)| \leq b(G)^2$.*

Proof. Let $U = \mathbf{F}(G)/\Phi(G))$ and $\overline{G} = G/\mathbf{F}(G)$. U is a faithful and completely reducible $\overline{G}$-module by Gaschütz's theorem [7, Theorem 1.12]. Let $V = \mathrm{Irr}(\mathbf{F}(G)/\Phi(G))$. V is a faithful and completely reducible $\overline{G}$-module by [7, Proposition 12.1]. By Corollary 2.4, there exists $\lambda \in V$ such that $\bar{I} = I_{\bar{G}}(\lambda) = \{\bar{g} \in \bar{G} \,|\lambda^{\bar{g}} = \lambda\}$ satisfies $|\bar{I}| \leq |\overline{G}|^{1/2}$. Consider λ as a character of $\mathbf{F}(G)$ with a kernel containing $\Phi(G)$. Let I be the preimage of $\bar{I}$ in G. Now $I = I_G(\lambda) = \{g \in G \,|\lambda^g = \lambda\}$. Take $\mu \in \mathrm{Irr}(I|\lambda)$. Now $\psi = \mu^G \in \mathrm{Irr}(G)$. Thus we have $|G : \mathbf{F}(G)| = |G : I||I : \mathbf{F}(G)| \leq |G : I|^2 \leq \psi(1)^2 \leq b(G)^2$. □

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