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# RIGID PROPERTIES OF QUASI-EINSTEIN METRICS

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ABSTRACT. In this paper we get some rigid results for $m-$dimensional quasi-Einstein metrics on complete Riemannian manifolds.

## 1. INTRODUCTION

Let $(M, \mathrm{g})$ be an $n-$dimensional complete Riemannian manifold. We call g Einstein if the Ricci curvature tensor satisfies

$$\mathrm{Ric} = \lambda \mathrm{g}$$

for some constant $\lambda$. The Einstein metric plays important roles in both differential geometry and physics.

Let $f$ be a smooth real-valued function on $M$. When considering the weighted measure $\mathrm{d}\mu = e^{-f}\, \mathrm{d}x$, one always use $m-$dimensional Bakry-Émery Ricci curvature

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess}\ f - \frac{\nabla f \otimes \nabla f}{m-n}$$

to replace the Ricci curvature, where $m \geq n$ and $m = n$ if and only if $h$ is a constant [1, 2]. There has been an active interest in the study of the weighted measure under conditions about the $m-$dimensional Bakry-Émery curvature. In particular, Li obtained several Liouville type theorems about the weighted Laplacian $\triangle_\mu$ in [3]. The author of [4] proved a sharp upper bound of the $L^2_\mu$ spectrum on complete noncompact Riemannian manifolds. Further results about weighted measure can be found in [3, 4] and the references therein.

Similar to the Einstein metric, we call a metric g $m-$dimensional quasi-Einstein with potential function $f$ if for some constant $\lambda$,

$$\mathrm{Ric}_{f,m} = -(m-1)\lambda \mathrm{g}. \tag{1.1}$$

This definition can be found in [5, 6]. A quasi-Einstein metric becomes Einstein when the potential function is constant. We also note that an $\infty-$dimensional quasi-Einstein metric means a gradient Ricci soliton. We call a quasi-Einstein metric shrinking, steady or expanding, respectively, when $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

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For a gradient Ricci soliton, i.e. an $\infty-$dimensional quasi-Einstein metric, after a rescaling of the metric, $R$ and $f$ satisfy

$$R+|\nabla f|^2-f=\mu. \tag{1.2}$$

By adding the constant $\mu$ to $f$, we can assume $\mu=0$, which means that $\mu$ can be neglected when studying a gradient Ricci soliton. From this observation, the authors of [7] and [8] proved that there does not exist a nontrivial expanding gradient Ricci soliton on a closed manifold. In Section 2, we give an identity similar to (1.2) for a quasi-Einstein metric on complete manifolds, by which we get some rigid results for quasi-Einstein metrics on closed manifolds.

In Section 3, we prove the gradient estimate of a potential function on a complete noncompact Riemannian manifold by using the weighted Laplacian comparison theorem and the maximum principle. Then we get a rigid result for a steady quasi-Einstein metric. In this section, we also consider the expanding quasi-Einstein metric on a noncompact manifold by using the weak maximum principle at infinity.

In [9], the author proved that any complete shrinking Ricci soliton has nonnegative scalar curvature. This fact is very useful in studying the geometry of complete gradient shrinking Ricci solitons [10, 11, 12]. The authors of [5] obtained some estimates of the scalar curvature for an $m-$dimensional quasi-Einstein metric on closed manifolds. In [10, 11], the authors considered the estimate of scalar curvature for a gradient Ricci soliton on noncompact manifolds. In Section 4, we get the lower estimate of the scalar curvature for expanding a quasi-Einstein metric on a noncompact manifold.

## 2. Rigid results on closed manifolds

In this section, we get some rigid results for quasi-Einstein metrics on closed manifolds. The following lemma is proved in [5]; the main idea comes from [13].

**Lemma 2.1.** *If* g *is an $m-$dimensional quasi-Einstein metric with potential function $f$ and constant $\lambda$, then we have*

$$\frac{1}{2}\nabla R=\frac{m-n-1}{m-n}\mathrm{Ric}(\nabla f)+\frac{1}{m-n}(R+(m-1)(n-1)\lambda)\nabla f \tag{2.1}$$

*and*

(2.2)

$$\begin{aligned}\frac{1}{2}\triangle R&-\frac{m-n+2}{2(m-n)}\nabla f\cdot\nabla R\\ &=-\frac{m-n-1}{m-n}|\mathrm{Ric}-\frac{1}{n}R\mathrm{g}|^2-\frac{m-1}{(m-n)n}(R+n(m-1)\lambda)(R+n(n-1)\lambda).\end{aligned}$$

Using Lemma 2.1, we can show a formula similar to (1.2).

