This paper concerns a generalization of a theorem of Dan Gottlieb [5] motivated by a question of Dusa McDuff [12]. Let \( f : M \to N \) be a map of closed oriented connected manifolds of the same dimension \( n \). We suppose that \( M \) and \( N \) are pointed and that \( f \) preserves the base point. We consider the mapping space \( \text{Map}(M, N; f) \) of continuous maps homotopic to \( f \), and we denote by \( \omega : \text{Map}(M, N; f) \to N \) the evaluation map at the base point \( \ast \in M \), \( \omega(g) = g(\ast) \). In [5] Gottlieb proves that if \( \pi_1(\omega) : \pi_1(\text{Map}(M, N; f)) \to \pi_1(N) \) is non-zero, then either the degree of \( f \), \( \deg f = 0 \), or the Euler characteristic \( \chi(N) = 0 \).

In this paper we consider what happens with the higher order homotopy groups when \( M \) and \( N \) are simply connected. Recall first that in the case \( f = id_N \), the space \( \text{Map}(M, N; f) \) is the monoid \( \text{aut}_1(N) \) of self-equivalences of \( N \) homotopic to the identity. Then denote the evaluation map by \( ev : \text{aut}_1 N \to N \). The image of \( \pi_q(ev) \) is called the \( q^{\text{th}} \) Gottlieb group of \( N \), \( G_q(N) \). The groups \( G_q(N) \otimes \mathbb{Q} \) and the map \( ev \) have been intensively studied in rational homotopy. In particular, for each integer \( q \), \( G_{2q}(N) \otimes \mathbb{Q} = 0 \) and \( \sum_q \dim G_q(N) \otimes \mathbb{Q} < n \) [2]. On the other hand, by a result of G. Lupton and the author, \( \tilde{H}_s(ev; \mathbb{Q}) = 0 \) if \( \chi(N) \neq 0 \) [4]. See also [10], [11], [6] for more recent results on Gottlieb groups.

Since the composition with \( f \) induces a continuous map \( j_N : \text{aut}_1 N \to \text{Map}(M, N; f) \) satisfying \( \omega \circ j_N = ev \), we have the inclusion \( G_*(N) \subset \text{Im} \pi_*(\omega) \). Our main theorem is essentially a converse to this result.

**Theorem 1.**

1. If \( \deg f \neq 0 \), then \( \text{Im} \pi_*(\omega) \otimes \mathbb{Q} = G_*(N) \otimes \mathbb{Q} \).
2. If \( \deg f \neq 0 \) and \( \chi(N) \neq 0 \), then the composition of \( \pi_*(\omega) \) with the Hurewicz map \( \text{hur}_N : \pi_*(N) \otimes \mathbb{Q} \to H_*(N; \mathbb{Q}) \),
   \[
   \pi_*(\text{Map}(M, N; f)) \otimes \mathbb{Q} \to H_*(N; \mathbb{Q}),
   \]
   is zero.
An important step in the proof of Theorem 1 is given by Theorem 2.

**Theorem 2.** Suppose \( \deg f \neq 0 \). Then the injection \( j_N : \text{aut}_1(N) \to \text{Map}(M, N; f) \) induces an injection on the rational homotopy groups,

\[
\pi_*(j_N) \otimes \mathbb{Q} : \pi_*(\text{aut}_1(N)) \otimes \mathbb{Q} \to \pi_*(\text{Map}(M, N; f)) \otimes \mathbb{Q}.
\]

Moreover \( \pi_*(j_N) \otimes \mathbb{Q} \) admits a retraction \( \sigma \) satisfying \( (\pi_*(\text{ev}) \otimes \mathbb{Q}) \circ \sigma = \pi_*(\omega) \otimes \mathbb{Q} \).

Now recall that a continuous map \( g : X \to Y \) is called a rational Gottlieb map if \( \pi_*(g) \otimes \mathbb{Q} \) maps \( G_*(X) \otimes \mathbb{Q} \) into \( G_*(Y) \otimes \mathbb{Q} \). For instance, when \( X \) is an odd-dimensional sphere \( S^q \), then \( g \) is a rational Gottlieb map if and only if \([g]\) belongs to the Gottlieb group \( G_*(Y) \otimes \mathbb{Q} \). We prove

**Theorem 3.** With the above notation, if \( \deg f \neq 0 \), then \( f \) is a rational Gottlieb map.

For the proofs, we use the machinery of rational homotopy theory as described for instance in [8]. We use more precisely the Poincaré duality model for manifolds given by Lambrechts and Stanley in [8] and the description of the rational homotopy groups of mapping spaces in terms of derivations given by Lupton and Smith in [9].

