MAPPING SPACES BETWEEN MANIFOLDS
AND THE EVALUATION MAP

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Abstract. Let $f : M \to N$ be a map between simply connected $n$-dimensional manifolds. We suppose that $\deg f \neq 0$. Then the injection of $\text{aut}_1(N)$ into the component $\text{Map}(M, N; f)$ of the mapping space containing $f$ induces an injection on the rational homotopy groups, and the evaluation at the base point $\text{map}(M, N; f) \to N$ is zero on the rational homotopy groups of even dimension.

This paper concerns a generalization of a theorem of Dan Gottlieb [5] motivated by a question of Dusa McDuff [12]. Let $f : M \to N$ be a map of closed oriented connected manifolds of the same dimension $n$. We suppose that $M$ and $N$ are pointed and that $f$ preserves the base point. We consider the mapping space $\text{Map}(M, N; f)$ of continuous maps homotopic to $f$, and we denote by $\omega : \text{Map}(M, N; f) \to N$ the evaluation map at the base point $* \in M$, $\omega(g) = g(*)$. In [5] Gottlieb proves that if $\pi_1(\omega) : \pi_1(\text{Map}(M, N; f)) \to \pi_1(N)$ is non-zero, then either the degree of $f$, $\deg f = 0$, or the Euler characteristic $\chi(N) = 0$.

In this paper we consider what happens with the higher order homotopy groups when $M$ and $N$ are simply connected. Recall first that in the case $f = \text{id}_N$, the space $\text{Map}(M, N; f)$ is the monoid $\text{aut}_1(N)$ of self-equivalences of $N$ homotopic to the identity. Then denote the evaluation map by $ev : \text{aut}_1 N \to N$. The image of $\pi_q(ev)$ is called the $q^{th}$ Gottlieb group of $N$, $G_q(N)$. The groups $G_q(N) \otimes \mathbb{Q}$ and the map $ev$ have been intensively studied in rational homotopy. In particular, for each integer $q$, $G_{2q}(N) \otimes \mathbb{Q} = 0$ and $\sum_q \dim G_q(N) \otimes \mathbb{Q} < n$ [2]. On the other hand, by a result of G. Lupton and the author, $\tilde{H}_s(ev; \mathbb{Q}) = 0$ if $\chi(N) \neq 0$ [4]. See also [10], [11], [6] for more recent results on Gottlieb groups.

Since the composition with $f$ induces a continuous map $j_N : \text{aut}_1 N \to \text{Map}(M, N; f)$ satisfying $\omega \circ j_N = ev$, we have the inclusion $G_s(N) \subset \text{Im} \pi_s(\omega)$. Our main theorem is essentially a converse to this result.

Theorem 1. (1) If $\deg f \neq 0$, then $\text{Im} \pi_s(\omega) \otimes \mathbb{Q} = G_s(N) \otimes \mathbb{Q}$.
(2) If $\deg f \neq 0$ and $\chi(N) \neq 0$, then the composition of $\pi_s(\omega)$ with the Hurewicz map $\text{hur}_N : \pi_s(N) \otimes \mathbb{Q} \to H_s(N; \mathbb{Q})$,
$$
\pi_s(\text{Map}(M, N; f)) \otimes \mathbb{Q} \to H_s(N; \mathbb{Q}),
$$
is zero.
An important step in the proof of Theorem 1 is given by Theorem 2.

**Theorem 2.** Suppose deg $f \neq 0$. Then the injection $j_N : \text{aut}_1(N) \to \text{Map}(M, N; f)$ induces an injection on the rational homotopy groups,

$$\pi_*(j_N) \otimes \mathbb{Q} : \pi_*(\text{aut}_1(N)) \otimes \mathbb{Q} \to \pi_*(\text{Map}(M, N; f)) \otimes \mathbb{Q}.$$  
Moreover $\pi_*(j_N) \otimes \mathbb{Q}$ admits a retraction $\sigma$ satisfying $\pi_*(ev) \otimes \mathbb{Q} \circ \sigma = \pi_*(\omega) \otimes \mathbb{Q}$.

Now recall that a continuous map $g : X \to Y$ is called a rational Gottlieb map if $\pi_*(g) \otimes \mathbb{Q}$ maps $G_*(X) \otimes \mathbb{Q}$ into $G_*(Y) \otimes \mathbb{Q}$. For instance, when $X$ is an odd-dimensional sphere $S^q$, then $g$ is a rational Gottlieb map if and only if $[g]$ belongs to the Gottlieb group $G_*(Y) \otimes \mathbb{Q}$. We prove

**Theorem 3.** With the above notation, if deg $f \neq 0$, then $f$ is a rational Gottlieb map.

For the proofs, we use the machinery of rational homotopy theory as described for instance in [3]. We use more precisely the Poincaré duality model for manifolds given by Lambrechts and Stanley in [8] and the description of the rational homotopy groups of mapping spaces in terms of derivations given by Lupton and Smith in [9].

