STABILITY OF SEMI-FREDHOLM PROPERTIES IN COMPLEX INTERPOLATION SPACES

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Abstract. In this paper, we show that for interpolation morphisms \( S \) and the complex interpolation method the set of all \( \theta \in (0, 1) \) such that \( S[\theta] \) is a semi-Fredholm operator is open and the nullities, deficiencies and the indices of \( S[\theta] \) are locally constant on this set.

1. Introduction

Let \( S = (S_0, S_1) \) be an interpolation morphism between compatible couples of Banach spaces. E. Albrecht proved the following for the complex interpolation method (see [Alb84, Corollary 4.4]): If the operator \( S[\lambda] \) is semi-Fredholm for a \( \lambda \in (0, 1) \), then there exists \( \delta > 0 \) such that for all \( \theta \in (0, 1) \) with \( |\lambda - \theta| < \delta \), the operator \( S[\theta] \) is semi-Fredholm and \( \text{ind} S[\theta] = \text{ind} S[\lambda] \), \( \text{nul} S[\theta] \leq \text{nul} S[\lambda] \) and \( \text{def} S[\theta] \leq \text{def} S[\lambda] \). In this paper, we will prove that the nullities and the deficiencies (and the indices) of \( S[\theta] \) are actually constant in some neighborhood of \( \lambda \).

There exist several papers on Fredholm properties of operators in various interpolation scales. For interpolation operators constructed with the real method, it is well-known that the set of parameters, for which these operators are semi-Fredholm, is open and the nullities, deficiencies and indices are locally constant on this set; see [Kra98, Corollary 4.6], [CS90, Theorem 3.1] and [AS, Theorem 3]. I.Ya. Shneiberg, see [Shn73] and [Shn74], was the first author who considered Fredholm properties of interpolation operators \( A_\alpha \) between interpolation spaces \( E_\alpha \) and \( F_\alpha \), \( \alpha \in [0, 1] \). Under certain conditions on the families, the following was shown in [Shn74, Lemma 7]: For \( \alpha_0 \in (0, 1) \) such that \( A_{\alpha_0} \) has positive minimum modulus, the operators \( A_\alpha \) have positive minimum moduli and they have constant deficiencies in a neighborhood of \( \alpha_0 \). These results were extended to the interpolation of
families of Banach spaces; see [CS91]. N. Kalton and M. Mitrea considered interpolating families of operators \( T_w \) between interpolation scales of quasi-Banach spaces \( X_w \) and \( Y_w \); see [KaMi98 Section 2]. One of their results is the following (see [KaMi98 Theorem 2.3, Lemma 2.8]): For an element \( w_0 \) of the parameter space such that \( T_{w_0} \) is surjective, the operators \( T_w \) are surjective and they have constant nullities in a neighborhood of \( w_0 \).

E. Albrecht proved the results mentioned at the beginning of this introduction by lifting the operator \( S_{[\theta]} \) to a graph (i.e. relation) and applying perturbation methods for relations. In [CS90, Corollary 2.6], this result was proved in the Fredholm case by different methods. N. Krugljak and M. Milman proved in [KrMi04, Sections 6 and 7] injectivity and surjectivity results for the orbital methods using corresponding results for quotient spaces.

In the proof of our results, we will use a lifting procedure different from the one described in [Alb84 Section 4]. In the first step, we consider the case of finite deficiencies. For an interpolation couple \( \vec{E} \), let \( \mathcal{F}(\vec{E}) \) denote the well-known class of analytic functions in the strip introduced by A.P. Calderón [BL76 Chapter 4]). Each interpolation morphism \( \vec{S} \) between two Banach couples \( \vec{E} \) and \( \vec{F} \) induces an operator \( S_{[\vec{F}]} : \mathcal{F}(\vec{E}) \rightarrow \mathcal{F}(\vec{F}) \) and an operator \( S_{[\lambda]} : E_{[\lambda]} \rightarrow F_{[\lambda]} \) for \( \lambda \in (0,1) \), where the interpolation spaces \( E_{[\lambda]} \) and \( F_{[\lambda]} \) are constructed with the complex method. Let \( M_{\vec{F}(\vec{E})} \) denote the (closed, unbounded) operator of multiplication with the independent variable in \( \mathcal{F}(\vec{F}) \). For \( S_{[\lambda]} \), we consider the operator

\[
T_{\vec{F},\lambda} := \text{row}(S_{\vec{F}}, \lambda \text{id}_{\mathcal{F}(\vec{F})} - M_{\mathcal{F}(\vec{F})}) \quad \text{with domain } \mathcal{F}(\vec{E}) \times \text{D}(M_{\mathcal{F}(\vec{F})}) \text{ in } \mathcal{F}(\vec{E}) \times \mathcal{F}(\vec{F})
\]