**Theorem 2.2.** *If* g *is an $m-$dimensional quasi-Einstein metric with potential function $f$ and constant $\lambda$, then for some constant $\mu$,*

$$R+\frac{m-n-1}{m-n}|\nabla f|^2-(m-2n)(m-1)\lambda=\mu\ e^{\frac{2}{m-n}f}. \tag{2.3}$$

*Proof.* From (1.1) and (2.1) we have

$$\nabla(R+\frac{m-n-1}{m-n}|\nabla f|^2)=\frac{2}{m-n}(R+\frac{m-n-1}{m-n}|\nabla f|^2)\nabla f-\frac{2(m-2n)(m-1)}{m-n}\lambda\nabla f,$$

which means that

$$\nabla[(R+\frac{m-n-1}{m-n}|\nabla f|^2)e^{-\frac{2}{m-n}f}-(m-2n)(m-1)\lambda e^{-\frac{2}{m-n}f}]=0,$$

so

$$(R+\frac{m-n-1}{m-n}|\nabla f|^2)e^{-\frac{2}{m-n}f}-(m-2n)(m-1)\lambda e^{-\frac{2}{m-n}f}$$

is constant, and (2.3) follows. □

Using (2.3), we get rigid results for $m-$dimensional quasi-Einstein metrics on closed manifolds.

**Theorem 2.3.** *1) All expanding $m-$dimensional quasi-Einstein metrics with potential function $f$ on closed manifolds are trivial in the sense that $f$ is constant. Moreover, the metric is Einstein.*

*2) All shrinking $m-$dimensional quasi-Einstein metrics with potential function $f$ on closed manifolds are trivial if $\mu \leq 0$.*

*3) All steady $m-$dimensional quasi-Einstein metrics with potential function $f$ on closed manifolds are trivial.*

*Proof.* Taking the trace of (1.1), we have

$$R+\triangle f-\frac{1}{m-n}|\nabla f|^2=-n(m-1)\lambda. \tag{2.4}$$

Then (2.3) and (2.4) tell us that

$$\triangle f-|\nabla f|^2+(m-1)(m-n)\lambda+\mu e^{\frac{2}{m-n}f}=0. \tag{2.5}$$

Let

$$h=e^{-\frac{2}{m-n}f}.$$

Then

$$\nabla f=-\frac{m-n}{2}\frac{\nabla h}{h},$$

$$\triangle f=-\frac{m-n}{2}\frac{h\triangle h-|\nabla h|^2}{h^2}.$$

(2.5) can be rewritten as

$$\triangle h+\frac{m-n-2}{2}\frac{|\nabla h|^2}{h}-2(m-1)\lambda h-\frac{2}{m-n}\mu=0. \tag{2.6}$$

Let

$$\tilde{h}=h+\frac{\mu}{(m-n)(m-1)\lambda}.$$

Then (2.6) becomes

$$\triangle\tilde{h}+\frac{m-n-2}{2}\frac{|\nabla\tilde{h}|^2}{h}-2(m-1)\lambda\tilde{h}=0.$$

If $\lambda>0$, from the maximum principle, we get that $\tilde{h}$ is constant, so $f$ is constant.

If $\lambda<0$ and $\mu\leq 0$, by (2.6), we have

$$\Delta h+\frac{m-n-2}{2}\frac{|\nabla h|^2}{h}\leq 0.$$

From the maximum principle, we deduce that $h$ is constant and that the $m$-dimensional quasi-Einstein metric is trivial.

When $\lambda = 0$, through integrating (2.5) against the measure $\mathrm{d}\mu = e^{-f}\ \mathrm{d}x$, it is easy to see that $\mu = 0$, which shows immediately that steady quasi-Einstein metrics are trivial.

□

*Remark* 2.4. Part 1) of Theorem 2.3 was first proved in [14].