1. A Convenien Model For $f$

A Poincaré duality model for a simply connected Poincaré duality complex $X$ of dimension $n$ is a commutative differential graded algebra $(A, d)$ quasi-isomorphic to the Sullivan minimal model of $X$ and satisfying Poincaré duality in dimension $n$. This means that there exists a linear map $\varepsilon : A^n \to \mathbb{Q}$ such that $\varepsilon(dA^{n-1}) = 0$ and such that the induced bilinear forms

$$A^k \otimes A^{n-k} \to \mathbb{Q}, \quad a \otimes b \mapsto \varepsilon(ab)$$

are non-degenerate. Such a map $\varepsilon$ is then called an orientation.

**Theorem 4** (Lambrecht, Stanley [8]).

1. If $(A, d)$ is a commutative differential graded algebra whose cohomology is a simply connected Poincaré duality algebra, then $(A, d)$ is weakly equivalent to a commutative differential graded algebra $(A', d)$ that is a simply connected Poincaré duality algebra.

2. Moreover, if $A$ is finite type, $A^0 = \mathbb{Q}$, $A^1 = 0$, $A^2 \subset \text{Ker}d$, and $n \geq 7$, then there is a quasi-isomorphism $\varphi : (A, d) \to (A', d)$.

We deduce the following generalization,

**Proposition 1.** Let $f : M \to N$ be a map between simply connected $n$-dimensional manifolds, $n \geq 7$. Suppose that $H^2(f)$ is injective. Then $f$ admits a Sullivan model of the form

$$\varphi : (A, d) \to (B, d),$$

where $(A, d)$ and $(B, d)$ are Poincaré duality algebras of dimension $n$.

**Proof.** Let $j : (\bigwedge V, d) \to (\bigwedge V \otimes \bigwedge W, D)$ be a relative minimal model for $f$ and let $(\bigwedge V, d) \to (A, d)$ be a quasi-isomorphism with $(A, d)$ a Poincaré duality model for $N$. By taking the tensor product $(A, d) \otimes (\bigwedge V, d)$, we get another model of $f$,

$$1 \otimes j : (A, d) \to (A \otimes \bigwedge W, D') := (A, d) \otimes (\bigwedge V, d) (\bigwedge V \otimes \bigwedge W, D).$$

Since $H^2(f)$ is injective, $(A \otimes \bigwedge W)^1 = 0$. By Theorem 4(2) there then exists a quasi-isomorphism $g : (A \otimes \bigwedge W, D') \to (B, d)$ where $(B, d)$ is a Poincaré duality model of $M$. We define $\varphi : (A, d) \to (B, d)$ by $\varphi = g \circ (1 \otimes j)$.

\[\square\]
Our next proposition explains the structure of the model \( \varphi \) when \( \deg f \neq 0 \).

**Proposition 2.** With the above notation, if \( \deg f \neq 0 \), then

1. the map \( \varphi \) is injective,
2. the graded vector space \( B \) can be decomposed as \( B = \varphi(A) \oplus Z \), where \( d(Z) \subset Z \) and \( Z \cdot \varphi(A) \subset Z \).

**Proof.** Denote by \( \omega \) and \( \omega' \) rational fundamental classes of \( A \) and \( B \) with \( \varphi(\omega) = \omega' \), and denote by \( \varepsilon' \) the associated orientation of \( B \). In particular, \( \varepsilon(a) = \varepsilon'(\varphi(a)) \) for \( a \in A \). If \( \varphi(a) = 0 \) for some \( a \), then \( \varphi(\omega) = 0 \) because there is an element \( a' \) such that \( aa' = \omega \), and so \( \varphi \) is injective.

