SINGULAR HYPERSURFACES
POSSESSING INFINITELY MANY STAR POINTS

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ABSTRACT. We prove that a component \( \Lambda \) of the closure of the set of star points on a hypersurface of degree \( d \geq 3 \) in \( \mathbb{P}^N \) is linear. Afterwards, we focus on the case where \( \Lambda \) is of maximal dimension and the case where \( X \) is a surface.

1. INTRODUCTION

Consider a hypersurface \( X \) of degree \( d \geq 3 \) in a projective space \( \mathbb{P}^N \) defined over the field \( \mathbb{C} \) of complex numbers. A smooth point on \( X \) is called a star point if and only if the intersection of \( X \) with the embedded tangent space \( T_P(X) \) is a cone with vertex \( P \). As explained in [2], this notion is a generalisation of total inflection points on plane curves. It is also a generalisation of the classical notion of an Eckardt point on a smooth cubic surface in \( \mathbb{P}^3 \) (see [3, 9, 10]). Star points on smooth hypersurfaces have been studied in [2], where it is proven that such hypersurfaces contain only finitely many star points. Star points on singular cubic surfaces in \( \mathbb{P}^3 \) have been examined in [1, 10, 11]. In particular, in [11] the author investigates a singular cubic surface containing infinitely many star points and formulates some general expectations about it.

In Section 2 of this article, we prove that each component \( \Lambda \) of the closure of the set of star points of \( X \) is a linear subspace of dimension \( 0 \leq \lambda \leq N - 2 \). Moreover, each smooth point \( P \) in \( \Lambda \) has the same tangent space \( \Pi \), and \( \Pi \cap X \) is a cone with vertex \( \Pi \). Note that for smooth hypersurfaces, we always have \( \lambda = 0 \). In Section 3 we study the case where \( \Lambda \) is of maximal dimension, i.e. \( \lambda = N - 2 \). We prove that either all such \((N - 2)\)-dimensional linear spaces \( \Lambda \) belong to the same hyperplane in \( \mathbb{P}^N \) or they contain a common \((N - 3)\)-dimensional linear space. In the first case, there are at most \( d \) such linear spaces \( \Lambda \); in the second case, there are at most \( 3d \) such linear spaces \( \Lambda \). To finish the article, in Section 4 we consider surfaces in \( \mathbb{P}^3 \) having a line \( L \) of star points. We show that in general such a line contains \( d - 1 \) singularities of \( X \) of type \( A_{d-1} \). Conversely, the presence of such singularities implies that \( L \) is a line of star points. We also make some remarks in more special cases. On the one hand, our results give contradictions to the expectations in [1]; on the other hand, we place them in a much more general situation. Finally, we show that star points on a line of star points can be considered as a limiting case.
of a special type of isolated star points. This generalises results of [12] for cubic surfaces.

2. Main theorem

We work over the field \( \mathbb{C} \) of complex numbers.

**Definition 2.1.** Let \( X \) be an irreducible reduced hypersurface of degree \( d \geq 3 \) in \( \mathbb{P}^N \) and let \( P \) be a smooth point on \( X \). We say that \( P \) is a star point on \( X \) if and only if \( T_P(X) \cap X \) is a cone with vertex \( P \).

**Example 2.2.** Assume that \( X \) is an irreducible reduced hypersurface of degree \( d \geq 3 \) in \( \mathbb{P}^N \) and that \( \Pi \) is a hyperplane in \( \mathbb{P}^N \) such that the scheme \( \Pi \cap X \) is a cone with vertex \( \Lambda \) (here \( \Lambda \) is a linear subspace of \( \Pi \)). In case \( \Lambda \) is not contained in \( \text{Sing}(X) \), then all points of \((X \setminus \text{Sing}(X)) \cap \Lambda\) are star points of \( X \). In particular, \( \Lambda \) is contained in the closure of the set of star points on \( X \). In case \( P \in \Lambda \) is a smooth point of \( X \), then \( T_P(X) = \Pi \); hence all those star points have the same tangent space to \( X \).

**Example 2.3.** Let \( V \) be a plane in \( \mathbb{P}^N \) and let \( \Gamma \) be an irreducible plane curve of degree \( d \) in \( V \). Consider an \((N - 3)\)-dimensional linear subspace \( L \) of \( \mathbb{P}^N \) such that \( L \cap V = \emptyset \) and let \( X \) be the cone on \( \Gamma \) with vertex \( L \). If \( Q \) is a total inflection point of \( T \) with tangent line \( T \), then each point \( P \in (Q, L) \setminus L \) is a star point of \( X \) and \( T_P(X) = (T, L) \). In particular, \((Q, L)\) is contained in the closure of the locus of star points on \( X \) and all those star points have the same tangent space to \( X \).