*Remark* 2.5. When $m$ is finite, a manifold with $m-$dimensional quasi-Einstein metric and $\lambda < 0$ is automatically compact [1].

*Remark* 2.6. We recall Proposition 3.3 in [5], where the authors got a rigid result for a quasi-Einstein metric under the assumption of Ricci curvature. Theorem 2.3 shows that the constant $\mu$ is important when studying the rigidity of shrinking $m-$dimensional quasi-Einstein metrics. It is natural to ask whether a shrinking quasi-Einstein metric is Einstein on a closed manifold when $\mu > 0$. I conjecture the answer is no, but, we need a counterexample.

*Remark* 2.7. A large class, namely when $m$ are integers, of rigidity results for closed steady or expanding quasi-Einstein metrics have been proved in [6].

## 3. Rigid results on noncompact manifolds

In this section, we prove several rigid results for a quasi-Einstein metric on noncompact manifolds. We first introduce the weighted Laplacian comparison theorem, which can be found in [2].

**Lemma 3.1.** *Let $(M, \mathrm{g})$ be an $n-$dimensional complete Riemannian manifold, $f$ be a real-valued smooth function on $M$ and $\triangle_\mu = \triangle - \nabla f \cdot \nabla$ be the weighted Laplacian, we also assume that the $m-$dimensional Bakry-Émery Ricci curvature on $M$ is bounded by*

$$\mathrm{Ric}_{f,m} \geq -(m-1)K$$

*with constant $K \geq 0$ and $r(x) = \mathrm{dist}(O, x)$ is the distance function determined by a fixed point $O$. If $a_K$ is a solution of the Riccati equation*

$$\begin{cases} \frac{\partial a_K}{\partial r} = -(m-1)K - \frac{a_K^2}{m-1} \\ \lim\limits_{r\to 0+} r a_K = m-1, \end{cases}$$

*then at $x \notin \mathrm{Cut}(O)$,*

$$\triangle_\mu r \leq a_K(r).$$

*In particular, if $K > 0$,*

$$\begin{aligned} \triangle_\mu r &\leq (m-1)\sqrt{K}\coth\sqrt{K}r \\ &\leq \frac{m-1}{r}(1+\sqrt{K}r). \end{aligned}$$

Using the maximum principle [15, 16, 17], we can get the gradient estimate of the potential function $f$, of independent interest.

**Theorem 3.2.** *Letting $M$ be an $n-$dimensional noncompact Riemannian manifold, we assume that the metric $\mathrm{g}$ satisfies (1.1) with potential function $f$ and constant $\lambda \geq 0$, where $\mu$ is the constant in (2.3).*

*1) If $\mu$ is nonpositive, then*

$$|\nabla f|^2(x) \leq (m-n)(m-1)\lambda$$

*holds for any $x \in M$.*

*2) If $\mu > 0$ and $f$ is bounded by $f \leq C$, then*

$$|\nabla f|^2(x) \leq (m-n)(m-1)\lambda + 2\mu e^{\frac{2C}{m-n}}$$

*holds for any $x \in M$.*

*Proof.* Consider a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$ (see the proof of Theorem 5.1 in [3]),

$$\theta(t) = \begin{cases} 1 & 0 \leq t \leq 1, \\ 0 & t \geq 2, \end{cases}$$

so that

$$-10\theta^{\frac{1}{2}} \leq \theta' \leq 0, \theta'' \geq -10. \tag{3.1}$$

For some constant $R_0 > 0$, define the smooth cutoff function $\varphi : M \to \mathbf{R}$ by

$$\varphi(x,t) = \theta(\frac{r(x)}{R_0}).$$

Then

$$\nabla\varphi = \frac{\theta'\nabla r}{R_0} \tag{3.2}$$

and

$$\begin{aligned} \triangle_\mu\varphi &= \triangle\varphi - \nabla\varphi\cdot\nabla f \\ &= \frac{\theta''}{R_0^2} + \frac{\theta'\triangle_\mu r}{R_0} \\ &\geq \frac{\theta''}{R_0^2} + \frac{(m-1)\theta'(1+\sqrt{\lambda}R_0)}{R_0^2}, \end{aligned} \tag{3.3}$$

where we have used Lemma 3.1. Let

$$G = \varphi|\nabla f|^2.$$

Then

$$\triangle_\mu G = \triangle_\mu\varphi|\nabla f|^2 + 2\nabla\varphi\cdot\nabla|\nabla f|^2 + \varphi\triangle_\mu|\nabla f|^2.$$