Denote

\[
Z = \{ x \in B \mid \varepsilon'(x \cdot \varphi(A)) = 0 \}.
\]

Choose a homogeneous basis \( h_1, \ldots, h_m \) of \( A \). Using the Poincaré duality of \( A \), there is another family \( \{ h^*_j \} \subset A \) such that \( \varepsilon(h^*_j \cdot h_i) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol. Then \( \varphi(h_1), \ldots, \varphi(h_m) \) and \( \varphi(h^*_1), \ldots, \varphi(h^*_m) \) are homogeneous bases of \( \varphi(A) \). If \( b \in B \), the element

\[
b' = b - \sum_j \varepsilon'(b \cdot \varphi(h_j)) \varphi(h^*_j)
\]

is in \( Z \). This shows that \( B = \varphi(A) \oplus Z \). Now for an element \( z \in Z \),

\[
\varepsilon'(dz \cdot \varphi(a)) = \varepsilon'(d(z \varphi(a))) - (-1)^{|z|} \varepsilon'(z \cdot d\varphi(a)) = 0.
\]

This shows that \( d(Z) \subset Z \). We also have \( Z \cdot \varphi(A) \subset Z \) because \( \varepsilon' \left[(z \cdot \varphi(a)) \cdot \varphi(A)\right] = 0 \).

\[\square\]

**2. Model for mapping spaces**

In this section we recall the derivation model for the rational homotopy groups of a mapping space \( \text{Map}(X, Y; f) \), where \( X \) and \( Y \) are simply connected CW complexes and \( \dim X < \infty \). Recall that under those hypothesis, \( \text{Map}(M, N; f) \) is a nilpotent space whose rationalization is obtained by taking the composition with a rationalization of \( N, \ell : N \rightarrow N_0 \) [7].

Let \( \ell : (\wedge V, d) \rightarrow (A, d) \) be a model for \( f \), with \( (\wedge V, d) \) a Sullivan (non-necessarily minimal) model for \( Y \) and \( (A, d) \) a connected model for \( X \). We consider the differential graded vector space \( (\text{der}^\ell(\wedge V, A), \delta) \) of \( \ell \)-derivations, where \( \text{der}^\ell(\wedge V, A)_m \) is the vector space of linear maps of degree \( m \), \( \theta : (\wedge V)^* \rightarrow A^{* - m} \) for which \( \theta(xy) = \theta(x)\ell(y) + (-1)^{|x|}\ell(x)\theta(y) \). The differential \( \delta \) is defined as usual by

\[
\delta \theta = d \circ \theta + (-1)^{m+1} \theta \circ d.
\]

From now on, we restrict to \( \text{Der}^\ell(\wedge V, A) \), the positive \( \ell \)-derivations,

\[
\text{Der}^\ell(\wedge V, A)_r = \begin{cases} 
\text{der}^\ell(\wedge V, A)_r, & \text{if } r > 1, \\
Z \text{der}^\ell(\wedge V, A)_1, & \text{if } r = 1,
\end{cases}
\]

where \( Z \) denotes the space of cycles.

It is well known that if \( g : (\wedge W, d) \rightarrow (\wedge V, d) \) is a quasi-isomorphism, then the induced map \( \text{Der}^\ell(\wedge V, A) \rightarrow \text{Der}^{\ell \circ g}(\wedge W, A) \) is a quasi-isomorphism. In the same way, if \( h : (A, d) \rightarrow (A', d) \) is a quasi-isomorphism, then the induced map \( \text{Der}^\ell(\wedge V, A) \rightarrow \text{Der}^{h \circ \ell}(\wedge V, A') \) is also a quasi-isomorphism (see for instance [1],
Theorem 2.8). The following theorem follows then directly from \[9\], where the theorem is proved when \((\wedge V, d)\) is minimal and \((A, d)\) is the minimal model of \(X\).

**Theorem 5** (\[9\], \[1\], Theorem 3.8). With the above notation,

1. we have natural isomorphisms of graded vector spaces

\[
H_*(\text{Der}^\ell(\wedge V, A)) \cong \pi_*(\text{Map}_*(X, Y; f)) \otimes \mathbb{Q},
\]

\[
H_*(\text{Der}^\ell(\wedge V, A^+)) \cong \pi_*(\text{Map}_*(X, Y; f)) \otimes \mathbb{Q},
\]

where \(\text{Map}_*(X, Y; f)\) denotes the subspace of \(\text{Map}(M, N; f)\) consisting of pointed maps.

