RESTRICTION OF HYPERGEOMETRIC $\mathcal{D}$-MODULES
WITH RESPECT TO COORDINATE SUBSPACES

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Abstract. We compute the restriction of an $A$-hypergeometric $\mathcal{D}$-module with respect to a coordinate subspace under certain genericity conditions on the parameter.

1. Introduction

Let $A \in \mathbb{Z}^{d \times n}$ be a matrix of rank $d$ with integer entries $a_{ij}$, $i = 1, \ldots, d$, $j = 1, \ldots, n$. For $u \in \mathbb{Z}^n$, let $u_+, u_- \in \mathbb{N}^n$ be such that $u = u_+ - u_-$, and write $\square_u$ for the element $\partial^{u_+} - \partial^{u_-}$ of $R_A = \mathbb{C}[\partial_1, \ldots, \partial_n]$. Here and elsewhere we use multi-index notation: $\partial^v = \prod_{i=1}^n v_i$ for $v \in \mathbb{N}^n$. The toric ideal associated with $A$ is the prime binomial ideal

$$I_A = \langle \square_u : u \in \mathbb{Z}^n, Au = 0 \rangle \subseteq R_A.$$ 

Identifying $\partial_j$ with the partial derivation operator $\frac{\partial}{\partial x_j}$, let $D \supseteq R_A$ be the Weyl algebra of order $n$, i.e. the $\mathbb{C}$-algebra of linear partial differential operators with coefficients in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Following [GGZ87] and [GZK89], the hypergeometric ideal associated with a matrix $A$ as above and a vector of complex parameters $\beta \in \mathbb{C}^d$ is the left ideal

$$H_A(\beta) := DI_A + D(E_A - \beta) \subseteq D,$$

where $E_A - \beta$ is the sequence of Euler operators

$$E_i - \beta_i = \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i, \quad i = 1, \ldots, d.$$ 

The (global) hypergeometric $\mathcal{D}$-module associated with $(A, \beta)$ is by definition the quotient

$$M_A(\beta) := D/H_A(\beta).$$

Let $\mathcal{D}$ be the sheaf of linear partial differential operators with holomorphic coefficients in $\mathbb{C}^n$. To the pair $(A, \beta)$ one may associate the corresponding analytic
hypergeometric $\mathcal{D}$-module, denoted by $\mathcal{M}_A(\beta)$, which is the quotient of $\mathcal{D}$ modulo the left $\mathcal{D}$-ideal generated by $H_A(\beta)$.

The restriction functor is a useful tool for the study of the irregularity of a holonomic $\mathcal{D}$-module $\mathcal{M}$ (see, for example, the Cauchy–Kovalevskaya theorem for Gevrey series [LM02 Cor. 2.2.4]). There is an algorithm due to Oaku–Takayama (see Oak97 OT01) for the effective computation of the restriction of holonomic $D$-modules to linear subspaces by means of Gröbner basis calculations in the Weyl algebra. This algorithm is employed in the proof of the explicit restriction formulas in [CJT03 Thm. 4.4] and [FFCJ11 Thm. 4.2] that hold for certain classes of hypergeometric systems. The purpose of our paper is, for a sufficiently general parameter $\beta \in \mathbb{C}^d$, to generalize these formulas to the case of arbitrary $A$ by using the Euler–Koszul functor developed in [MMW05]. We also show by example that there are parameters for which our formula does not hold (see Example 2.8).

2. Explicit restriction formula for $\mathcal{M}_A(\beta)$

**Notation 2.1.** We denote by $a_j \in \mathbb{Z}^d$ the $j$-th column of $A$, $j = 1, \ldots, n$. For any subset $\tau \subseteq \{1, \ldots, n\}$ we shall write $x_\tau = (x_i)_{i \in \tau}$, $\partial_\tau = (\partial_i)_{i \in \tau}$, $R_\tau = \mathbb{C}[\partial_\tau]$, and $D_\tau = \mathbb{C}[x_\tau][\partial_\tau]$. If $I_\tau$ is the toric ideal associated with the submatrix $A_\tau$ consisting of the columns indexed by $\tau$, then $R_\tau/I_\tau = S_\tau$ is isomorphic to the semigroup ring $S_\tau = \mathbb{C}[x^{a_i}_i : i \in \tau] \subseteq \mathbb{C}[t_1^{\geq 1}, \ldots, t_d^{\geq 1}]$.