and range in \( \mathcal{F}(\vec{F}) \). From the results in Section 4, it follows that \( T_{\vec{F},\lambda} \) is semi-Fredholm with finite deficiency if and only if \( S_{[\lambda]} \) is semi-Fredholm with finite deficiency and the deficiencies coincide. Our main results now follow from the famous Punctured Neighborhood Theorem of T. Kato; see [Kat66 Theorem IV-5.31]. In the second step, the result for the finite nullity is proved by duality; here we use the characterization of dual spaces of interpolation spaces constructed with the complex method presented in [KPS82 Lemma IV-1.7].

The paper is organized as follows: In Section 2 we introduce the spaces \( \mathcal{F}(\vec{E}) \) and \( \mathcal{K}(\vec{E}^\prime) \) for a compatible couple \( \vec{E} \); these are spaces of analytic functions on the strip. In Section 3, we introduce operators induced by \( \vec{S} \) and multiplication operators in \( \mathcal{F}(\vec{E}) \) and \( \mathcal{K}(\vec{E}^\prime) \). In Section 4 we describe an abstract lifting procedure, which will be applied in Section 5 to obtain our main results; we show that a nonjumping version of the punctured neighborhood theorem holds.

We will use the notation and definitions of [Kat66] for all relevant concepts for linear operators between Banach spaces. For the convenience of the reader, we recall the definition of semi-Fredholm operators (cf. [Kat66 IV-5.1]): A closed linear operator \( T \) between Banach spaces \( X \) and \( Y \) is said to be semi-Fredholm if its range is closed and at least one of the spaces \( N(T) \) or the quotient \( Y/R(T) \) has finite dimension; here \( N(T) \) and \( R(T) \) denote the kernel and the range of \( T \), respectively. In the following, \( \text{null} \, T \) and \( \text{def} \, T \) denote the dimension of the kernel of \( T \) and \( Y/R(T) \), respectively.
2. Complex interpolation methods

For a compatible couple $\vec{E} := (E_0, E_1)$ (i.e. the Banach spaces $E_0$ and $E_1$ can be continuously embedded into a Hausdorff topological vector space), let $E_\Delta := (E_0, E_1)_\Delta := E_0 \cap E_1$ and $E_\Sigma := (E_0, E_1)_\Sigma := E_0 + E_1$ be provided with the norms

$$\|x_\Delta\|_{E_\Delta} := \max\{\|x_\Delta\|_{E_0}, \|x_\Delta\|_{E_1}\},$$

$$\|x_\Sigma\|_{E_\Sigma} := \inf_{x_0 \in E_0, x_1 \in E_1} \{\|x_0\|_{E_0} + \|x_1\|_{E_1}\}.$$ 

It is well-known that $E_\Delta$ and $E_\Sigma$ are Banach spaces.

Now, we recall the definition of the complex interpolation method (see [BL76, Chapter 4]). Let $S_0 := \{z \in \mathbb{C} : \Re z \in (0,1)\}$. Its closure is denoted by $\Sigma$. The space $\mathcal{F}(\vec{E})$ is the set of all bounded and continuous functions $f : \Sigma \to E_\Sigma$ such that $f$ is analytic on $S_0$, $f(j + it) \in E_j$ for all $t \in \mathbb{R}$, the map $t \mapsto f(j + it)$ is continuous with respect to the norm on $E_j$ and $\|f(j + it)\|_{E_j} \to 0$ for $|t| \to \infty$, where $j \in \{0, 1\}$. If we provide $\mathcal{F}(\vec{E})$ with the norm

$$\|f\|_{\mathcal{F}(\vec{E})} := \max_{t \in \mathbb{R}} \|f(it)\|_{E_0}, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_{E_1},$$

then $\mathcal{F}(\vec{E})$ is a Banach space.

For $\lambda \in S_0$, let $(E_{\lambda}, \|\cdot\|_{\lambda})$ denote the Banach space constructed with the complex interpolation method, i.e.

$$E_{\lambda} := \{x \in E_\Sigma : \exists f \in \mathcal{F}(\vec{E}) \text{ with } f(\lambda) = x\},$$

$$\|x\|_{\lambda} := \inf_{f \in \mathcal{F}(\vec{E})} \{\|f\|_{\mathcal{F}(\vec{E})} : f(\lambda) = x\}.$$ 

The space

$$\mathcal{N}_{\mathcal{F}(\vec{E}), \lambda} := \{f \in \mathcal{F}(\vec{E}) : f(\lambda) = 0\}$$

is closed in $\mathcal{F}(\vec{E})$. The quotient space $\mathcal{F}(\vec{E})/\mathcal{N}_{\mathcal{F}(\vec{E}), \lambda}$ is isometrically isomorphic to $E_{\lambda}$.