2. Denote by \(\alpha : A \to \mathbb{Q}\) the augmentation. Then the composition with \(\alpha\) induces a morphism of complexes that is a model for \(\pi_*(\omega) \otimes \mathbb{Q}\); i.e., we have a commutative diagram

\[
\begin{array}{ccc}
H_*(\text{Der}^\ell(\wedge V, A)) & \overset{\cong}{\longrightarrow} & H_*(\text{Der}^\alpha(\wedge V, \mathbb{Q})) \\
\uparrow & & \uparrow \cong \\
\pi_*(\text{Map}(X, Y; f)) \otimes \mathbb{Q} & \overset{\pi_*(\omega)}{\longrightarrow} & \pi_*(N) \otimes \mathbb{Q}.
\end{array}
\]

3. **Proof of Theorem 2**

We first consider the case \(n \geq 7\) and we use the notation of section 1. In particular, \(\varphi : (A, d) \to (B, d)\) is a model for \(f\), and \((A, d)\) and \((B, d)\) are Poincaré duality models for \(N\) and \(M\). Let \(\psi : (\wedge V, d) \to (A, d)\) be a surjective Sullivan model for \((A, d)\), and then let \(\rho = \varphi \circ \psi : (\wedge V, d) \to (B, d)\).

Recall that \(B = \varphi(A) \oplus Z\) with \(dZ \subset Z\) and \(Z \cdot \varphi(A) \subset Z\). Then \(B = \rho(\wedge V) \oplus Z\) with \(Z \cdot \rho(\wedge V) \subset Z\). Define a linear map

\[
\Phi : \text{Der}^\rho(\wedge V, B) \to \text{Der}^\psi(\wedge V, A)
\]

by \(\Phi(\theta) = p \circ \theta\), where \(p : B \to \varphi(A) \cong A\) is the linear projection with kernel \(Z\).

**Lemma 1.** The morphism \(p \circ \theta\) is a \(\psi\)-derivation.

**Proof.** Write \(\theta(a) = x + z\) with \(z \in Z\) and \(x = \varphi(p\theta(a))\) and \(\theta(b) = x' + z'\) with \(z' \in Z\) and \(x' = \varphi(p\theta(b))\). Then,

\[
p\theta(ab) = p(\theta(a) \cdot \rho(b) + (-1)^{|a||\theta|} \rho(a) \cdot \theta(b))
\]

\[
= p(x\rho(b) + (-1)^{|a||\theta|} \rho(a)x')
\]

\[
= p(\varphi(p\theta(a)) \cdot \psi(b) + (-1)^{|\theta||a|} \varphi(\psi(a)) \cdot p\theta(b))
\]

\[
= p\theta(a) \cdot \psi(b) + (-1)^{|a||\psi|} \psi(a) \cdot p\theta(b).
\]

**Lemma 2.** \(\Phi\) is a morphism of complexes.

**Proof.**

\[
\delta(p\theta) = dp\theta - (-1)^{|\theta|} (p\theta)d = p(d\theta) - (-1)^{|\theta|} p(\theta d) = p(\delta\theta).
\]

Since \(A\) is a subalgebra of \(B\), the injection of \(A\) into \(B\) induces a morphism of complexes

\[
\Psi : \text{Der}^\psi(\wedge V, A) \to \text{Der}^\rho(\wedge V, B)
\]

such that \(\Phi \circ \Psi = id\text{Der}^\psi(\wedge V, A)\).
Proof of Theorem 2 when $n \geq 7$. When $n \geq 7$, Theorem 5 directly yields that the induced injection $\Psi : \operatorname{Der}^n(\bigwedge V, A) \to \operatorname{Der}^n(\bigwedge V, B)$ induces in homology the morphism $\pi_*(j_N) \otimes \mathbb{Q}$. A retraction $\sigma$ to $\pi_*(j_N) \otimes \mathbb{Q}$ is given by $H_*(\Phi)$.

Now the commutativity of the diagram

$\begin{array}{c}
\operatorname{Der}^n(\bigwedge V, B) \\
\phi \downarrow \\
\operatorname{Der}^n(\bigwedge V, A)
\end{array} \xrightarrow{\sim} \begin{array}{c}
\operatorname{Der}^n(\bigwedge V, B) \\
\phi \downarrow \\
\operatorname{Der}^n(\bigwedge V, A)
\end{array}$

implies in homology that $(\pi_*(\psi) \otimes \mathbb{Q}) \circ \sigma = \pi_*(\omega) \otimes \mathbb{Q}$. □

Proof of Theorem 2 for $n \leq 6$. Let define $g = f \times \text{id}_{S^n} : M \times S^n \to N \times S^n$. If $f' \sim f$, then $f' \times \text{id} \sim f \times \text{id}$. This defines a continuous map

$e : \operatorname{Map}(M, N; f) \to \operatorname{Map}(M \times S^n, N \times S^n; f \times \text{id})$.