1. A Convenient Model for \( f \)

A Poincaré duality model for a simply connected Poincaré duality complex \( X \) of dimension \( n \) is a commutative differential graded algebra \( (A, d) \) quasi-isomorphic to the Sullivan minimal model of \( X \) and satisfying Poincaré duality in dimension \( n \). This means that there exists a linear map \( \varepsilon : A^n \to \mathbb{Q} \) such that \( \varepsilon(dA^{n-1}) = 0 \) and such that the induced bilinear forms

\[
A^k \otimes A^{n-k} \to \mathbb{Q}, \quad a \otimes b \mapsto \varepsilon(ab)
\]

are non-degenerate. Such a map \( \varepsilon \) is then called an orientation.

**Theorem 4** (Lambrecht, Stanley [8]).

1. If \( (A, d) \) is a commutative differential graded algebra whose cohomology is a simply connected Poincaré duality algebra, then \( (A, d) \) is weakly equivalent to a commutative differential graded algebra \( (A', d) \) that is a simply connected Poincaré duality algebra.

2. Moreover, if \( A \) is finite type, \( A^0 = \mathbb{Q} \), \( A^1 = 0 \), \( A^2 \subset \text{Kerd} \), and \( n \geq 7 \), then there is a quasi-isomorphism \( \varphi : (A, d) \to (A', d) \).

We deduce the following generalization,

**Proposition 1.** Let \( f : M \to N \) be a map between simply connected \( n \)-dimensional manifolds, \( n \geq 7 \). Suppose that \( H^2(f) \) is injective. Then \( f \) admits a Sullivan model of the form

\[
\varphi : (A, d) \to (B, d),
\]

where \( (A, d) \) and \( (B, d) \) are Poincaré duality algebras of dimension \( n \).

**Proof.** Let \( j : (\wedge V, d) \to (\wedge V \otimes \wedge W, D) \) be a relative minimal model for \( f \) and let \( (\wedge V, d) \to (A, d) \) be a quasi-isomorphism with \( (A, d) \) a Poincaré duality model for \( N \). By taking the tensor product \( (A, d) \otimes (\wedge V, d) \), we get another model of \( f \),

\[
1 \otimes j : (A, d) \to (A \otimes \wedge W, D') := (A, d) \otimes (\wedge V, d) (\wedge V \otimes \wedge W, D).
\]

Since \( H^2(f) \) is injective, \( (A \otimes \wedge W)^1 = 0 \). By Theorem 4(2) there then exists a quasi-isomorphism \( g : (A \otimes \wedge W, D') \to (B, d) \) where \( (B, d) \) is a Poincaré duality model of \( M \). We define \( \varphi : (A, d) \to (B, d) \) by \( \varphi = g \circ (1 \otimes j) \).

\( \square \)
Our next proposition explains the structure of the model \( \varphi \) when \( \deg f \neq 0 \).

**Proposition 2.** With the above notation, if \( \deg f \neq 0 \), then

1. the map \( \varphi \) is injective,
2. the graded vector space \( B \) can be decomposed as \( B = \varphi(A) \oplus Z \), where \( d(Z) \subset Z \) and \( Z \cdot \varphi(A) \subset Z \).

**Proof.** Denote by \( \omega \) and \( \omega' \) rational fundamental classes of \( A \) and \( B \) with \( \varphi(\omega) = \omega' \), and denote by \( \varepsilon' \) the associated orientation of \( B \). In particular, \( \varepsilon(a) = \varepsilon'(\varphi(a)) \) for \( a \in A \). If \( \varphi(a) = 0 \) for some \( a \), then \( \varphi(\omega) = 0 \) because there is an element \( a' \) such that \( aa' = \omega \), and so \( \varphi \) is injective.

Denote
\[
Z = \{ x \in B \mid \varepsilon'(x \cdot \varphi(A)) = 0 \}.
\]
Choose a homogeneous basis \( h_1, \ldots, h_m \) of \( A \). Using the Poincaré duality of \( A \), there is another family \( \{ h_j^* \} \subset A \) such that \( \varepsilon(h_j^* \cdot h_i) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol. Then \( \varphi(h_1), \ldots, \varphi(h_m) \) and \( \varphi(h_1^*), \ldots, \varphi(h_m^*) \) are homogeneous bases of \( \varphi(A) \). If \( b \in B \), the element
\[
b' = b - \sum_j \varepsilon'(b \cdot \varphi(h_j)) \varphi(h_j^*)
\]
is in \( Z \). This shows that \( B = \varphi(A) \oplus Z \). Now for an element \( z \in Z \),
\[
\varepsilon'(dz \cdot \varphi(a)) = \varepsilon'(d(z \varphi(a))) - (-1)^{|z|} \varepsilon'(z \cdot d \varphi(a)) = 0.
\]
This shows that \( d(Z) \subset Z \). We also have \( Z \cdot \varphi(A) \subset Z \) because \( \varepsilon' [(z \cdot \varphi(a)) \cdot \varphi(A)] = 0 \).