**Remark 2.4.** It should be noted that Example 2.3 is a special case of Example 2.2. Here \( \Pi = (T, L) \) and \( \Pi \cap X \) is a divisor on \( \Pi \) equal to \( d \Lambda \) with \( \Lambda = (Q, L) \); hence it is a cone with vertex \( \Lambda \).

The following theorem implies that Example 2.2 is the typical example.

**Theorem 2.5.** Let \( X \) be an irreducible reduced hypersurface of degree \( d \) in \( \mathbb{P}^N \) and let \( \Lambda \) be a component of the closure of the locus of star points on \( X \). Then \( \Lambda \) is a linear subspace of \( \mathbb{P}^N \) of some dimension \( 0 \leq \lambda \leq N - 2 \), and there is a hyperplane \( \Pi \) in \( \mathbb{P}^N \) containing \( \Lambda \) such that \( \Pi \cap X \) is a cone in \( \Pi \) with vertex \( \Lambda \). In particular, \( \Pi \) is the tangent space to \( X \) at all smooth points of \( X \) contained in \( \Lambda \). Moreover, in case \( \dim(\Lambda) = N - 2 \), then \( \Pi \cap X \) is equal to \( d \Lambda \) as a divisor on \( \Pi \).

**Corollary 2.6.** The set of hyperplanes that do occur as tangent spaces to \( X \) at star points is finite.

**Proof of Theorem 2.5.** Let \((\mathbb{P}^N)^*\) be the dual projective space of hyperplanes in \( \mathbb{P}^N \). We are going to make use of the Gauss map \( \gamma : X \dashrightarrow (\mathbb{P}^N)^* \), which maps a smooth point \( P \) of \( X \) to the tangent hyperplane \( T_P(X) \) of \( X \) at \( P \). The tangent map \( d_P \gamma : T_P(X) \rightarrow T_{\gamma(P)}((\mathbb{P}^N)^*) \) can be described by the second fundamental form (see [4] Section 2.4.1)

\[
\Pi_P : T_P(X) \times T_P(X) \rightarrow N_P(X),
\]

where \( N_P(X) \) is the normal line to \( X \) at \( P \). To be precise, for \( \Pi_P \), the subspaces \( T_P(X) \) and \( N_P(X) \) of \( \mathbb{P}^N \) should be replaced by their inverse images in \( \mathbb{C}^{N+1} \). Consider its kernel \( K_P = \{ v \in T_P(X) \mid \Pi_P(v, w) = 0 \text{ for all } w \in T_P(X) \} \). If \( F = 0 \) is the equation of \( X \), then (see [4] Section 2.4.2)

\[
K_P = \{ v \in T_P(X) \mid v \cdot \text{Hess}_P F \cdot w^T = 0 \text{ for all } w \in T_P(X) \},
\]
where \( \text{Hess}_p F = \left[ \frac{\partial^2 F}{\partial x_i \partial x_j} (P) \right]_{i,j} \) is the Hessian matrix of \( F \) at \( P \). It is easy to see that in case \( P \) is a star point of \( X \), it follows that \( K_P = T_P(X) \). Indeed, using a coordinate transformation if necessary, we may assume that \( P = (1:0:\ldots:0) \) and \( T_P(X) = Z(X_1) \). Since \( P \) is a star point, the defining polynomial \( F \) of \( X \) is of the form \( X_1 g(X_0,\ldots,X_N) + h(X_2,\ldots,X_N) \). Hence, the Hessian is of the form

\[
\text{Hess}_p F = \begin{bmatrix}
0 & *_{0,1} & 0 & \ldots & 0 \\
*_{1,0} & *_{1,1} & *_{1,2} & \ldots & *_{1,N} \\
0 & *_{2,1} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & *_{N,1} & 0 & \ldots & 0
\end{bmatrix}
\]

and \( v \cdot \text{Hess}_p F \cdot w^T = 0 \) for all \( v, w \in T_P(X) \).