By (1.1), (2.3) and the Ricci identity

$$f_{ijj} = f_{jji} + R_{ij}f_j,$$

we have

$$\begin{aligned} \triangle_\mu|\nabla f|^2 &= \triangle|\nabla f|^2 - \nabla|\nabla f|^2\cdot\nabla f \\ &= 2|\nabla^2 f|^2 + 2\mathrm{Ric}(\nabla f,\nabla f) + 2\nabla f\cdot\nabla\triangle f - \nabla|\nabla f|^2\cdot\nabla f \\ &= 2|\nabla^2 f|^2 + 2[\frac{1}{m-n}\nabla f\otimes\nabla f - \mathrm{Hess}f - (m-1)\lambda \mathrm{g}](\nabla f,\nabla f) \\ &\quad +2\nabla f\cdot\nabla[-\mu e^{\frac{2}{m-n}f} - (m-1)(m-n)\lambda + |\nabla f|^2] - \nabla f\cdot\nabla|\nabla f|^2 \\ &= 2|\nabla^2 f|^2 - 2(m-1)\lambda|\nabla f|^2 + \frac{2}{m-n}|\nabla f|^4 - \frac{4\mu}{m-n}e^{\frac{2}{m-n}f}|\nabla f|^2. \end{aligned}$$

If $\mu \le 0$, then

$$\triangle_\mu |\nabla f|^2 \ge -2(m-1)\lambda |\nabla f|^2 + \frac{2}{m-n}|\nabla f|^4.$$

Noting that

$$\nabla G = \nabla\varphi |\nabla f|^2 + \varphi \nabla |\nabla f|^2,$$

we have

$$\nabla |\nabla f|^2 = \frac{\nabla G}{\varphi} - \frac{G\nabla\varphi}{\varphi^2}.$$

So

$$\triangle_\mu G \ge \frac{\triangle_\mu \varphi G}{\varphi} + \frac{2\nabla G\cdot\nabla\varphi}{\varphi} - \frac{2|\nabla\varphi|^2 G}{\varphi^2} - 2(m-1)\lambda G + \frac{2G^2}{(m-n)\varphi}.$$

We assume that $G$ achieves maximal value at $x_0$. Then

$$\nabla G = 0$$

and

$$\triangle_\mu G \le 0$$

hold at $x_0$. Using (3.1), (3.2) and (3.3), we conclude that at $x_0$,

$$\begin{aligned} 0 &\ge \frac{\theta'' G}{R_0^2} + \frac{(m-1)\theta'(1+\sqrt{\lambda}R_0)}{R_0^2}G - \frac{2|\theta'|^2 G}{\theta R_0^2} - 2(m-1)\lambda\theta G + \frac{2G^2}{m-n} \\ &\ge \frac{2G^2}{m-n} - [2(m-1)\lambda + \frac{210 + 10(m-1)(1+\sqrt{\lambda}R_0)}{R_0^2}]G. \end{aligned}$$

So for $x \in B(O, R_0)$,

$$\begin{aligned} |\nabla f|^2(x) &= |\nabla f|^2(x)\varphi(x) = G(x) \le G(x_0) \\ &\le (m-n)(m-1)\lambda + \frac{105(m-n) + 5(m-n)(m-1)(1+\sqrt{\lambda}R_0)}{R_0^2}. \end{aligned}$$

Letting $R_0 \to \infty$, we get that on $M$,

$$|\nabla f|^2(x) \le (m-n)(m-1)\lambda.$$

If $\mu > 0$ and $f \le C$, then

$$\triangle_\mu |\nabla f|^2 \ge -2(m-1)\lambda|\nabla f|^2 + \frac{2}{m-n}|\nabla f|^4 - \frac{4\mu}{m-n}e^{\frac{2}{m-n}C}|\nabla f|^2.$$