2. **Model for mapping spaces**

In this section we recall the derivation model for the rational homotopy groups of a mapping space \( \text{Map}(X, Y; f) \), where \( X \) and \( Y \) are simply connected CW complexes and \( \dim X < \infty \). Recall that under those hypothesis, \( \text{Map}(M, N; f) \) is a nilpotent space whose rationalization is obtained by taking the composition with a rationalization of \( N \), \( \ell : N \to N_0 \) ([7]).

Let \( \ell : (\wedge V, d) \to (A, d) \) be a model for \( f \), with \( (\wedge V, d) \) a Sullivan (non-necessarily minimal) model for \( Y \) and \( (A, d) \) a connected model for \( X \). We consider the differential graded vector space \( (\text{der}^\ell(\wedge V, A), \delta) \) of \( \ell \)-derivations, where \( \text{der}^\ell(\wedge V, A)_m \) is the vector space of linear maps of degree \( m \), \( \theta : (\wedge V)^* \to A^{* - m} \) for which \( \theta(xy) = \theta(x)\ell(y) + (-1)^{m|y|}\ell(x)\theta(y) \). The differential \( \delta \) is defined as usual by
\[
\delta \theta = d \circ \theta + (-1)^{m+1} \theta \circ d.
\]
From now on, we restrict to \( \text{Der}^\ell(\wedge V, A) \), the positive \( \ell \)-derivations,
\[
\text{Der}^\ell(\wedge V, A)_r = \begin{cases} 
\text{der}^\ell(\wedge V, A)_r, & \text{if } r > 1, \\
Z \text{der}^\ell(\wedge V, A)_1, & \text{if } r = 1,
\end{cases}
\]
where \( Z \) denotes the space of cycles.

It is well known that if \( g : (\wedge W, d) \to (\wedge V, d) \) is a quasi-isomorphism, then the induced map \( \text{Der}^\ell(\wedge V, A) \to \text{Der}^{\ell g}(\wedge W, A) \) is a quasi-isomorphism. In the same way, if \( h : (A, d) \to (A', d) \) is a quasi-isomorphism, then the induced map \( \text{Der}^\ell(\wedge V, A) \to \text{Der}^{h \circ \ell}(\wedge V, A') \) is also a quasi-isomorphism (see for instance [1],
Theorem 2.8). The following theorem follows then directly from [9], where the theorem is proved when \((\wedge V,d)\) is minimal and \((A,d)\) is the minimal model of \(X\).

**Theorem 5** ([9], [1], Theorem 3.8). With the above notation,

1. we have natural isomorphisms of graded vector spaces
   \[
   H_*(\text{Der}^f(\wedge V,A)) \cong \pi_*(\text{Map}_*(X,Y;f)) \otimes Q;
   
   H_*(\text{Der}^f(\wedge V,A^+)) \cong \pi_*(\text{Map}_*(X,Y;f)) \otimes Q,
   \]
   where \(\text{Map}_*(X,Y;f)\) denotes the subspace of \(\text{Map}(M,N;f)\) consisting of pointed maps.

2. Denote by \(\alpha : A \to Q\) the augmentation. Then the composition with \(\alpha\) induces a morphism of complexes that is a model for \(\pi_*(\omega) \otimes Q\); i.e., we have a commutative diagram
   \[
   H_*(\text{Der}^f(\wedge V,A)) \xrightarrow{\tilde{\alpha}} H_*(\text{Der}^\alpha(\wedge V,Q)) \cong H_*(\text{Hom}(V,Q)) \cong \pi_*(N) \otimes Q.
   \]

### 3. Proof of Theorem 2

We first consider the case \(n \geq 7\) and we use the notation of section 1. In particular, \(\varphi : (A,d) \to (B,d)\) is a model for \(f\), and \((A,d)\) and \((B,d)\) are Poincaré duality models for \(N\) and \(M\). Let \(\psi : (\wedge V,d) \to (A,d)\) be a surjective Sullivan model for \((A,d)\), and then let \(\rho = \varphi \circ \psi : (\wedge V,d) \to (B,d)\).