Now let \( \Lambda \) be a component of the closure of the locus of star points on \( X \). Consider the restriction \( \gamma|_{\Lambda} : \Lambda \to (\mathbb{P}^N)^* \). The previous argument implies that \( d_P(\gamma|_{\Lambda}) = 0 \) for \( P \in \Lambda \) in general; hence \( \gamma|_{\Lambda} \) is constant at the smooth points. In case \( X \) is smooth, this implies that \( X \) has finitely many star points because \( \gamma \) is a finite morphism in that case. In general, we find a hyperplane \( \Pi \in (\mathbb{P}^N)^* \) such that \( T_P(X) = \Pi \) for each \( P \in \Lambda \) smooth on \( X \). So for \( P \in \Lambda \) general, \( \Pi \cap X \) is a cone with vertex \( V \) containing \( P \). Hence \( V \) is a linear subspace of \( \mathbb{P}^N \) containing \( \Lambda \) and each point \( P \in V \) smooth on \( X \) is a star point of \( X \). This implies \( V = \Lambda \), finishing the proof. \( \square \)

**Example 2.7.** Let \( X \) be a hypersurface in \( \mathbb{P}^N \) and assume that a linear subspace \( \Lambda \) of dimension \( 0 \leq \lambda \leq N - 2 \) is a component of the closure of the set of star points on \( X \). Let \( \Pi \) be the tangent space \( T_P(X) \) of a smooth point \( P \in \Lambda \). We can choose projective coordinates \( (X_0 : X_1 : \ldots : X_N) \) on \( \mathbb{P}^N \) such that \( \Lambda \) has equation \( X_{\lambda+1} = \ldots = X_N = 0 \) and \( \Pi \) has equation \( X_{\lambda+1} = 0 \). We can write the defining polynomial \( f \) of \( X \) as \( X_{\lambda+1} h + g \), where \( g \) is independent of the variable \( X_{\lambda+1} \). The intersection of \( X \) with the hyperplane \( \Pi \) is a cone with vertex \( \Lambda \). On the other hand, it is defined by \( g = 0 \); hence \( g \) is independent of \( X_0, \ldots, X_\lambda \). So we get that the polynomial \( f \) is of the form

\[
X_{\lambda+1} h(X_0,\ldots,X_N) + g(X_{\lambda+2},\ldots,X_N).
\]

We conclude that there exists a family of hypersurfaces of dimension

\[
\binom{N + d - 1}{N} + \binom{N + d - \lambda - 2}{N - \lambda - 2} - 1
\]

such that each member \( X \) has star points in \( \Lambda \setminus (\Lambda \cap \text{Sing}(X)) \). If \( X \) is a general element of the family, its singular locus is given by \( X_{\lambda+1} = \ldots = X_N = h(X_0,\ldots,X_N) = 0 \), so \( \text{Sing}(X) \) is a hypersurface of degree \( d - 1 \) in \( \Lambda \).

**3. Extremal Case**

In this section, we consider the case where \( \lambda \) is maximal, i.e. \( \lambda = N - 2 \) (see Theorem 2.5). Assume there do exist two such linear subspaces \( \Lambda_1 \) and \( \Lambda_2 \) of \( X \). Let \( \Pi_1 \) and \( \Pi_2 \) be the corresponding hyperplanes in \( \mathbb{P}^N \). Then we have that \( \dim(\Pi_2 \cap \Lambda_1) = N - 3 \). Since \( \Pi_2 \cap \Lambda_1 \subset \Pi_2 \cap X = \Lambda_2 \), we get that \( \dim(\Lambda_1 \cap \Lambda_2) = N - 3 \) and therefore \( \dim((\Lambda_1,\Lambda_2)) = N - 1 \).
Assume $\Lambda_4$ is another such linear subspace of dimension $N - 2$ and assume $\Lambda_3$ does not contain $\Lambda_1 \cap \Lambda_2$. Since $\dim(\Lambda_1 \cap \Lambda_3) = N - 3$ and $\dim(\Lambda_2 \cap \Lambda_3) = N - 3$, it follows that $\Lambda_3 \subset (\Lambda_1, \Lambda_2)$.

**Proposition 3.1.** Let $X$ be an irreducible reduced hypersurface in $\mathbb{P}^N$ and let $S$ be the set of $(N - 2)$-dimensional linear subspaces $\Lambda$ of $X$ such that a general element of $\Lambda$ is a star point of $X$. If $S$ has at least two elements, then one of the following two possibilities holds:

1. There exists a linear subspace $L$ of dimension $N - 3$ such that all such $\Lambda \in S$ do contain $L$.

2. There exists a linear subspace $H$ of dimension $N - 1$ such that all such $\Lambda \in S$ are contained in $H$.

**Remark 3.2.** Cones over plane curves do give examples of Case 1 in Proposition 3.1. If we are in Case 1 and if $S$ has at least $d$ elements, the hypersurface $X$ will automatically be a cone over a plane curve with vertex $L$. Indeed, assume there are $d$ such linear spaces $\Lambda_1, \ldots, \Lambda_d \in S$. They correspond to $d$ different tangent hyperplanes $\Pi_1, \ldots, \Pi_d$. Take $P \in L$ general and $Q \in X$ general. It is enough to prove that $\langle P, Q \rangle \subset X$. Choose a plane $V$ containing $P$ and $Q$ such that $L_i = \Pi_i \cap V$ are $d$ different lines through $P$. Let $\gamma = V \cap X$ be considered as a divisor on $V$. Since $\Pi_i \cap X = d\Lambda_i$ as a divisor on $\Pi_i$ and $L_i$ is a line on $\Pi_i$ through $P$, it follows that $D_i = dP$ as a divisor on $L_i$ is a closed subscheme of $\gamma$. Let $V'$ be the blowing-up of $V$ at $P$, let $E$ be the associated exceptional divisor on $V'$ and let $L'_i$ (resp. $\gamma'$) be the proper transform of $L_i$ (resp. $\gamma$) on $V'$. In case the multiplicity of $\gamma$ would be smaller than $d$, we have that $\gamma'$ contains $L'_i \cap E$ for $1 \leq i \leq d$. Since the lines $L_i$ are $d$ different lines, it follows that $\gamma'$ contains $d$ different points of $E$, which contradicts the assumption. This implies that the multiplicity of $\gamma$ at $P$ is equal to $d$; hence $\langle P, Q \rangle \subset \gamma \subset X$ since $Q \in \gamma$.

It follows that in this case the set $S$ contains at most $3d$ linear subspaces, since a plane curve of degree $d$ has at most $3d$ total inflection points (see e.g. [3], IV, Ex. 2.3(e))). Equality holds for the cone over the Fermat curve of degree $d$.

**Remark 3.3.** In Case 2 the set $S$ contains at most $d$ such linear subspaces, since $H \cap X$ is a divisor of degree $d$ in $H$.

**Example 3.4.** Assume that $X$ is a hypersurface satisfying Case 2 and that $S$ has $d$ elements $\Lambda_1, \ldots, \Lambda_d \subset H$ corresponding to the tangent spaces $\Pi_1, \ldots, \Pi_d$. We can choose projective coordinates $(X_0 : \ldots : X_N)$ on $\mathbb{P}^N$ such that $H$ has equation $X_N = 0$ and $\Pi_i$ has equation $l_i(X_0, \ldots, X_N) = 0$; hence $\Lambda_i$ is defined by $X_N = l_i = 0$. One can see that the equation of $X$ is of the form

$$f(X_0, \ldots, X_N) = \prod_{i=1}^{d} l_i(X_0, \ldots, X_N) + \alpha X_N^d = 0$$

with $\alpha \in \mathbb{C}$; hence there is a 1-dimensional family of hypersurfaces having star points in the general points of $\Lambda_1, \ldots, \Lambda_d$.

4. **Surfaces in $\mathbb{P}^3$ with a Line of Star Points**

In this section, we consider the special case where $N = 3$ and $\lambda = 1$, so $X$ is a surface of degree $d \geq 3$ in $\mathbb{P}^3$ and $\Lambda$ is a line on $X$ such that the smooth points of $X$ contained in $\Lambda$ are star points of $X$. We can take projective coordinates
\(X_0 : X_1 : X_2 : X_3\) on \(\mathbb{P}^3\) such that \(\Lambda\) has equation \(X_2 = X_3 = 0\) and \(\Pi\) has equation \(X_2 = 0\). From Example 2.7 it follows that the equation of \(X\) can be written as \(f \equiv X_2h(X_0, X_1, X_2, X_3) + g(X_3) = 0\). We may take \(g(X_3) \equiv X_3^d\). If we write \(h(X_0, X_1, X_2, X_3)\) as

\[
X_3L_3(X_0, X_1, X_2, X_3) + X_2L_2(X_0, X_1, X_2) + L(X_0, X_1),
\]

the intersection \(\text{Sing}(X) \cap \Lambda\) is defined by \(X_2 = X_3 = L(X_0, X_1) = 0\). Assume \(\text{Sing}(X) \cap \Lambda\) consists of \(d - 1\) different points. We can choose the coordinates so that \(P_0 = (1 : 0 : 0 : 0)\) is one of those points, so \(L(1, 0) = 0\); hence \(L(X_0, X_1) = X_1L_1(X_0, X_1)\) and \(L_1(1, 0) \neq 0\). Consider the affine coordinates \((x_1, x_2, x_3)\) on the chart \(X_0 \neq 0\) as local coordinates around \(P_0\) (so \(x_i = X_i/X_0\)). Using these coordinates, \(X\) is defined by

\[
f \equiv x_2(x_3l_3(x_1, x_2, x_3) + x_2l_2(x_1, x_2) + x_1l_1(x_1)) + x_3^d,
\]