Consider for $Y_\tau = \{x_i = 0 : i \notin \tau\}$ the natural inclusion

$$i_\tau : Y_\tau \hookrightarrow X = \mathbb{C}^n.$$

We say that $\beta \in \mathbb{C}^d$ is *generic* if it is outside a hyperplane arrangement depending on $A$, and that $\beta$ is *very generic* if it is outside a locally finite arrangement of countably many hyperplanes.

With this notation we will prove:

**Theorem 2.2.** Suppose one of the following conditions holds:

(i) $\beta \in \mathbb{C}^d$ is generic and $A_\tau = \mathbb{Q}_{\geq 0}A$ (i.e., the real positive cones spanned by the columns of $A$ and $A_\tau$ respectively agree);

(ii) $\beta \in \mathbb{C}^d$ is very generic and $\text{rank}(A_\tau) = d$.

Choose a collection $\Omega \subseteq N\mathfrak{A}$ of coset representatives for $\mathbb{Z}\mathfrak{A}/\mathbb{Z}\tau$. Then the (derived) restriction of $\mathcal{M}_A(\beta)$ with respect to $Y_\tau$ is given by

$$(2.1) \quad L^{-k}i^*_{\mathcal{T}}\mathcal{M}_A(\beta) \simeq \begin{cases} \bigoplus_{\lambda \in \Omega} \mathcal{M}_{A_\tau}(\beta - \lambda) & \text{if } k = 0; \\ 0 & \text{if } k > 0. \end{cases}$$

**Proof.** The left multiplications by $x_i$, $i \notin \tau$, form commuting endomorphisms of left $\mathcal{T}_\tau$-modules (but not of left $\mathcal{D}_A$-modules). By [MT04 Prop. 3.1], $Li^*_{\mathcal{T}}\mathcal{M}_A(\beta)$ is quasi-isomorphic to the Koszul complex $K_\bullet(x_\tau : i \notin \tau; \mathcal{M}_A(\beta))$ as a complex of left $\mathcal{T}_\tau$-modules. Moreover, because of the flatness of the fibers of $\mathcal{D}_A$ (resp. $\mathcal{T}_\tau$) over their algebraic counterparts and since affine spaces are $D$-affine, it suffices to prove Theorem 2.2 using the global sections $D_A$ (resp. $D_\tau$) instead of the sheaves $\mathcal{D}_A$ (resp. $\mathcal{T}_\tau$).

Define a $\mathbb{Z}^d$-grading on $R_A \subseteq D_A$ by

$$\deg(\partial_i) = -a_i = -\deg(x_i), \quad i = 1, \ldots, n.$$
For any \( \mathbb{Z}^d \)-graded right \( \mathcal{R} \)-module \( N \), the Euler–Koszul complex \( K_\bullet(E_\mathcal{A} - \beta, N) \) (see [MMW05 Def. 4.2]) is the Koszul complex of left \( D_\mathcal{A} \)-modules defined by the sequence \( E_\mathcal{A} - \beta \) of commuting endomorphisms on the left \( D_\mathcal{A} \)-module \( D_\mathcal{A} \otimes_{\mathcal{R}} N \). This complex is concentrated in homological degrees \( d \) to \( 0 \). The \( i \)-th Euler–Koszul homology of \( N \) is

\[
\mathcal{H}_i(E_\mathcal{A} - \beta; N) = \mathcal{H}_i(K_\bullet(E_\mathcal{A} - \beta, N)).
\]

By [MMW05 Theorem 6.6], \( K_\bullet(E_\mathcal{A} - \beta, S_\mathcal{A}) \) is a resolution of \( M_\mathcal{A}(\beta) \) for generic \( \beta \).