Let $\vec{F}$ be a second compatible couple. Assume $S_0 : E_0 \to F_0$ and $S_1 : E_1 \to F_1$ are everywhere defined, linear and bounded operators. If $S_0 x = S_1 x$ for all $x \in E_\Delta$ (where the values are considered in $F_\Sigma$), then the pair $\vec{S} := (S_0, S_1)$ is said to be an interpolation morphism between $\vec{E}$ and $\vec{F}$. As usual, its norm is

$$\|\vec{S}\|_{\text{Mor}} = \|(S_0, S_1)\|_{\text{Mor}} := \max\{\|S_0\|, \|S_1\|\}.$$ 

For an interpolation morphism $\vec{S}$ between $\vec{E}$ and $\vec{F}$, the operators $S_{\Sigma} := (S_0, S_1)_\Sigma : E_\Sigma \to F_\Sigma$, $S_\Delta := (S_0, S_1)_\Delta : E_\Delta \to F_\Delta$ and $S_{\lambda} : E_{\lambda} \to F_{\lambda}$ are defined by

$$S_{\Sigma}(x_0 + x_1) := (S_0, S_1)_\Sigma(x_0 + x_1) := S_0 x_0 + S_1 x_1, \quad x_0 \in E_0, \ x_1 \in E_1, \ x \in E_\Sigma,$$

$$S_\Delta x_\Delta := (S_0, S_1)_\Delta x_\Delta := S_\Sigma x_\Delta = S_0 x_\Delta = S_1 x_\Delta, \quad x_\Delta \in E_\Delta,$$

$$S_{\lambda} x := S_{\Sigma} x, \quad x \in E_{\lambda}.$$ 

All these operators are everywhere defined, linear and bounded with $\|S_\Delta\|, \|S_{\Sigma}\| \leq \|\vec{S}\|_{\text{Mor}}$ and $\|S_{\lambda}\| \leq \|S_0\|^{1-\lambda} \|S_1\|^{\lambda}$.

The following observation plays an important role in the proof of our main result. For $\beta \in \mathbb{R}$, the operator in $\mathcal{F}(\vec{E})$ defined by $f \mapsto f(\cdot - i\beta)$ is an isometric isomorphism. Thus $E_{\lambda} = E_{\lambda + i\beta}$ with equal norms and $S_{\lambda} = S_{\lambda + i\beta}$ for all $\beta \in \mathbb{R}$ and $\lambda \in S_0$. 

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Now, we introduce the space described implicitly in [KPS82, Lemma IV-1.7] to characterize the dual space of an interpolation space constructed with the complex interpolation method.

Note that if \( \vec{E} \) is a compatible couple such that \( E_\Delta \) is dense in both \( E_0 \) and \( E_1 \), then \( \vec{E}' := (E_0', E_1') \) is a compatible couple and it follows that \( (E_0', E_1')_\Delta = E_\Sigma' \) and \( (E_0', E_1')_\Sigma = E_\Delta' \) with equal norms; see [BL76, Theorem 2.7.1]. In this case, \( \vec{E} \) is said to be a regular couple.

**Definition 2.1.** Let \( \vec{E} \) be a regular couple. We define \( \mathcal{K}(\vec{E}') \) as the set of all triples \((\psi, \psi_0, \psi_1)\) such that the following properties are fulfilled:

- \( \psi : S_0 \rightarrow (E_0', E_1')_\Sigma \) is analytic and bounded,
- \( \psi_j : \mathbb{R} \rightarrow E_j' \) is bounded for \( j \in \{0, 1\} \),
- for all \( x \in E_\Delta \) and \( j \in \{0, 1\} \), we have for almost all \( t \in \mathbb{R} \) that \( \langle x, \psi_j(t) \rangle \) is the nontangential limit (see [GM05]) of \( \langle x, \psi(z) \rangle \) for \( z \rightarrow j + it \).