So

$$\begin{aligned} \triangle_\mu G \quad &\ge \quad \frac{\triangle_\mu\varphi G}{\varphi} + \frac{2\nabla G\cdot\nabla\varphi}{\varphi} - \frac{2|\nabla\varphi|^2 G}{\varphi^2} \\ &\quad - 2(m-1)\lambda G + \frac{2G^2}{(m-n)\varphi} - \frac{4\mu G}{m-n}e^{\frac{2C}{m-n}}. \end{aligned}$$

We assume that $G$ obtains a maximal value at $x_0$. Then

$$0 \ge \frac{2G^2}{m-n} - [2(m-1)\lambda + \frac{4\mu}{m-n}e^{\frac{2C}{m-n}} + \frac{210 + 10(m-1)(1+\sqrt{\lambda}R_0)}{R_0^2}]G$$

holds at $x_0$, which means that

$$G(x_0) \le (m-n)(m-1)\lambda + 2\mu e^{\frac{2C}{m-n}} + \frac{105(m-n) + 5(m-n)(m-1)(1+\sqrt{\lambda}R_0)}{R_0^2}.$$

So for $x \in B(O, R_0)$,

$$\begin{aligned} & |\nabla f|^2(x) = |\nabla f|^2(x)\varphi(x) = G(x) \leq G(x_0) \\ \leq \quad & (m-n)(m-1)\lambda + 2\mu e^{\frac{2C}{m-n}} + \frac{105(m-n) + 5(m-n)(m-1)(1+\sqrt{\lambda}R_0)}{R_0^2}. \end{aligned}$$

Letting $R_0 \to \infty$, we get that on $M$,

$$|\nabla f|^2(x) \leq (m-n)(m-1)\lambda + 2\mu e^{\frac{2C}{m-n}}.$$

Hence Theorem 3.2 follows. □

Using Theorem 3.2, we get a rigid result for a steady quasi-Einstein metric on a noncompact manifold, which has been proved in [18].

**Theorem 3.3.** *For a steady $m-$dimensional quasi-Einstein metric with potential function $f$, if $\mu \leq 0$, then $f$ is constant and the metric is Einstein.*

*Remark* 3.4. When $M$ is noncompact and g is steady quasi-Einstein, Jeffrey Case proved in [18] that $\mu \geq 0$ holds with equality holding if and only if g is Ricci flat.

We say that the weak maximum principle at infinity for $\triangle_\mu$ holds if given a $C^2$ function $u$,

$$\sup_M u = u^* < +\infty.$$

Then there exists a sequence $\{x_n\} \subset M$ such that

$$u(x_n) > u^* - \frac{1}{n}$$

and

$$\triangle_\mu u(x_n) \leq \frac{1}{n}.$$

The following lemma was given in [17].

**Lemma 3.5.** *Let $(M, g)$ be a complete weighted manifold satisfying the volume growth condition*

$$\int_1^\infty \frac{r}{\log \mu(B_r)}\, dr = +\infty. \tag{3.4}$$

*Then the weak maximum principle at infinity for the weighted Laplacian $\triangle_\mu$ holds on $M$, where $\mu(B_r)$ denotes the weighted measure of a geodesic ball with radius $r$.*

Using the weak maximum principle at infinity, we get a rigid result for expanding an $m-$dimensional quasi-Einstein metric on a noncompact manifold.

**Theorem 3.6.** *For an expanding $m-$dimensional quasi-Einstein metric on a noncompact manifold with potential function $f$, if*

$$\sup_M |\nabla f|^2 < \frac{(m-n)^2(m-1)\lambda}{m}, \tag{3.5}$$

*then $f$ is constant and the metric is Einstein.*

*Proof.* From Lemma 3.2 in [5],

$$\frac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f) + \frac{2}{m-n}|\nabla f|^2 \triangle f,$$

so

$$\frac{1}{2}\triangle_\mu|\nabla f|^2 = |\nabla^2 f|^2 - \frac{1}{m-n}|\nabla f|^4 + (m-1)\lambda|\nabla f|^2 + \frac{2}{m-n}|\nabla f|^2\triangle f.$$

Noting that

$$\begin{aligned}|\nabla^2 f|^2 + \frac{2}{m-n}|\nabla f|^2\triangle f &\geq \frac{1}{n}(\triangle f)^2 + \frac{2}{m-n}|\nabla f|^2\triangle f\\ &\geq -\frac{n}{(m-n)^2}|\nabla f|^4.\end{aligned}$$