Set \( C_\bullet,\bullet = K_\bullet(x_i: i \notin \tau; E_\mathcal{A} - \beta, S_\mathcal{A}) \), a double complex as follows from [MMW05 Lemma 4.3] and

\[
\begin{align*}
\bullet \quad & x_i \cdot (D_\mathcal{A} \otimes_{\mathcal{R}} S_\mathcal{A})_\alpha \subseteq (D_\mathcal{A} \otimes_{\mathcal{R}} S_\mathcal{A})_{\alpha + \alpha_i}, \\
\bullet \quad & x_i \cdot (E_\mathcal{A} - \beta - \alpha) = (E_\mathcal{A} - \beta - \alpha - \alpha_i)x_i, \\
\bullet \quad & K_\bullet(E_\mathcal{A} - \beta, S_\mathcal{A}) = \bigoplus_{\alpha \in \mathbb{Z}^d} K_\bullet(E_\mathcal{A} - \beta - \alpha, (D_\mathcal{A} \otimes_{\mathcal{R}} S_\mathcal{A})_\alpha),
\end{align*}
\]

showing that all the squares in \( C_\bullet,\bullet \) are commutative.

Denote by \( \tau \) the complement \( A \setminus \tau \). Let \( \pi \) be the natural projection of \( C_\bullet,\bullet \) to \( K_\bullet(x_i: i \notin \tau; M_\mathcal{A}(\beta)) \) and let \( \eta \) be the natural projection of \( C_\bullet,\bullet \) to \( D_\mathcal{A}/x_\tau D_\mathcal{A} \otimes_{D_\mathcal{A}} K_\bullet(E_\mathcal{A} - \beta, S_\mathcal{A}) \). Consider the induced morphisms:

\[
(2.2) \quad \text{Tot}(C_\bullet,\bullet) \xrightarrow{\pi} \text{Tot}(K_\bullet(x_i: i \notin \tau; M_\mathcal{A}(\beta))) = K_\bullet(x_i: i \notin \tau; M_\mathcal{A}(\beta))
\]

and

\[
(2.3) \quad \text{Tot}(C_\bullet,\bullet) \xrightarrow{\eta} \text{Tot}(D_\mathcal{A}/x_\tau D_\mathcal{A} \otimes_{D_\mathcal{A}} K_\bullet(E_\mathcal{A} - \beta, S_\mathcal{A})).
\]

Remark 2.3. By [MMW05 Theorem 6.6], \( K_\bullet(E_\mathcal{A} - \beta, S_\mathcal{A}) \) is a resolution of \( M_\mathcal{A}(\beta) \) if and only if \( \beta \) is not rank-jumping. Thus, for such \( \beta \), \( \pi \) and \( \eta \) are quasi-isomorphisms and \( H_\bullet \text{Tot}(C_\bullet,\bullet) = \text{Tor}^{D_\mathcal{A}}_\bullet(D_\mathcal{A}/x_\tau D_\mathcal{A}, M_\mathcal{A}(\beta)) \).

Notation 2.4. For a toric \( S_\mathcal{A} \)-module \( N \) we write \( K_\bullet(E_\tau - \beta, N) \) for the Koszul complex of left \( D_\tau \)-modules defined by the sequence \( E_\tau - \beta \) on the (\textit{a fortiori} weakly toric) \( S_\tau \)-module \( N \).

Using the isomorphism (see [MMW05])

\[
(D_\mathcal{A} \otimes_{\mathcal{R}} S_\mathcal{A}) \xrightarrow{\sim} \mathbb{C}[x_\tau] \otimes_{\mathbb{C}} (D_\tau \otimes_{\mathcal{R}} S_\mathcal{A}),
\]

\[
x_\tau^\mu x_\tau^\nu \partial_\tau^\mu \partial_\tau^\nu \otimes m \mapsto x_\tau^\mu \otimes (x_\tau^\nu \partial_\tau^\nu) \otimes \partial_\tau^\mu m,
\]

one may now identify \( D_\mathcal{A}/x_\tau D_\mathcal{A} \otimes_{D_\mathcal{A}} K_\bullet(E_\mathcal{A} - \beta, S_\mathcal{A}) \) with \( K_\bullet(E_\tau - \beta, S_\mathcal{A}) \) as complexes of \( D_\tau \)-modules. We have thus proved