The norm on the space \( \mathcal{K}(\vec{E}') \) is

\[
\| (\psi, \psi_0, \psi_1) \|_{\mathcal{K}(\vec{E}')} := \max \left\{ \sup_{t \in \mathbb{R}} \| \psi_0(t) \|_{E_0'}, \sup_{t \in \mathbb{R}} \| \psi_1(t) \|_{E_1'} \right\}.
\]

The authors would like to thank J.B. Garnett and S. Kaijser for valuable hints concerning the following statements. They suggested proving these statements by using the conformal map from the strip to the upper half plane, changing variables and using the corresponding results for the half plane (see [GM05]).

Let \( \vec{E} \) be a regular couple and \((\psi, \psi_0, \psi_1) \in \mathcal{K}(\vec{E}')\).

- \( \langle x, \psi_j(\cdot) \rangle \) is almost everywhere equal to a measurable function for \( j \in \{0, 1\} \) and all \( x \in E_\Delta \).
- The Poisson formula holds for the strip \( S_0 \); i.e., it is possible to obtain \( \langle x, \psi(\cdot) \rangle \) from the boundary functions \( \langle x, \psi_0(\cdot) \rangle \) and \( \langle x, \psi_1(\cdot) \rangle \) via the Poisson kernels for all \( x \in E_\Delta \).
- It follows for all \( z \in S_0 \) that

\[
|\langle x, \psi(z) \rangle| \leq \max \left\{ \sup_{t \in \mathbb{R}} |\langle x, \psi_0(t) \rangle|, \sup_{t \in \mathbb{R}} |\langle x, \psi_1(t) \rangle| \right\}.
\]

From the last inequality, we obtain

\[
|\psi(z)|_{(E_0', E_1')_\Sigma} \leq \max \left\{ \sup_{t \in \mathbb{R}} \| \psi_0(t) \|_{E_0'}, \sup_{t \in \mathbb{R}} \| \psi_1(t) \|_{E_1'} \right\}.
\]

Let the space \( H^\infty(S_0, (E_0', E_1')_\Sigma) \) be the set of all bounded and analytic functions \( \psi : S_0 \rightarrow (E_0', E_1')_\Sigma \) provided with the norm

\[
\| \psi \|_{H^\infty(S_0, (E_0', E_1')_\Sigma)} := \sup_{z \in S_0} \| \psi(z) \|_{(E_0', E_1')_\Sigma}
\]

and, for \( j \in \{0, 1\} \), let the space \( L^\infty(\mathbb{R}, E_j') \) be the set of all measurable and essentially bounded functions \( \psi : \mathbb{R} \rightarrow E_j' \) provided with the norm

\[
\| \psi \|_{L^\infty(\mathbb{R}, E_j')} := \text{ess sup}_{t \in \mathbb{R}} \| \psi(t) \|_{E_j'}.
\]

Since \( (\mathcal{K}(\vec{E}'), \| \cdot \|_{\mathcal{K}(\vec{E}')} \) is closed in \( H^\infty(S_0, (E_0', E_1')_\Sigma) \times L^\infty(\mathbb{R}, E_0') \times L^\infty(\mathbb{R}, E_1') \), the space \( (\mathcal{K}(\vec{E}'), \| \cdot \|_{\mathcal{K}(\vec{E}')} \) is a Banach space. It follows that \( F(\vec{E}') \subseteq \mathcal{K}(\vec{E}') \).
For convenience, we sometimes consider \((\psi, \psi_0, \psi_1) \in K(E')\) as a map from \(S\) into \((E_0', E_1')_\Sigma\); i.e.

\[
(\psi, \psi_0, \psi_1)(z) := \begin{cases} 
\psi(z) & \text{for } z \in S_0, \\
\psi_0(t) & \text{for } z = it, \\
\psi_1(t) & \text{for } z = 1 + it,
\end{cases}
\]

Let \(\lambda \in S_0\). Set

\[
N_{K(E'), \lambda} := \{(\psi, \psi_0, \psi_1) \in K(E') : \psi(\lambda) = 0\}.
\]

Since \(\|\psi(\lambda)\|_{(E_0', E_1')} \leq \|(\psi, \psi_0, \psi_1)\|_{K(E')}\) by (1), we conclude that \(N_{K(E'), \lambda}\) is closed in \(K(E')\).

In [KPS82 Lemma IV-1.7 and Theorem IV-1.6], it is proved that the spaces \(K(E')/N_{K(E'), \lambda}\) and \(E_\beta\)' are isometrically isomorphic. Again, we observe that the spaces \(K(E')/N_{K(E'), \lambda}\) and \(K(E')/N_{K(E'), \lambda + i\beta}\) are isometrically isomorphic for all \(\beta \in \mathbb{R}\) and \(\lambda \in S_0\).

3. Multiplication operators and induced operators
in \(F(E)\) and \(K(E')\)

The operators introduced in this section will be essential for the proof of the main result in Section 5.