Hence

$$\frac{1}{2}\triangle_\mu|\nabla f|^2 \geq (m-1)\lambda|\nabla f|^2 - \frac{m}{(m-n)^2}|\nabla f|^4.$$

By the proof of Lemma 3.1 in [2] (we can also refer to Section 10 in [3]), we deduce that (3.4) is right for a noncompact manifold with expanding $m-$dimensional quasi-Einstein metric. Noting that $\sup_M|\nabla f|^2 < +\infty$, by Lemma 3.5 there exists a sequence $\{x_k\} \subset M$ such that

$$|\nabla f|^2(x_k) \geq \sup_M|\nabla f|^2 - \frac{1}{k}$$

and

$$\triangle_\mu|\nabla f|^2(x_k) \leq \frac{1}{k}.$$

Hence,

$$\frac{m}{(m-n)^2}|\nabla f|^4 - (m-1)\lambda|\nabla f|^2 + \frac{1}{2k} \geq 0$$

holds at $x_k$. By (3.5), we conclude that for $k$ large enough,

$$|\nabla f|^2(x_k) \leq \frac{(m-n)^2}{2m}[(m-1)\lambda - \sqrt{(m-1)^2\lambda^2 - \frac{2m}{k(m-n)^2}}],$$

which means that

$$\sup_M|\nabla f|^2 - \frac{1}{k} \leq \frac{(m-n)^2}{2m}[(m-1)\lambda - \sqrt{(m-1)^2\lambda^2 - \frac{2m}{k(m-n)^2}}].$$

Let $k \to \infty$, we get that

$$\sup_M|\nabla f|^2 \leq 0.$$

Hence $f$ is constant. □

*Remark* 3.7. If $m = \infty$, we recover Theorem 11 in [19].

## 4. Lower bound of scalar curvature

It was proved in [9] that any complete shrinking Ricci soliton has nonnegative scalar curvature. We also note that when $m$ is finite, a manifold with $m-$dimensional shrinking quasi-Einstein metric is automatically compact [1]. In this section, we get the lower bound estimate of scalar curvature for an expanding or steady quasi-Einstein metric on a noncompact manifold.

**Theorem 4.1.** *Letting $M$ be an $n-$dimensional noncompact Riemannian manifold, the metric $\mathrm{g}$ is $m-$dimensional quasi-Einstein with potential function $f$ and constant $\lambda \geq 0$. If $m-n \geq 1$, $\mu \leq 0$ or $\mu > 0$ and $f$ is bounded by $f \leq C$, then*

$$R(x) \geq -n(m-1)\lambda \tag{4.1}$$

*holds for any $x \in M$.*

*Proof.* Consider a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$,

$$\theta(t) = \begin{cases} 1 & 0 \leq t \leq 1, \\ 0 & t \geq 2, \end{cases}$$

so that

$$-10\theta^{\frac{1}{2}} \leq \theta' \leq 0, \theta'' \geq -10. \tag{4.2}$$

For $R_0 > 0$, define a smooth cutoff function $\varphi : M \to \mathbf{R}$ by

$$\varphi(x,t) = \theta(\frac{r(x)}{R_0}).$$

Let $\bar{G} = \varphi R$. Then

$$\nabla R = -\frac{\bar{G}\nabla\varphi}{\varphi^2} + \frac{\nabla\bar{G}}{\varphi}.$$

By (2.2),

$$\begin{aligned} \triangle_\mu \bar{G} &= \triangle_\mu\varphi R + 2\nabla\varphi\cdot\nabla R + \varphi\triangle_\mu R \\ &= \triangle_\mu\varphi R + 2\nabla\varphi\cdot\nabla R + \varphi(\triangle R - \nabla R\cdot\nabla f) \\ &= \frac{\triangle_\mu\varphi\bar{G}}{\varphi} - \frac{2|\nabla\varphi|^2\bar{G}}{\varphi^2} + \frac{2\nabla\varphi\cdot\nabla\bar{G}}{\varphi} - \frac{2\nabla f\cdot\nabla\varphi\bar{G}}{(m-n)\varphi} \\ &\quad + \frac{2\nabla f\cdot\nabla\bar{G}}{m-n} - \frac{2(m-n-1)\varphi}{m-n}|\mathrm{Ric} - \frac{1}{n}R\mathrm{g}|^2 \\ &\quad - \frac{2(m-1)\varphi}{(m-n)n}[\frac{\bar{G}}{\varphi} + n(m-1)\lambda][\frac{\bar{G}}{\varphi} + n(n-1)\lambda]. \end{aligned}$$