Lemma 2.5. If \( \beta \) is not rank-jumping for \( A \) and \( \text{rank}(A_\tau) = \text{rank}(A) \), then

\[
\text{L}i_*^\tau M_\mathcal{A}(\beta) \simeq K_\bullet(E_\tau - \beta, S_\mathcal{A})
\]

as complexes of left \( D_\tau \)-modules. \( \square \)

Remark 2.6. For all \( \beta \in \mathbb{C}^d \), \( i_*^\tau M_\mathcal{A}(\beta) \simeq \mathcal{H}_0(E_\tau - \beta, S_\mathcal{A}) \) because \( i_*^\tau \) is right exact.

It is clear that \( \mathbb{Q}_{\geq 0} A = \mathbb{Q}_{\geq 0} A_\tau \) implies that \( \text{rank}(A_\tau) = \text{rank}(A) = d \) and this last condition is equivalent to \( [ZA : ZA_\tau] < +\infty \).
Consider \( S_A = \mathbb{C}[NA] \) and \( S_\tau = \mathbb{C}[NA_\tau] \). Then the assumption \( Q_{\geq 0}A = Q_{\geq 0}A_\tau \) guarantees that \( S_A \) is a finitely generated \( \mathbb{Z}^d \)-graded \( S_\tau \)-module and so it is a toric \( R_\tau \)-module (see Definition 4.5. and Example 4.7 in [MMW05]). If we do not assume \( Q_{\geq 0}A = Q_{\geq 0}A_\tau \) but only \( \text{rank}(A_\tau) = \text{rank}(A) = d \), then \( S_A \) is a weakly toric \( R_\tau \)-module (see [SW09]).

Choose a collection of coset representatives \( \Omega \subseteq NA \) for \( \mathbb{Z}A/\mathbb{Z}A_\tau \). Then, with \( S_\tau(\lambda) = t^\lambda S_\tau \), there is a short exact sequence
\[
0 \to \bigoplus_{\lambda \in \Omega} S_\tau(\lambda) \to S_A \to Q \to 0,
\]
where no shifted copy of \( \mathbb{N}\tau \) is contained in \( \mathbb{N}A \setminus \bigcup_{\lambda \in \Omega} (\lambda + \mathbb{N}\tau) \). In particular, \( \dim(Q) < d \).

Assume from now on that condition (i) is in force. As \( S_A \) is then a finite integral extension of \( S_\tau \), \( S_\tau \)-modules are toric over \( A \) precisely when they are toric over \( \tau \).

We consider the long exact sequence of Euler–Koszul homology over \( D_\tau \) associated with the sequence (2.5).

By [MMW05] Proposition 5.3, vanishing of \( H_i(E_\tau - \beta, Q) \) for all \( i \geq 0 \) is equivalent to \(-\beta \notin q\deg(Q)\), where \( q\deg(Q) \) is the Zariski closure in \( \mathbb{C}^d \) of the set of the \( \mathbb{Z}^d \)-degrees of \( Q \). As \( \dim(Q) < d \), generic \( \beta \) will be outside \(-q\deg(Q)\). Hence for generic \( \beta \) we have
\[
H_i(E_\tau - \beta, S_A) \cong H_i(E_\tau - \beta, \bigoplus_{\lambda \in \Omega} S_\tau(\lambda)) = \bigoplus_{\lambda \in \Omega} H_i(E_\tau - \beta, S_\tau(\lambda)) \cong \bigoplus_{\lambda \in \Omega} H_i(E_\tau - \beta + \lambda, S_\tau)(\lambda)
\]
for all \( i \geq 0 \), and \( H_0(E_\tau - \beta + \lambda, S_\tau) = M_A(\beta - \lambda) \). Furthermore, for generic \( \beta \), \( \beta - \lambda \) is not rank-jumping for \( A_\tau \), and so \( H_k(E_\tau - \beta + \lambda, S_\tau) = 0 \) for all \( k > 0 \) and for all \( \lambda \in \Omega \); this implies that \( L^{-k}i^*_\tau M_A(\beta) = 0 \) for all \( k > 0 \).