Let \(\bar{E}\) be a compatible couple. We define in \(F(\bar{E})\) the operator \(M_{F(\bar{E})}\) as the operator of multiplication by the independent variable, namely,

\[
D(M_{F(\bar{E})}) := \{f \in F(\bar{E}) : \{z \mapsto zf(z)\} \in F(\bar{E})\},
\]

\[
(M_{F(\bar{E})}f)(z) := zf(z) \quad \text{for all } z \in S.
\]

The operator \(M_{F(\bar{E})}\) is linear and closed but not everywhere defined; indeed, let \(0 \neq x \in E_\lambda\), \(q \in \mathbb{C} \setminus S\) and \(g(z) := -\overline{z^q}\) for all \(z \in S\). Then \(g \in F(\bar{E})\) but \(g \notin D(M_{F(\bar{E})})\).

This operator was used in the proof of [AM00 Theorem 4] to show that the local uniqueness-of-resolvent condition is always fulfilled. In [Gü08 Theorem 4.4], it is shown that this can be proved using the operator \(\lambda \text{id}_{F(\bar{E})} - M_{F(\bar{E})}\) and results of the perturbation theory stated in [For06 p. 58].

**Proposition 3.1.** Let \(\bar{E}\) be a compatible couple and \(\lambda \in S_0\). The operator \(\lambda \text{id}_{F(\bar{E})} - M_{F(\bar{E})}\) is linear, injective, closed and the range \(R(\lambda \text{id}_{F(\bar{E})} - M_{F(\bar{E})})\) is equal to the closed space \(N_{F(\bar{E}), \lambda}\).

**Proof.** We only prove that \(R(\lambda \text{id}_{F(\bar{E})} - M_{F(\bar{E})}) \supseteq N_{F(\bar{E}), \lambda}\); the other assertions are clear. Let \(g \in N_{F(\bar{E}), \lambda}\). Since \(\lambda \in S_0\), there exists a function \(f : S \to F_2\) such that \(g(z) = (\lambda - z) f(z)\) for \(z \in S\). From \(\lambda \in S_0\) and \(g \in F(\bar{E})\), it follows easily that \(f \in D(\lambda \text{id}_{F(\bar{E})} - M_{F(\bar{E})})\) and \(g = (\lambda \text{id}_{F(\bar{E})} - M_{F(\bar{E})}) f \in R(\lambda \text{id}_{F(\bar{E})} - M_{F(\bar{E})})\).

Let \(\bar{S}\) be an interpolation morphism between the compatible couples \(\bar{E}\) and \(\bar{F}\). The linear operator \(S_F : F(\bar{E}) \to F(\bar{F})\), defined by

\[
(S_F f)(z) := S_2(f(z)) \quad \text{for all } z \in S,
\]
is everywhere defined and bounded with \( \| S_F \| \leq \| \tilde{S} \|_{\text{Mor}} \). It follows that \( S_F (N_{F(\tilde{E})}, \lambda) \subseteq N_{F(\tilde{E})}, \lambda \) for \( \lambda \in S_0 \). We call \( S_F \) the operator induced by \( \tilde{S} \). If \( S_0 \) is injective, then \( S_F \) is injective.

Let \( \lambda \in S_0 \). The linear operator \( S_{F, \lambda} : F(\tilde{E})/N_{F(\tilde{E})}, \lambda \rightarrow F(\tilde{F})/N_{F(\tilde{E})}, \lambda \), defined by

\[
S_{F, \lambda}(f + N_{F(\tilde{E})}, \lambda) := S_F f + N_{F(\tilde{E})}, \lambda,
\]

is well defined and bounded with \( \| S_{F, \lambda} \| \leq \| S_F \| \). If \( S_0 \) is injective, then \( S_{F, \lambda} \) is injective.

The fact that \( F(\tilde{E})/N_{F(\tilde{E})}, \lambda \) and \( E[\lambda] \) (correspondingly \( F(\tilde{F})/N_{F(\tilde{E})}, \lambda \) and \( F[\lambda] \)) are isometrically isomorphic shows that

\[
\text{def } S_{F, \lambda} = \text{def } S[\lambda].
\]