We assume that for some $R_0 > 0$, the minimal value of $\bar{G}$ on $B(O, R_0)$ at $x_0$ is negative. Then

$$\nabla\bar{G} = 0$$

and

$$\triangle_\mu\bar{G} \geq 0$$

hold at $x_0$. Hence at $x_0$,

$$\begin{aligned} 0 \leq \;& \triangle_\mu\varphi\bar{G} - \frac{2|\nabla\varphi|^2\bar{G}}{\varphi} - \frac{2\nabla f\cdot\nabla\varphi\bar{G}}{m-n} \\ & - \frac{2(m-1)}{(m-n)n}[\bar{G} + n(m-1)\lambda\varphi][\bar{G} + n(n-1)\lambda\varphi]. \end{aligned} \tag{4.3}$$

For $R_0 > 0$ large enough, we define

$$\sigma(R_0) = \frac{\inf\{R(x)|x \in B(O, R_0)\}}{\inf\{R(x)|x \in B(O, 2R_0)\}}.$$

It is easy to see that

$$\bar{G}(x_0) = \varphi(x_0)R(x_0) \leq \inf\{R(x)|x \in B(O, R_0)\}$$

and

$$\bar{G}(x_0) = \varphi(x_0)R(x_0) \geq \varphi(x_0)\inf\{R(x)|x \in B(O, 2R_0)\}.$$

Using the assumption that $\inf\{R(x)|x \in B(O, R_0)\} < 0$, we conclude that

$$\sigma(R_0) \leq \varphi(x_0) \leq 1. \tag{4.4}$$

By Lemma 3.1, (4.2), (4.3) and (4.4), we conclude that at $x_0$,

$$\begin{aligned}\frac{\theta'' + (m-1)\theta'(1+\sqrt{\lambda}R_0)}{R_0^2}\bar{G} &\geq -\frac{2|\nabla f||\theta'|}{(m-n)R_0}\bar{G} + \frac{2|\theta'|^2}{\theta R_0^2}\bar{G} \\ &+ \frac{2(m-1)}{(m-n)n}[\bar{G}^2 + n(m+n-2)\lambda\bar{G} + n^2(m-1)(n-1)\lambda^2\sigma^2(R_0)].\end{aligned}$$

By Theorem 3.2, we conclude that at $x_0$,

$$\bar{G}^2 + n(m+n-2)\lambda\bar{G} + n^2(m-1)(n-1)\lambda^2\sigma^2(R_0) \leq \frac{C_1 + C_2R_0}{R_0^2}\bar{G},$$

where $C_1, C_2$ are constants depending only on $m, n, \lambda, \mu, C$. Hence for $x \in B(O, R_0)$,

$$R(x) = \varphi(x)R(x) = \bar{G}(x) \geq \bar{G}(x_0) \geq \frac{1}{2}[A - \sqrt{A^2 - 4B}] \tag{4.5}$$

with

$$A = \frac{C_1 + C_2R_0}{R_0^2} - n(m+n-2)\lambda$$

and

$$B = n^2(m-1)(n-1)\lambda^2\sigma^2(R_0).$$

Noting that $0 \leq \sigma(R_0) \leq 1$, (4.5) means that $R$ is bounded from below. Hence

$$\lim_{R_0 \to \infty} \sigma(R_0) = 1,$$

which means that

$$\lim_{R_0 \to \infty} A = -n(m+n-2)\lambda$$

and

$$\lim_{R_0 \to \infty} B = n^2(m-1)(n-1)\lambda^2.$$

By (4.5), we deduce that for $x \in M$,

$$R(x) \geq -\frac{1}{2}[n(m+n-2)\lambda + n(m-n)|\lambda|],$$

and (4.1) follows. □

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