In case (ii) is in force, the argument is similar, using weakly toric modules, which arise since \( S_A \) is not a finite \( S_\tau \)-module in this case. The main vanishing tool is then [SW09] Thm. 5.4 instead of [MMW05] Prop. 5.3. This concludes the proof of Theorem 2.2.

\[\square\]

**Remark 2.7.** If \( M \) is a holonomic \( \mathcal{D} \)-module and \( Y_\tau \) is noncharacteristic for \( M \), then \( L^{-k}i^*_\tau M = 0 \) for all \( k > 0 \) and the holonomic rank of \( M \) coincides with the holonomic rank of \( i^*_\tau M \) (see, for example, [MT04]). In our case, \( Y_\tau \) is noncharacteristic for \( \mathcal{M}_A(\beta) \) if and only if \( \tau \) contains all the nonzero vertices of the convex hull \( \Delta_A \) of \( \{a_1, a_2, a_3, a_4\} \subseteq \mathbb{R}^d \).

Let us show that formula (2.1) may fail if \( \beta \) is not sufficiently generic.

**Example 2.8.** (1) Let \( A = (a_0 \ a_1 \ a_2 \ a_3) \) with \( a_i = \binom{i}{1} \) for \( i = 0,1,3,4 \). In this case, by [ST98],
\[
\text{rank } M_A(\beta) = \begin{cases} 
\text{vol}_{\mathbb{Z}}(\Delta_A) = 4 & \text{if } \beta \in \mathbb{C}^2 \setminus \binom{1}{1}, \\
5 & \text{if } \beta = \binom{1}{1}.
\end{cases}
\]
Consider \( \tau = \{0,1,4\} \) and \( Y_\tau = \{x_3 = 0\} \), so \( [\mathbb{Z}A : \mathbb{Z}\tau] = 1 \) and \( \mathbb{Q}A = \mathbb{Q}+\tau \). According to Remark 2.7 the holonomic rank of \( i^*_\tau M_A(\beta) \) is 5 for \( \beta = \binom{1}{2} \).
On the other hand, the toric ideal $I_x$ associated with $A_x$ is principal and thus Cohen–Macaulay. It follows that the holonomic rank of $\mathcal{M}_{A_x}(\beta)$ is $\text{vol}_{x_\beta}(\Delta_x) = 4$ for all $\beta \in \mathbb{C}^2$ (see [MMW05 Corollary 9.2]). This implies that for $\beta = \left(\frac{1}{2}\right)$, $i_x^*\mathcal{M}_{A}(\beta)$ cannot be isomorphic to $\mathcal{M}_{A_x}(\beta')$ for any $\beta' \in \mathbb{C}^2$.

(2) Considering $A = (a_0, a_1, a_2, a_3, a_4)$ with notation as above and restricting to $x_2 = 0$, one has $Q_+A = Q_+\tau$ and $ZA = Z\tau$, but the restriction of $\mathcal{M}_{A}(\left(\frac{1}{2}\right))$ is not $\mathcal{M}_{\mathbb{C}^2}(\left(\frac{1}{2}\right))$ since once again the ranks disagree, this time the original GKZ-system having smaller rank (the rational quartic is arithmetically Cohen–Macaulay). However, by [Sai01 Thm. 6.3], $\mathcal{M}_{A}(\beta) \simeq \mathcal{M}(\left(\frac{1}{2}\right))$ for $\beta \in NA$ and restricting $\mathcal{M}(\beta)$ to $\tau$ for such $\beta$ leads to $\mathcal{M}_\tau(\beta)$ unless $\beta = \left(\frac{1}{2}\right)$.

Remark 2.9. We do not know a pair $(A, \tau)$ for which the conclusion of our theorem fails on a set that is not a finite subspace arrangement. Indeed, while it seems clear that $\beta$ not rank-jumping for both $A$ and $\tau$ is relevant, we believe that the question whether $\beta \in -\text{qdeg}(Q)$ is a red herring. The situation appears the same as the duality question discussed in [Wal07].

References


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