FAITHFUL REPRESENTATIONS OF ASSOCIATION SCHEMES

AKIHIDE HANAKI

(Communicated by Jim Haglund)

Abstract. Every character of an association scheme can be considered as a faithful character of some quotient scheme. Also we will show that a faithful character of an association scheme determines a thin closed subset which is cyclic as a finite group.

1. Introduction

Let $G$ be a finite group and let $\chi$ be a character of $G$. We can consider that $\chi$ is a faithful character of $G/\text{Ker}(\chi)$. If $G$ has a faithful irreducible character, then the center of $G$ is cyclic. These are well known facts in group representation theory. We will generalize them to characters of association schemes.

Every character of an association scheme can be considered as a faithful character of some quotient scheme (Theorem 2.1). Also we will show that a faithful character of an association scheme determines a thin closed subset which is cyclic as a finite group (Theorem 3.1).

Let $(X,S)$ be an association scheme in the sense of [7] or [3]. We will denote the valency of $s \in S$ by $n_s$. Let $T$ be a closed subset of $S$. Put $e_T = n_T^{-1} \sum_{t \in T} \sigma_t$. Then $e_T$ is an idempotent of $\mathbb{C}S$. It is known that $\mathbb{C}(S//T) \cong e_T \mathbb{C}S e_T$ as algebras by $\sigma_{sT} \mapsto (n_s/n_s) e_T \sigma_s e_T$ (see [4]).

We denote the identity matrix by $E$.

2. Faithful representations

Let $(X,S)$ be an association scheme, and let $\Phi : \mathbb{C}S \rightarrow M_n(\mathbb{C})$ be a representation of $(X,S)$ affording a character $\varphi$. Define

$$K(\Phi) = \{ s \in S \mid \Phi(\sigma_s) = n_s E \}$$

and

$$K(\varphi) = \{ s \in S \mid \varphi(\sigma_s) = n_s \varphi(1) \}.$$

It is known that $K(\Phi) = K(\varphi)$ (see [11] section 3). Note that $K(\Phi)$ is closed but not necessarily normal. We say that $\Phi$ or $\varphi$ is faithful if $K(\Phi) = \{1\}$.

If $K(\Phi)$ is a normal closed subset of $S$, then there is a natural algebra epimorphism $\pi : \mathbb{C}S \rightarrow \mathbb{C}(S//K(\Phi))$ and $\Phi$ can be considered as a representation of $S//K(\Phi)$. But $K(\Phi)$ is not necessarily normal, and the natural map $\pi : \mathbb{C}S \rightarrow \mathbb{C}(S//K(\Phi))$ is not an algebra homomorphism in general (see [6]).
Theorem 2.1. Let \((X, S)\) be an association scheme, and let \(\Phi : \mathbb{C}S \to M_n(\mathbb{C})\) be a representation of \((X, S)\). Suppose that \(T\) is a closed subset contained in \(K(\Phi)\). Then we can define a representation \(\Phi' : \mathbb{C}(S/T) \to M_n(\mathbb{C})\) by \(\Phi'(\sigma_T) = (n_{sT}/n_s)\Phi(\sigma_s)\). Moreover, \(\Phi'\) is faithful if \(T = K(\Phi)\).

Proof. If \(\Psi\) is an irreducible component of \(\Phi\), then \(K(\Phi) \subseteq K(\Psi)\). Thus, without loss of generality, we may suppose that \(\Phi\) is irreducible.

Let \(\varphi\) be the character afforded by \(\Phi\). Let \(e_{\varphi}\) be the primitive central idempotent of \(\mathbb{C}S\) corresponding to \(\varphi\). By the assumption on \(T\), we have \(e_{\varphi}e_T = e_{\varphi} = e_T e_{\varphi}\).

We will show that \(\Phi'\) is well-defined. Suppose \(s^T = u^T\). We have

\[
\frac{1}{n_s} \Phi(\sigma_s) = \Phi(e_{\varphi} \frac{1}{n_s} \sigma_s e_{\varphi}) = \Phi(e_{\varphi} e_T \frac{1}{n_s} \sigma_s e_T e_{\varphi}) = \Phi(e_{\varphi} e_T \frac{1}{n_u} \sigma_u e_T e_{\varphi}) = \frac{1}{n_u} \Phi(\sigma_u).
\]

This means that \(\Phi'\) is well-defined.