Similarly, we introduce a multiplication operator in \( K(\tilde{E}) \). Let \( \tilde{E} \) be a regular couple. We define \( M_{K(\tilde{E})} : K(\tilde{E}) \supseteq D(M_{K(\tilde{E})}) \rightarrow K(\tilde{E}) \) with

\[
D(M_{K(\tilde{E})}) := \{ (\psi, \psi_0, \psi_1) \in K(\tilde{E}) : \{ z \mapsto z(\psi, \psi_0, \psi_1)(z) \} \in K(\tilde{E}) \},
\]

\[
M_{K(\tilde{E})}(\psi, \psi_0, \psi_1)(z) := z(\psi, \psi_0, \psi_1)(z) = \begin{cases} 
\psi(z) & \text{for } z \in S_0, \\
\psi_0(t) & \text{for } z = it, \\
\psi_1(t) & \text{for } z = 1 + it,
\end{cases}
\]

The operator \( M_{K(\tilde{E})} \) is linear and closed but not everywhere defined. Indeed, for \( x' \in (E_0', E_1')_\Delta \), the triple \( (\psi, \psi_0, \psi_1) \) defined by \( (\psi, \psi_0, \psi_1)(z) := x' \) for all \( z \in S \) is an element of \( K(\tilde{E}) \) but not an element of the domain of \( M_{K(\tilde{E})} \).

The proof of the next proposition is similar to the proof of Proposition 3.1.

**Proposition 3.2.** Let \( \tilde{E} \) be a regular couple and \( \lambda \in S_0 \). The operator \( \lambda \text{id}_{K(\tilde{E})} - M_{K(\tilde{E})} \) is linear, injective, closed and the range \( R(\lambda \text{id}_{K(\tilde{E})} - M_{K(\tilde{E})}) \) is equal to the closed space \( N_{K(\tilde{E})}, \lambda \).

Let \( \tilde{F} \) be an interpolation morphism between the regular couples \( \tilde{E} \) and \( \tilde{F} \). Then \((S_0', S_1')\) is an interpolation morphism between the regular couples \((F_0', F_1')\) and \((E_0', E_1')\). Indeed, we have for \( y' \in F_0' \cap F_1' \) and \( x \in E_\Delta \) that

\[
\langle x, S_0'y' \rangle = \langle S_0x, y' \rangle = \langle S_1x, y' \rangle = \langle x, S_1'y' \rangle.
\]

Since \( E_\Delta \) is dense in \( E \), we conclude that \( S_0'y' = S_1'y' \). It follows that \( (S_0', S_1')_\Sigma = S_\lambda' \) and \( (S_0', S_1')_\Delta = S_\Sigma' \).

The linear operator \( S'_K : K(\tilde{F}) \rightarrow K(\tilde{E}) \), defined by

\[
S'_K(\psi, \psi_0, \psi_1) := ((S_0', S_1')_\Sigma \circ \psi, S_0' \circ \psi_0, S_1' \circ \psi_1),
\]

is everywhere defined and bounded with \( \| S'_K \| \leq \| (S_0', S_1') \|_{\text{Mor}} \). It follows that \( S'_K (N_{K(\tilde{F})}, \lambda) \subseteq N_{K(\tilde{F})}, \lambda \) for \( \lambda \in S_0 \). We call \( S'_K \) the operator induced by \((S_0', S_1')\).

If \((S_0', S_1')_\Sigma \) is injective, then \( S'_K \) is injective.

Let \( \lambda \in S_0 \). We define \( S'_{K, \lambda} : K(\tilde{F})/N_{K(\tilde{F})}, \lambda \rightarrow K(\tilde{E})/N_{K(\tilde{E})}, \lambda \) by setting

\[
S'_{K, \lambda}(\psi, \psi_0, \psi_1) = S'_K(\psi, \psi_0, \psi_1) + N_{K(\tilde{F})}, \lambda.
\]

This operator is everywhere defined, linear and bounded with \( \| S'_{K, \lambda} \| \leq \| S'_K \| \). If \((S_0', S_1')_\Sigma \) is injective, then \( S'_{K, \lambda} \) is injective.
Similarly, let $q_{Y_0}T = q_{Y_0}U\text{proj}_{X \times Z, X} = U_0q_{X_0}\text{proj}_{X \times Z, X}$ on $D(T)$; i.e. the following diagram is commutative on $D(T)$. (The operator $i_{X_0 \times Z}$ denotes the embedding corresponding to the inclusion.)
The following properties hold:

(a) $T$ is closed since $U$ is bounded and $V$ is closed,
(b) $q_{y_0}\{R(T)\} = R(q_{y_0}T) = R(U_0\text{proj}_{X\times Z,X}) = R(U_0)$,
(c) $N(q_{y_0}) = Y_0 = R(V) \subseteq R(T)$,
(d) $N(q_{y_0}T) = N(\text{row}(q_{y_0}U,0_D(V))) = N(q_{y_0}U) \times D(V)$,
(e) $N(U_0) = \{q_{x_0}x : x \in X \text{ and } 0 = U_0q_{x_0}x = q_{y_0}Ux\} = q_{x_0}\{N(U_0q_{x_0})\}$