Recall that \(B = \varphi(A) \oplus Z\) with \(dZ \subset Z\) and \(Z : \varphi(A) \subset Z\). Then \(B = \rho(\wedge V) \oplus Z\) with \(Z : \rho(\wedge V) \subset Z\). Define a linear map
   \[
   \Phi : \text{Der}^\rho(\wedge V,B) \to \text{Der}^\psi(\wedge V,A)
   \]
by \(\Phi(\theta) = p \circ \theta\), where \(p : B \to \varphi(A) \cong A\) is the linear projection with kernel \(Z\).

**Lemma 1.** The morphism \(p \circ \theta\) is a \(\psi\)-derivation.

**Proof.** Write \(\theta(a) = x + z\) with \(z \in Z\) and \(x = \varphi(p\theta(a))\) and \(\theta(b) = x' + z'\) with \(z' \in Z\) and \(x' = \varphi(p\theta(b))\). Then,
   \[
   p\theta(ab) = p(\theta(a) \cdot \rho(b) + (-1)^{|a||\theta|}\rho(a) \cdot \theta(b))
   = p(x\rho(b) + (-1)^{|a||\theta|}\rho(a)x')
   = p(\varphi(p\theta(a)) \cdot \psi(b) + (-1)^{|\theta||a|}\varphi(p\theta(a)) \cdot p\theta(b))
   = p\theta(a) \cdot \psi(b) + (-1)^{|a||\psi|}\psi(a) \cdot p\theta(b).
   \]

**Lemma 2.** \(\Phi\) is a morphism of complexes.

**Proof.**
   \[
   \delta(p\theta) = dp\theta - (-1)^{|\theta|}(p\theta)d = p(d\theta) - (-1)^{|\theta|}p(\theta d) = p(\delta \theta).
   \]

Since \(A\) is a subalgebra of \(B\), the injection of \(A\) into \(B\) induces a morphism of complexes
   \[
   \Psi : \text{Der}^\psi(\wedge V,A) \to \text{Der}^\rho(\wedge V,B)
   \]
such that \(\Phi \circ \Psi = id_{\text{Der}^\psi(\wedge V,A)}\).
Proof of Theorem 2 when \( n \geq 7 \). When \( n \geq 7 \), Theorem 5 directly yields that the induced injection \( \Phi : \text{Der}^p(\wedge V, A) \to \text{Der}^p(\wedge V, B) \) induces in homology the morphism \( \pi_*([j_N] \otimes Q) \). A retraction \( \sigma \) to \( \pi_*([j_N] \otimes Q) \) is given by \( H_*(\Phi) \).

Now the commutativity of the diagram
\[
\begin{array}{ccc}
\text{Der}^p(\wedge V, B) & \xrightarrow{\hat{\alpha}} & \text{Der}^{\alpha \circ \hat{\alpha}}(\wedge V, Q) \\
\Phi \downarrow & & \| \\
\text{Der}^p(\wedge V, A) & \xrightarrow{\hat{\alpha}} & \text{Der}^{\alpha \circ \hat{\alpha}}(\wedge V, Q)
\end{array}
\]

implies in homology that \( (\pi_*(\text{ev}) \otimes Q) \circ \sigma = \pi_*(\omega) \otimes Q \).

\[\blacksquare\]

Proof of Theorem 2 for \( n \leq 6 \). Let define \( g = f \times \text{id}_{S^6} : M \times S^6 \to N \times S^6 \). If \( f' \sim f \), then \( f' \times \text{id} \sim f \times \text{id} \). This defines a continuous map
\[ e : \text{Map}(M, N; f) \to \text{Map}(M \times S^6, N \times S^6; f \times \text{id}) \, .\]

On the other hand, if \( g' : M \times S^6 \to N \times S^6 \) is homotopic to \( f \times \text{id} \), then we consider the composition
\[ M \xrightarrow{h} M \times S^6 \xrightarrow{g'} N \times S^6 \xrightarrow{p} N \, , \]
where \( * \in S^6 \) is the base point, \( h(m) = (m, *) \) and \( p(r, x) = r \). Since \( pg'h \sim f \), this gives a continuous map
\[ r : \text{Map}(M \times S^6, N \times S^6; f \times \text{id}) \to \text{Map}(M, N; f) \, .\]

Clearly \( r \circ e = \text{id} \); i.e., \( \text{Map}(M, N; f) \) is a retract of \( \text{Map}(M \times S^6, N \times S^6; f \times \text{id}) \).