where \(l_i\) is the polynomial corresponding to the form \(L_i\). The transformation defined by

\[
\begin{cases}
y_1 = x_3l_3(x_1, x_2, x_3) + x_2l_2(x_1, x_2) + x_1l_1(x_1), \\
y_2 = x_2, \\
y_3 = x_3
\end{cases}
\]

is a transformation of local coordinates since \(l_1(0) \neq 0\). The equation of \(X\) in the local coordinate system \((y_1, y_2, y_3)\) is \(y_1y_2 + y_3^d\). Using a second transformation of local coordinates defined by

\[
\begin{cases}
z_1 = \frac{y_1 + y_2}{2}, \\
z_2 = \frac{(y_1 - y_2)}{2}, \\
z_3 = y_3,
\end{cases}
\]

the equation of \(X\) becomes \(z_1^2 + z_2^2 + z_3^d = 0\); hence \(P_0\) is an \(A_{d-1}\)-singularity of \(X\). We conclude that the \(d - 1\) points in \(\text{Sing}(X) \cap \Lambda\) are \(A_{d-1}\)-singularities if they are pairwise different. The following proposition states that the converse also holds.

**Proposition 4.1.** Let \(X \subset \mathbb{P}^3\) be a surface of degree \(d \geq 3\). Assume \(\Lambda\) is a line on \(X\) such that \(\Lambda \not\subset \text{Sing}(X)\) and assume \(\Lambda\) contains \(d - 1\) \(A_{d-1}\)-singularities of \(X\). Then a general point of \(\Lambda\) is a star point of \(X\).

**Proof.** We can choose projective coordinates on \(\mathbb{P}^3\) so that \(\Lambda\) is given by \(X_2 = X_3 = 0\). The equation of \(X\) can be written as

\[
X_3g(X_0, X_1, X_3) + X_2(X_3L_3(X_0, X_1, X_2, X_3) + X_2L_2(X_0, X_1, X_2) + L(X_0, X_1)).
\]

The intersection \(\text{Sing}(X) \cap \Lambda\) is defined by

\[
X_2 = X_3 = g(X_0, X_1, 0) = L(X_0, X_1) = 0
\]

and contains \(d - 1\) points, so there exists a complex number \(\alpha\) such that \(g(X_0, X_1, 0) \equiv \alpha L(X_0, X_1)\). The tangent space \(\Pi\) at a smooth point on \(\Lambda\) has equation \(X_2 + \alpha X_3 = 0\). We can change the coordinates on \(\mathbb{P}^3\) so that \(\alpha = 0\); hence \(g(X_0, X_1, 0) \equiv 0\). Write \(X_3g(X_0, X_1, X_3) = X_3^m G(X_0, X_1, X_3)\), where \(m\) is maximal. Note that \(m \geq 2\) and we need to prove that \(m = d\). Let \(\Gamma\) be the curve defined by \(X_2 = G(X_0, X_1, X_3) = 0\); hence \(X \cap \Pi = m\Gamma + \Gamma\) as divisors on \(\Pi\). Since \(\Gamma \cap \Lambda\) consists of at most \(d - m < d - 1\) points, we can take a point \(P \in \text{Sing}(X) \cap \Lambda\) such that \(P \not\subset \Gamma \cap \Lambda\). We can also take the coordinates on \(\mathbb{P}^3\) such that \(P = (1 : 0 : 0 : 0)\). It is easy to see that \(P\) is an \(A_{m-1}\)-singularity; hence \(m = d\). \(\square\)
having infinitely many star points. However, this is not true. Indeed, as a trivial
that paper, the author expects this is the only case giving rise to cubic surfaces
having infinitely many star points. However, this is not true. Indeed, as a trivial
example, one can consider a cone $X$ over a cubic curve $\Gamma$. In this case, a smooth
inflection point on $\Gamma$ corresponds to a line of star points with exactly one singular
point of $X$ and this singularity is not even rational. More generally, assume $\Lambda$
contains only one singular point of a cubic surface $X$. By taking projective coordinates
$(X_0 : X_1 : X_2 : X_3)$ on $\mathbb{P}^3$ as before (such that $\Lambda$ has equation $X_2 = X_3 = 0$ and $\Pi$
has equation $X_2 = 0$), we can write the equation of the surface $X$ as
\[ f \equiv X_2(X_3L_3(X_0, X_1, X_2, X_3) + X_2L_2(X_0, X_1, X_2) + L(X_0, X_1)) + X_3^2 = 0, \]
where $L$ is a square. We may assume that $L(X_0, X_1) = X_1^2$; hence the only singular
point of $X$ on $\Lambda$ is $P = (1 : 0 : 0 : 0)$. The intersection of the plane $X_3 = 0$ and the
surface consists of the line $\Lambda$ and the conic with equation $X_3 = X_2L_2(X_0, X_1, X_2) + \frac{1}{2}X_3^2 = 0$. This conic is non-singular if $L_2(1, 0, 0) \neq 0$. In that case, the surface $X$
has a unique singular point on $\Lambda$ (of type $A_5$ if $L_3(0, 0, 0, 1) \neq 0$ and type $E_6$ if
$L_3(0, 0, 0, 1) = 0$); hence this case is also different from the example in $\Pi$. Of
course, both examples are specialisations of the example in $\Pi$. In that way, the
statement of $\Pi$ can be adjusted and generalised as follows.