On the other hand, if $g' : M \times S^n \to N \times S^n$ is homotopic to $f \times \text{id}$, then we consider the composition

$M \xrightarrow{h} M \times S^n \xrightarrow{g'} N \times S^n \xrightarrow{p} N$,

where $* \in S^n$ is the base point, $h(m) = (m, *)$ and $p(r, x) = r$. Since $pg'h \sim f$, this gives a continuous map

$r : \operatorname{Map}(M \times S^n, N \times S^n; f \times \text{id}) \to \operatorname{Map}(M, N; f)$.

Clearly $r \circ e = \text{id}$; i.e., $\operatorname{Map}(M, N; f)$ is a retract of $\operatorname{Map}(M \times S^n, N \times S^n; f \times \text{id})$.

Since $\dim (N \times S^n) \geq 7$, we have a map $\sigma$ making commutative the diagram

$\begin{array}{c}
\pi_*(\operatorname{Map}(M, N; f)) \otimes \mathbb{Q} \\
\pi_*(e) \downarrow \\
\pi_*(\operatorname{Map}(M \times S^n, N \times S^n, f \times \text{id})) \otimes \mathbb{Q}
\end{array} \xrightarrow{\pi_*(\omega)} \begin{array}{c}
\pi_*(N) \otimes \mathbb{Q} \\
\pi_*(e) \downarrow \\
\pi_*(N \times S^n) \otimes \mathbb{Q}
\end{array}$

where $h'(n) = (n, *)$.

In the same way, $\operatorname{aut}_1(N)$ is a retract of $\operatorname{aut}_1(N \times S^n)$. Denote by $i : \operatorname{aut}_1(N) \to \operatorname{aut}_1(N \times S^n)$ the natural injection. The retraction $\tau$ associates to a map $h : N \times S^n \to N \times S^n$ the composition

$N \xrightarrow{h} N \times S^n \xrightarrow{h} N \times S^n \xrightarrow{p} N$.

By construction we have $\psi \circ \tau = p \circ \psi$ and a commutative diagram

$\begin{array}{c}
\operatorname{aut}_1(N) \\
\downarrow j_N \\
\operatorname{Map}(M, N; f)
\end{array} \xrightarrow{i} \begin{array}{c}
\operatorname{aut}_1(N \times S^n) \\
\downarrow j_{N \times S^n} \\
\operatorname{Map}(M \times S^n, N \times S^n, f \times \text{id})
\end{array}$

Then $\sigma' = (\pi_*(\tau) \otimes \mathbb{Q}) \circ \sigma \circ (\pi_*(e) \otimes \mathbb{Q})$,

$\pi_*(\operatorname{Map}(M, N; f)) \otimes \mathbb{Q} \to \pi_*(\operatorname{Map}(M \times S^n, N \times S^n; f \times \text{id})) \otimes \mathbb{Q}$

is a retraction of $\pi_*(j_N) \otimes \mathbb{Q}$, and by construction $(\pi_*(\psi) \otimes \mathbb{Q}) \circ \sigma' = \pi_*(\omega) \otimes \mathbb{Q}$. □

Proof of Theorem 1. Theorem 2 shows that $\operatorname{Im} \pi_*(\omega) \otimes \mathbb{Q} \subset G_*(N) \otimes \mathbb{Q}$. This proves Theorem 1(1). If $\chi(N) \neq 0$, $\hat{H}_*(\psi; \mathbb{Q}) = 0$, and so by Theorem 2, $\hat{h}_* \circ (\pi_*(\omega) \otimes \mathbb{Q}) = \hat{h}_* \circ (\pi_*(\psi) \otimes \mathbb{Q}) \circ \sigma = \hat{H}_*(\psi; \mathbb{Q}) \circ \hat{h}_*(\psi; \mathbb{Q}) \circ \sigma = 0$. □
Proof of Theorem 3. Denote by $i_M : aut_1 M \to \text{Map}(M, N; f)$ the map associated to the composition with $f$. The commutativity of the diagram ($q \geq 1$)

\[
\begin{array}{ccc}
\pi_q(aut_1(M)) \otimes \mathbb{Q} & \longrightarrow & \pi_q(\text{Map}(M, N; f)) \otimes \mathbb{Q} \\
\downarrow \pi_q(ev) & & \downarrow \pi_q(ev) \\
\pi_q(M) \otimes \mathbb{Q} & \longrightarrow & \pi_q(N) \otimes \mathbb{Q}
\end{array}
\]

shows that $f$ maps $G_*(M) \otimes \mathbb{Q}$ into $G_*(N) \otimes \mathbb{Q}$. □

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