Since \( \dim(\wedge N) \geq 7 \), we have a map \( \sigma \) making commutative the diagram
\[
\begin{array}{ccc}
\pi_*(\text{Map}(M, N; f)) \otimes Q & \xrightarrow{\pi_*(\omega)} & \pi_*(N) \otimes Q \\
\pi_*(\text{Map}(M \times S^6, N \times S^6, f \times \text{id})) \otimes Q & \xrightarrow{\pi_*(\omega)} & \pi_*(N \times S^6) \otimes Q \\
\sigma & & \| \\
\pi_*(\text{aut}_1(N \times S^6)) \otimes Q & \xrightarrow{\pi_*(\text{ev})} & \pi_*(N \times S^6) \otimes Q,
\end{array}
\]
where \( h'(n) = (n, *) \).

In the same way, \( \text{aut}_1(N) \) is a retract of \( \text{aut}_1(N \times S^6) \). Denote by \( i : \text{aut}_1(N) \to \text{aut}_1(N \times S^6) \) the natural injection. The retraction \( \tau \) associates to a map \( h : N \times S^6 \to N \times S^6 \) the composition
\[ N \xrightarrow{h'} N \times S^6 \xrightarrow{h} N \times S^6 \xrightarrow{p} N \, .\]

By construction we have \( \text{ev} \circ \tau = p \circ \text{ev} \) and a commutative diagram
\[
\begin{array}{ccc}
\text{aut}_1(N) & \xrightarrow{\iota} & \text{aut}_1(N \times S^6) \\
\downarrow j_N & & \downarrow j_{N \times S^6} \\
\text{Map}(M, N; f) & \xrightarrow{i} & \text{Map}(M \times S^6, N \times S^6, f \times \text{id}) \, .
\end{array}
\]

Then \( \sigma' = (\pi_*(\tau) \otimes Q) \circ \sigma \circ (\pi_*(e) \otimes Q) \),
\[ \pi_*(\text{Map}(M, N; f)) \otimes Q \to \pi_*(\text{Map}(M \times S^6, N \times S^6; f \times \text{id})) \otimes Q \to \pi_*(\text{aut}_1(N \times S^6)) \otimes Q \to \pi_*(\text{aut}_1(N)) \otimes Q \]
is a retraction of \( \pi_*(j_N) \otimes Q \), and by construction \( (\pi_*(\text{ev}) \otimes Q) \circ \sigma' = \pi_*(\omega) \otimes Q \).

\[\blacksquare\]

Proof of Theorem 1. Theorem 2 shows that \( \text{Im} \pi_*(\omega) \otimes Q \subset G_*(N) \otimes Q \). This proves Theorem 1(1). If \( \chi(N) \neq 0 \), \( H_*\text{ev}; Q) = 0 \), and so by Theorem 2, \( h_{\text{ur}} \circ (\pi_*(\omega) \otimes Q) = h_{\text{ur}} \circ (\pi_*(\text{ev}) \otimes Q) \circ \sigma = h_{\text{ur}}\text{ev}; Q) \circ h_{\text{ur}}\text{aut}_1(N) \circ \sigma = 0 \).

\[\blacksquare\]
Proof of Theorem 3. Denote by $i_M : aut_1 M \to \text{Map}(M, N; f)$ the map associated to the composition with $f$. The commutativity of the diagram ($q \geq 1$)
\[
\begin{array}{cccc}
\pi_q(aut_1(M)) \otimes \mathbb{Q} & \xrightarrow{\pi_q(i_M)} & \pi_q(\text{Map}(M, N; f)) \otimes \mathbb{Q} & \xrightarrow{\sigma} & \pi_q(aut_1(N)) \otimes \mathbb{Q} \\
\downarrow \pi_q(ev) & & \downarrow \pi_q(ev) & & \downarrow \pi_q(ev) \\
\pi_q(M) \otimes \mathbb{Q} & \xrightarrow{\pi_q(f)} & \pi_q(N) \otimes \mathbb{Q} & & \pi_q(N) \otimes \mathbb{Q}
\end{array}
\]
shows that $f$ maps $G_*(M) \otimes \mathbb{Q}$ into $G_*(N) \otimes \mathbb{Q}$. □

ACKNOWLEDGMENT

The author would like to thank the referee for helpful comments and suggestions.

REFERENCES


Institut Mathématique, Université Catholique de Louvain, 2, Chemin du Cyclotron, 1348 Louvain-La-Neuve, Belgium