Assume that $V$ is injective. Then $V_0$ is injective and we have

(g) $N(T) = \{(x,z) \in X \times D(V) : Ux = -Vz\} = \{(x,-V^{-1}Ux) : Ux \in Y_0\}$,
(h) $N(T_0) = \{(x,z) \in X_0 \times D(V_0) : U_0x_0 = -V_0z\} = \{(x,-V_0^{-1}U_0x_0) : x_0 \in X_0\}$.

**Proposition 4.1.** Let $X,Y,Z$ be Banach spaces and $X_0$, $Y_0$ be closed subspaces of $X$ and $Y$, respectively. Assume that $U : X \to Y$ is everywhere defined, linear and bounded with $U\{X_0\} \subseteq Y_0$ and $V : Z \supseteq D(V) \to Y$ is linear and closed with $R(V) = Y_0$. Then for $T = \text{row}(U,V)$, the following properties hold:

(i) The set $R(T)$ is closed if and only if $R(U_0)$ is closed.
(ii) The spaces $Y/R(T)$ and $(Y/Y_0)/R(U_0)$ are algebraically isomorphic. If $R(T)$ (or equivalently $R(U_0)$) is closed; see (i)), then these spaces are isometrically isomorphic.
(iii) Assume $V$ is injective. Then the spaces $N(T)/N(T_0)$ and $N(U_0)$ are isomorphic with equivalent norms.

**Proof.** (i) Assume $R(T)$ is closed. Since $R(q_{y_0})$ is closed, we obtain that $q_{y_0}\{R(T)\} = R(U_0)$ is closed from (b), (c) above and [Kat58, Lemma 331].

Conversely, assume $R(U_0)$ is closed. Let $\{y_n\}_{n \in \mathbb{N}} \subseteq R(T)$, $y \in Y$ be such that $y_n \to y$. Then $q_{y_0}y_n \to q_{y_0}y$, and, by (b) above, it follows that $q_{y_0}y_n \in q_{y_0}\{R(T)\} = R(U_0)$. Since $R(U_0)$ is closed, we obtain $q_{y_0}y \in R(U_0)$. Thus $y \in R(T) + Y_0 = Y_0$ by (b) and (c) above.

(ii) The operator $q_{y_0} : Y \to Y_0$ is everywhere defined, linear, surjective and $q_{y_0}\{R(T)\} = R(U_0)$ by (b) above. Then it follows directly that its induced operator from $Y/R(T)$ into $(Y/Y_0)/R(U_0)$ is everywhere defined, linear and surjective. The injectivity follows from (b) above.

(iii) We consider the operator from $N(T)$ into $N(U_0)$ defined by

$$(x,z) \to q_{x_0}x, \quad (x,z) \in N(T)$$

This operator is well defined and surjective with the kernel $\{(x,-V^{-1}Ux) : x \in X_0\} = N(T_0)$. Moreover, it is the restriction of the bounded operator $q_{x_0}\text{proj}_{X\times Z,X}$ to the spaces $N(T)$ and $N(U_0)$. \qed
In the following section, we will need the corollary below, itself a direct consequence of Proposition 4.1.

**Corollary 4.2.** Under the assumptions of Proposition 4.1, we have the following equivalence:

\[
deft = \text{def row}(U, V) = n < \infty \iff \text{def } U/0 = n < \infty.
\]

5. Main result and corollaries

Our main result is the following ‘Punctured Neighborhood Theorem’ for the complex interpolation method. The proof of this theorem relies on the famous Punctured Neighborhood Theorem of T. Kato; see [Kat66, Theorem IV-5.31] (cf. [Kat58, Theorem 3 and Theorem 5]). We use the concept of semi-Fredholm operators as in [Kat66, IV-5.1]; see also the end of Section 1.

**Theorem 5.1.** Let \( \bar{S} \) be an interpolation morphism between the compatible couples \( \vec{E} \) and \( \vec{F} \).

(i) Assume \( \lambda \in \mathbb{S}_0 \) such that \( \text{def } S[\lambda] < \infty \). Then there exists \( \delta > 0 \) such that

\[
\text{def } S[\lambda] = \text{def } S[\theta]
\]

for all \( \theta \in \mathbb{D}_{\lambda, \delta} \cap \mathbb{S}_0 \), where \( \mathbb{D}_{\lambda, \delta} := \{ z \in \mathbb{C} : |z - \lambda| < \delta \} \).

