PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 9, September 2011, Pages 3217–3218
S 0002-9939(2011)10788-3
Article electronically published on March 2, 2011

SEPARABLE SUBGROUPS HAVE BOUNDED PACKING

WEN-YUAN YANG

(Communicated by Alexander N. Dranishnikov)

Abstract. In this paper, we prove that separable subgroups have bounded packing in ambient groups. The notion bounded packing was introduced by Hruska and Wise, and, in particular, our result confirms a conjecture of theirs which states that each subgroup of a virtually polycyclic group has the bounded packing property.

1. Introduction

Bounded packing was introduced for a subgroup of a countable group in Hruska-Wise [3]. Roughly speaking, this property gives a finite upper bound on the number of left cosets of the subgroup that are pairwise close in G. Precisely,

Definition. Let G be a countable group with a left invariant proper metric d. A subgroup H has *bounded packing* in G (with respect to d) if for each positive constant D, there is a natural number $N = N(G, H, D)$, such that for any collection $\mathcal{C}$ of N left H-cosets in G, there exist at least two H-cosets $gH, g'H \in \mathcal{C}$ satisfying $d(gH, g'H) > D$.

Remark. Bounded packing of a subgroup is independent of the choice of the left invariant proper metric d. Equivalently, bounded packing says that for each positive constant D, every collection of left H-cosets in G with pairwise distance at most D has a uniform bound $N = N(G, H, D)$ on their cardinality.

This paper aims to give a proof of the following.

Theorem. *If H is a separable subgroup of a countable group G, then H has bounded packing in G.*

A subgroup H of a group G is *separable* if H is an intersection of finite index subgroups of G. A group is called *subgroup separable* or *LERF* if every finitely generated subgroup is separable. For example, Hall showed that free groups are LERF in [1]. It follows from a theorem of Mal'cev [4] that polycyclic (and in particular finitely generated nilpotent) groups are LERF. A group is called *slender* if every subgroup is finitely generated. Polycyclic groups are also slender by a result

Received by the editors August 24, 2010.
2010 *Mathematics Subject Classification.* Primary 20F65, 20F67.
Key words and phrases. Bounded packing, separable, polycyclic group.
The author is supported by the China-funded Postgraduates Studying Abroad Program for Building Top University. This research was supported by the National Natural Science Foundation of China (No. 11081059).

of Hirsch [2]. Therefore we have the following corollary, which gives a positive answer to [3, Conjecture 2.14].

Corollary. *Let P be virtually polycyclic. Then each subgroup of P has bounded packing in P.*

2. Proof of the Theorem

We define the norm $|g|_d$ of an element $g \in G$ as the distance $d(1, g)$.

Proof of the Theorem. By the definition of bounded packing, it suffices to show, for each positive constant D, that there is a uniform bound on the cardinality of every collection of left H-cosets in G with pairwise distance at most D.

Given such a collection $\mathcal{A}$ satisfying $d(gH, g'H) < D$ for any gH, $g'H \in \mathcal{A}$. Without loss of generality, we can assume H belongs to $\mathcal{A}$, up to a translation of $\mathcal{A}$ by an appropriate element of G. Since $d(H, gH) < D$ for each $gH \in \mathcal{A}$, there exists an element h in H such that $d(1, hgH) < D$. Hence we conclude that the collection $\mathcal{A} \setminus \{H\}$ lies in the finite union of double cosets HgH with $|g|_d < D$ and $g \in G \setminus H$.

Since d is a left invariant proper metric on G, the set $F = \{g \in G \setminus H : |g|_d < D\}$ is finite. Since H is separable in G, we can take a finite index subgroup K of G such that $H < K$ and $F \subset G \setminus K$.

We claim that no two different left H-cosets of $\mathcal{A}$ lie in the same left K-coset. By way of contradiction, we suppose that there are two H-cosets gkH, $gk'H \in \mathcal{A}$ in the same coset gK such that $d(gkH, gk'H) < D$. By a similar argument as above, we get that $k^{-1}k'H$ belongs to a double coset Hg_0H with $|g_0|_d < D$. Moreover, we note that $g_0 \in F$. Since we have $k^{-1}k'H = hg_0H$ for some $h \in H$, it is easy to see that g_0 belongs to K. But by the choice of K, we know that g_0 belongs to $G \setminus K$. This is a contradiction. Our claim is proved.

Since K is of finite index in G, the cardinality of each $\mathcal{A}$ is upper bounded by $[G : K]$. Thus for each D, we have obtained a uniform bound on every $\mathcal{A}$. Hence H has bounded packing in G. □

Acknowledgement

The author would like to sincerely thank Professor Leonid Potyagailo for his comments and interest in this work.

References

1. M. Hall, *Coset representations in free groups.* Trans. Amer. Math. Soc., **67** (1949), 421–432. MR0032642 (11:322e)
2. K. A. Hirsch, *On infinite soluble groups. I.* Proc. London Math. Soc., **s2-44(1)** (1938), 53–60.
3. G. Hruska and D. Wise, *Packing subgroups in relatively hyperbolic groups.* Geom. Topol., **13(4)** (2009), 1945–1988. MR2497315 (2010c:20053)
4. A. I. Mal'cev, *On homomorphisms onto finite groups.* American Mathematical Society Translations, Ser. 2, vol. 119, 1983, 67–79.

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, People's Republic of China

Current address: U.F.R. de Mathematiques, Universite de Lille 1, 59655 Villeneuve D'Ascq Cedex, France

E-mail address: wyang@math.univ-lille1.fr