We show that \(\Phi'\) is an algebra homomorphism. We use the isomorphism \(\mathbb{C}(S/T) \cong e_T \mathbb{C}S e_T\) and identify them. Then \(\Phi'(e_T \sigma_s e_T) = \Phi(\sigma_s)\). We have

\[
\Phi'((e_T \sigma_s e_T)(e_T \sigma_u e_T)) = \Phi(\sigma_s e_T \sigma_u) = \Phi(\sigma_s e_T) \Phi(\sigma_u) = \Phi(\sigma_s) \Phi(\sigma_u) = \Phi'((e_T \sigma_s e_T) \Phi(\sigma_u) = \Phi(\sigma_s) \Phi(\sigma_u) = \Phi'(e_T \sigma_s e_T) \Phi'(e_T \sigma_u e_T).
\]

Finally, we will show that \(\Phi'\) is faithful if \(T = K(\Phi)\). Suppose \(s^T \in K(\Phi')\). Then \(E = n_{sT}^{-1} \Phi'(\sigma_T) = n_s^{-1} \Phi(\sigma_s)\). So \(s \in K(\Phi)\) and \(s^T = 1^T\). Now \(\Phi'\) is faithful.

\[\square\]

Corollary 2.2. Let \((X, S)\) be an association scheme, and let \(\varphi\) be a character of \((X, S)\). Suppose that \(T\) is a closed subset contained in \(K(\varphi)\). Then \(n_u^{-1} \varphi(\sigma_u) = n_s^{-1} \varphi(\sigma_s)\) for any \(u \in TsT\).

Proof. This is obtained by the fact that \(\Phi'\) in Theorem 2.1 is well-defined. \[\square\]

3. Faithful representations and closed subsets

Let \((X, S)\) be an association scheme, and let \(\Phi : \mathbb{C}S \to M_n(\mathbb{C})\) be a representation of \((X, S)\) affording a character \(\varphi\). Define

\[Z(\varphi) = \{s \in S \mid |\varphi(\sigma_s)| = n_s \varphi(1)\}.
\]

Then \(Z(\varphi)\) is a closed subset of \(S\) containing \(K(\varphi)\) (see [2] Proposition 3.2 and 3.3). For \(s \in S\), \(s \in Z(\varphi)\) if and only if \(\Phi(\sigma_s) = \varepsilon_s n_s E\) for some root of unity \(\varepsilon_s\).

The following theorem is a generalization of [3] Theorem 2.32 (a)].

Theorem 3.1. Let \(\varphi\) be a faithful character of \((X, S)\). Then \(Z(\varphi)\) is thin and cyclic as a finite group.

In the rest of this section, \((X, S)\) is an association scheme and \(\Phi : \mathbb{C}S \to M_n(\mathbb{C})\) is a faithful representation of \((X, S)\) affording a character \(\varphi\). For \(u \in Z(\varphi)\) we define a root of unity \(\varepsilon_u\) by \(\Phi(\sigma_u) = \varepsilon_u n_u E\) or equivalently by \(\varphi(\sigma_u) = \varepsilon_u n_u \varphi(1)\).

We need a lemma.

Lemma 3.2. If \(u, v \in Z(\varphi)\) and \(u \neq v\), then \(\varepsilon_u \neq \varepsilon_v\). Moreover, \(Z(\varphi)\) is thin.
Proof. For $u, v \in Z(\varphi)$, suppose $\varepsilon_u = \varepsilon_v$. Then $\Phi(\sigma_u)\Phi(\sigma_v^*) = n_u n_v^* E$. Now $w \in K(\varphi) = \{1\}$ for any $w \in uv^*$. So $uv^* = \{1\}$. This means that $u = v$ and $u$ is thin.

Proof of Theorem 3.1. If $\xi$ is an irreducible constituent of $\varphi$, then $Z(\varphi) \subseteq Z(\xi)$. So we may suppose that $\varphi$ is irreducible. We consider $Z(\varphi)$ as a finite group. Then, by Lemma 3.2, $\sigma_u \mapsto \varepsilon_u$ is an irreducible faithful character of an abelian group $Z(\varphi)$. So $Z(\varphi)$ is cyclic.

We remark that if $\varphi$ is a faithful irreducible character of a finite group $G$, then $Z(\varphi)$ is just the center of $G$. But for a character of an association scheme, it is not true in general.

Acknowledgement

The author would like to thank the referee for many helpful comments.

References


Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
E-mail address: hanaki@shinshu-u.ac.jp