(ii) Assume \( \vec{E} \) and \( \vec{F} \) are regular and \( \lambda \in \mathbb{S}_0 \) such that \( \text{def } S[\lambda]' < \infty \). Then there exists \( \delta > 0 \) such that

\[
\text{def } S[\lambda]' = \text{def } S[\theta]'
\]

for all \( \theta \in \mathbb{D}_{\lambda, \delta} \cap \mathbb{S}_0 \).

Proof. (i) For \( \theta \in \mathbb{S}_0 \), let \( T_{\vec{F}, \theta} := \text{row}(S_{\vec{F}}, \theta \text{id}_{\vec{F}}) - M_{\vec{F}}(\vec{F}) \) with domain \( \mathcal{F}(\hat{E}) \times \text{D}(M_{\vec{F}}(\vec{F})) \in \mathcal{F}(\hat{E}) \times \mathcal{F}(\hat{F}) \) and range in \( \mathcal{F}(\hat{F}) \). It follows that \( \text{def } S[\lambda] = \text{def } S_{\vec{F}, \lambda} \); see (2) in Section 3. From Proposition 3.1 and Corollary 4.2 (cf. the scheme in Section 4), it follows that \( T_{\vec{F}, \lambda} \) is semi-Fredholm and \( \text{def } T_{\vec{F}, \lambda} = \text{def } S_{\vec{F}, \lambda} \) is finite. We have

\[
T_{\vec{F}, \theta} = (\theta - \lambda)\text{row}(0, \text{id}_{\vec{F}}) + T_{\vec{F}, \lambda}
\]

for all \( \theta \in \mathbb{S}_0 \). Then we obtain from the well-known Punctured Neighborhood Theorem of T. Kato (see [Kat66, Theorem IV-5.31]) that there exist \( \delta > 0 \) and a non-negative integer \( r \) such that \( \text{def } (T_{\vec{F}, \theta}) = \text{def } (T_{\vec{F}, \lambda}) - r \) for all \( \theta \in (\mathbb{D}_{\lambda, \delta} \setminus \{\lambda\}) \cap \mathbb{S}_0 \). From Proposition 3.1 and Corollary 4.2 (cf. the scheme in Section 4), we conclude that \( \text{def } (S_{\vec{F}, \theta}) = \text{def } (S_{\vec{F}, \lambda}) - r \) for all \( \theta \in (\mathbb{D}_{\lambda, \delta} \setminus \{\lambda\}) \cap \mathbb{S}_0 \). Thus \( \text{def } S[\lambda] = \text{def } S[\lambda]' \) for all \( \theta \in (\mathbb{D}_{\lambda, \delta} \setminus \{\lambda\}) \cap \mathbb{S}_0 \) by (2) in Section 3. Since \( F[\lambda] = F[\lambda'+\frac{1}{2}] \) and \( S[\lambda] = S[\lambda'+\frac{1}{2}] \), it follows that \( r = 0 \).

(ii) In the same way as (i), we prove (ii) with Proposition 3.2 and (3) in Section 3 considering the operator row(\( S_\vec{F}', \theta \text{id}_{\vec{F}'} - M_{\vec{F}'} \)).
From Theorem 5.1 and [Alb84, Corollary 4.4], we then obtain the following corollary.

**Corollary 5.2.** Let \( \vec{S} \) be an interpolation morphism between the regular couples \( \vec{E} \) and \( \vec{F} \). Assume \( \lambda \in S_0 \) such that \( S[\lambda] \) is semi-Fredholm. Then there exists \( \delta > 0 \) such that

\[
\begin{align*}
\text{null } S[\lambda] &= \text{null } S[\theta], \\
\text{def } S[\lambda] &= \text{def } S[\theta], \\
\text{ind } S[\lambda] &= \text{ind } S[\theta]
\end{align*}
\]

for all \( \theta \in D_{\lambda, \delta} \cap S_0 \).

The previous corollary improves upon results of E. Albrecht stated in [Alb84, Corollary 4.4] concerning the weak inequalities of the nullities and deficiencies, in the sense that here we show that these inequalities are, in fact, equalities.

**Corollary 5.3.** Let \( \vec{S} \) be an interpolation morphism between the regular couples \( \vec{E} \) and \( \vec{F} \). The set

\[
\{ \theta \in S_0 : S[\theta] \text{ is semi-Fredholm} \}
\]

is open and the nullities, deficiencies and the indices of \( S[\cdot] \) are locally constant on this set.

**References**


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