**Theorem 4.3.** Let $X$ be a surface of degree $d \geq 3$ in $\mathbb{P}^3$ and assume there is an
irreducible curve $\Lambda$ on $X$ such that a general point of $\Lambda$ is a star point on $X$. Then
$\Lambda$ is a line in $\mathbb{P}^3$ and there exists a 1-parameter family $(X(t), \Lambda(t))$ with $X(0) = X$,
$\Lambda(0) = \Lambda$ and such that for $t \neq 0$, $X(t)$ is a surface of degree $d$ in $\mathbb{P}^3$ and $\Lambda(t)$ is a
line on $X(t)$ containing $d - 1$ $A_{d-1}$-singularities of $X(t)$. In particular, a general
point on $\Lambda(t)$ is a star point on $X(t)$.

In a particular case of a surface $X$ with a line $\Lambda$ of star points, we can say
something about the types of the singularities of $X$ on $\Lambda$ if there are less than $d - 1$
singular points of $X$ on $\Lambda$.

**Lemma 4.4.** Let $X$ be a surface in $\mathbb{P}^3$ of degree $d \geq 3$ defined by an equation of
the form
\[ f \equiv X_2(X_3X_0^{d-2} + X_1^2L(X_0, X_1)) + X_3^2 = 0, \]
with $L(1, 0) \neq 0$. Then $P = (1 : 0 : 0 : 0)$ is a singularity of $X$ of type $A_{d-1}$.

**Proof:** We are going to work with the affine coordinates $(x_1, x_2, x_3)$ on the chart
$X_0 \neq 0$ (so $x_i = X_i/X_0$). The equation of $X$ becomes
\[ f \equiv x_2(x_3 + x_1^0l(x_1)) + x_3^2 = 0, \]
with $l(x_1)$ the polynomial corresponding to the form $L(X_0, X_1)$. We are interested
in the singularity $P$ in the origin. Consider the transformation
\[
\begin{align*}
  y_1 &= x_1, \\
  y_2 &= x_2, \\
  y_3 &= x_3 + x_1^0l(x_1)
\end{align*}
\]
Proof. Lemma 4.4 implies that for\( f(y_1, y_2, y_3)\), the equation of \( X \) is given by
\[
f \equiv y_2y_3 + [y_3 - y_1^d l(y_1)]^d
\]
\[
\equiv y_2y_3 + \sum_{i=1}^{d} \binom{d}{i} y_3^i (-y_1^d l(y_1))^{d-i} + (-l(y_1))^{d} y_1^{d} y_{1,\alpha}^{\alpha} = 0.
\]
In the local coordinate system defined by
\[
\begin{aligned}
z_1 &= y_1, \\
z_2 &= y_2 + \sum_{i=1}^{d} \binom{d}{i} y_3^{i-1} (-y_1^d l(y_1))^{d-i}, \\
z_3 &= y_3,
\end{aligned}
\]
the equation of \( X \) becomes \( z_2z_3 + (-l(z_1))^{d} z_1^{\alpha} = 0 \). Since \( l(0) \neq 0 \), there exists a power series \( l'(x) \) such that \( (l'(x))^{\alpha} = -l(x) \). Finally, in the coordinate system defined by
\[
\begin{aligned}
w_1 &= l'(z_1)z_1, \\
w_2 &= \frac{z_2 + z_3}{2}, \\
w_3 &= \frac{l(z_2 - z_3)}{2},
\end{aligned}
\]
the surface \( X \) is locally given by \( w_1^{\alpha} + w_2^{d} + w_3^{d} = 0 \); hence \( P \) is a singularity of type \( A_{d,\alpha - 1} \). \( \square \)

**Proposition 4.5.** Let \( X \) be a surface in \( \mathbb{P}^3 \) of degree \( d \geq 3 \) defined by an equation of the form
\[
f \equiv X_2(X_3L_3(X_0, X_1, X_2, X_3) + X_2X_2L_2(X_0, X_1, X_2) + L(X_0, X_1)) + X_3^d = 0,
\]
where \( L \) is fixed and \( L_2, L_3 \) are general. Then each smooth point on the line \( \Lambda \) defined by \( X_2 = X_3 = 0 \) is a star point and each singular point \( P = (a : b : 0 : 0) \) on \( \Lambda \) is of type \( A_{k(P)} \), where \( k(P) = d\alpha - 1 \) and \( \alpha \) is the multiplicity of the root \( (a : b) \) of \( L = 0 \). So we have that
\[
(*) \quad \sum_{P \in \Lambda \setminus \text{Sing}(X)} \frac{k(P) + 1}{d} = d - 1.
\]

*Proof.* Lemma [1,4] implies that for \( L \) fixed, there exist polynomials \( L_2 \) and \( L_3 \) such that the singularities of \( X \) on the line \( \Lambda \) are all of the type \( A_k \) and satisfy formula [12]. From [7,8], it follows that such singularities are the least worse that can deform into \( d - 1 \) singularities of type \( A_{d-1} \); hence the generic statement follows. \( \square \)

The case of cubic surfaces is also investigated in [12]. Since in that paper (because of moduli reasons) only semi-stable surfaces are considered, for cubic surfaces with infinitely many star points only the example of [11] is obtained. In [12], the notion of star points is also introduced for singular points on a cubic surface. We can generalise this notion as follows.

**Definition 4.6.** Let \( X \) be an irreducible hypersurface of degree \( d \geq 3 \) in \( \mathbb{P}^N \). A point \( P \) on \( X \) is called a **star point** on \( X \) if there exists a hyperplane \( \Pi \) in \( \mathbb{P}^N \) such that \( \Pi \cap X \) (as a scheme) is a cone with vertex \( P \).
Of course, in the case of surfaces in \( \mathbb{P}^3 \), a point \( P \) is a star point on \( X \) if and only if there exists a plane \( \Pi \) in \( \mathbb{P}^3 \) such that the reduced scheme associated to \( \Pi \cap X \) is a union of lines through \( P \). This corresponds to the definition of a star point in \cite{12} §4. Note that for smooth points on hypersurfaces, the two definitions are equivalent.

In \cite{12} §5, the notion of a proper star point is introduced. The meaning of this notion is not completely clear because it is defined with respect to a family, but in the situation of e.g. \cite{12} Proposition 5.3 there is no family. Probably, what the author means is the following (we give the general definition).

**Definition 4.7.** A star point \( P \) on \( X \) is a proper star point if there exists a 1-parameter family \((X(t), P(t))\) such that \( X(0) = X, P(0) = P \) and \( P(t) \) is a smooth star point on \( X(t) \) for \( t \neq 0 \).

**Lemma 4.8.** Each star point on a hypersurface of degree \( d \geq 3 \) is proper.

**Proof.** Let \( P \) be a star point on \( X \subset \mathbb{P}^N \). Using coordinates on \( \mathbb{P}^N \), we can assume \( P = (1 : 0 : \ldots : 0) \) and \( \Pi \) is the hyperplane in \( \mathbb{P}^N \) with equation \( X_1 = 0 \) (such that \( \Pi \cap X \) is a cone with vertex \( P \)). From Example 2.7 it follows that \( X \) has equation \( X_1 h(X_0, \ldots, X_N) + g(X_2, \ldots, X_N) = 0 \) with \( h \) (resp. \( g \)) homogeneous of degree \( d - 1 \) (resp. \( d \)). If we take general homogeneous forms \( H \) of degree \( d - 1 \) and \( G \) of degree \( d \), the surface with equation \( X_1 H(X_0, \ldots, X_N) + G(X_2, \ldots, X_N) = 0 \) is smooth and \( P \) is a star point on it. Now consider \( X(t) \) to be the surface with equation \( X_1 (h + tH) + (g + tG) = 0 \).

This proof is clearly much shorter than the proof in \cite{12} §5. In that proof, the cases of \( P \) being a singular point of type \( A_2 \), \( P \) a star point on a line connecting two \( A_1 \)-singularities and \( P \) a star point on a line connecting two \( A_2 \)-singularities are handled separately. For the first two cases, a blowing-up of \( \mathbb{P}^2 \) in 6 points is used to obtain the result of the lemma. In the third case, a sharper statement is obtained: a star point on a line connecting two \( A_2 \)-singularities is the limit of a star point on a line connecting two \( A_1 \)-singularities. We are going to generalise this result.

**Theorem 4.9.** Let \( X \) be a surface of degree \( d \geq 3 \) in \( \mathbb{P}^3 \) and let \( \Lambda \not\subset \text{Sing}(X) \) be a line containing \( d - 1 \) singularities of type \( A_{d-1} \). Let \( P \) be a smooth star point of \( X \) on \( \Lambda \). Then there is a 1-parameter family \((X(t), \Lambda(t), P(t))\) with \((X(0), \Lambda(0), P(0)) = (X, \Lambda, P)\) such that for \( t \neq 0 \) the line \( \Lambda(t) \) contains \( d - 1 \) \( A_{d-2} \)-singularities of the surface \( X(t) \) in \( \mathbb{P}^3 \) of degree \( d \) and \( P(t) \) is a smooth star point of \( X(t) \) on \( \Lambda(t) \).

**Proof.** First we make a note on surfaces \( X \) of degree \( d \) in \( \mathbb{P}^3 \) containing a line \( \Lambda \not\subset \text{Sing}(X) \) such that \( \Lambda \) contains \( d - 1 \) singularities of type \( A_{d-2} \). In the proof of Proposition 4.11 instead of \( m = d \), one obtains \( m = d - 1 \); hence the equation of \( X \) can be written as

\[
X_3^{d-1} g(X_0, X_1, X_3) + X_2 h(X_0, X_1, X_2, X_3) = 0
\]

and \( g(X_0, X_1, 0) \neq 0 \). The intersection of \( X \) and the plane \( \Pi \) with equation \( X_2 = 0 \) is equal to \( (d - 1) \Lambda + L \), where the line \( L \) has equation \( X_2 = g(X_0, X_1, X_3) = 0 \). The set \( \text{Sing}(X) \cap \Lambda \) on the line \( \Lambda \) is given by \( L(X_0, X_1) = 0 \). In case the zero of \( g(X_0, X_1, 0) \) on \( \Lambda \) is also a zero of \( L(X_0, X_1) \), the corresponding point is an \( A_{d-1} \)-singularity; hence \( L \) intersects \( \Lambda \) at a smooth point \( P \) of \( X \). Clearly, the point is a star point of \( X \).
Now assume $X$ contains a line $\Lambda \subset \text{Sing}(X)$ containing $d-1$ singularities of type $A_{d-1}$. As in the proof of Proposition [1], using suited coordinates, the equation of $X$ can be written as

$$X_3^d + X_2(X_3L_3 + X_2L_2 + L(X_0, X_1)) = 0.$$

The $d-1$ $A_{d-1}$-singularities on $\Lambda$ are the zeroes of $L(X_0, X_1) = 0$. Now take a point $P = (\alpha_0 : \alpha_1 : 0 : 0)$ on $\Lambda$ with $L(\alpha_0, \alpha_1) \neq 0$, so $P$ is smooth on $X$. Let $g(X_0, X_1) := \alpha_1X_0 - \alpha_0X_1$ and consider the surface $X(t)$ with equation

$$X_3^{d-1}(X_3 + tg(X_0, X_1)) + X_2(X_3L_3 + X_2L_2 + L(X_0, X_1)) = 0.$$

Then $\Lambda(t) := \Lambda$ is a line on $X(t)$ not contained in $\text{Sing}(X(t))$ and for $t \neq 0$ the surface $X(t)$ has exactly $d-1$ $A_{d-2}$-singularities on $\Lambda$. From the previous discussion, the point $P(t) := P$ is smooth on $X(t)$.

Now consider a cubic surface $X$ in $\mathbb{P}^3$ and denote by $S$ the closure of the union of all 1-dimensional families of star points on $X$. From Proposition [3] it follows that there are only two possibilities: either $X$ is a cone on a cubic plane curve and then $S$ is the union of lines through the vertex of $X$ or there is a plane $H$ and $S$ is a union of lines in $H$. Consider the second case. We know each pair of $A_2$-singularities in $X$ gives rise to a line inside $S$, so all $A_2$-singularities are contained in one plane $H$. In case $X$ has at least three $A_2$-singularities, then $S$ needs to be the union of three different lines in $H$. If the intersection $P$ of two such lines would be a smooth point of $X$, since $X \cap H$ is singular at $P$, we would obtain $H = T_P(X)$. But $T_P(X) \cap X$ should be a cone (since $P$ is a star point); hence $P$ is not smooth on $X$. Therefore $S$ consists of non-concurrent lines and the intersection points are $A_2$-singularities. This shows that $X$ has at most three $A_2$-singularities. This is in accordance with known results; see e.g. [9].

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References


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