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LAW OF LARGE NUMBERS UNDER THE NONLINEAR EXPECTATION

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Abstract. In this paper, we propose a class of nonlinear expectations induced by backward stochastic differential equations and reflected backward stochastic differential equations and prove the law of large numbers under the nonlinear expectation.

1. Introduction

The nonlinear backward stochastic differential equation (BSDE, for short) was firstly introduced by Pardoux and Peng [8]. The theory of BSDE has wide applications to many fields such as financial mathematics, stochastic control, and partial differential equations (see [5]). In [4], El Karoui et al. generalized the results to BSDE's with reflection and introduced the notion of reflected backward stochastic differential equations (RBSDEs, for short). Their work linked a mixed optimal stopping-optimal stochastic control problem and a parabolic partial differential equation. Cvitanic and Karatzas [3] generalized the above results to the case of two reflecting barrier processes and establish the connection between RBSDE and certain stochastic games of stopping (Dynkin games).

One of the applications of BSDE, a nonlinear expectation, called the g-expectation, was introduced via BSDE by Peng [9]. The g-expectation is monotone and constant-preserving, but not additive. In the setting of the g-expectation, many results of classic probability theory can be extended to some more general cases (see [1], [2], [6], [7], [9] and the references therein). In many cases, however, there are no constant-preserving expectations. For example, in the Black-Scholes market, the pricing process generated by the Black-Scholes formula, which can be viewed as a special nonlinear expectation, is not constant-preserving. For both BSDE and RBSDE, we will introduce new nonlinear expectations which are not constant-preserving and study the weak law of large numbers (LLN, for short) under our nonlinear expectations.

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The rest of the paper is organized as follows. In Section 2, the LLN problem under the nonlinear expectation is formulated. Section 3 introduces some preliminaries and lemmas. In Section 4, the LLN under the nonlinear expectation induced by BSDE or RBSDE respectively is established.

2. Formulation of LLN problem

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ be a filtered complete probability space on which a standard Brownian motion $\{B_t; 0 \leq t \leq T\}$ is defined, where $(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration generated by (B_t) and augmented with $\mathcal{P}$-null sets of $\mathcal{F}$. Let ξ be an $\mathcal{F}_T$-measurable 1-dimensional random variable. Let us now introduce some spaces which will be used frequently in the sequel:

$$L^2 = \{\xi; \xi \text{ is an } \mathcal{F}_T - \text{measurable random variable s.t. } E(\xi^2) < \infty\};$$

$$H^2 = \{\varphi_t; \{\varphi_t, 0 \leq t \leq T\} \text{ is a predictable process s.t. } E\int_0^T |\varphi_t|^2 < +\infty\};$$

$$\mathcal{S}^2 = \{\varphi_t; \{\varphi_t, 0 \leq t \leq T\} \text{ is a predictable process s.t. } E(\sup_{0\leq t\leq T} |\varphi_t|^2) < +\infty\}.$$

Pardoux and Peng [8] introduced the following nonlinear BSDE:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dB_s, 0 \leq t \leq T, \tag{2.1}$$

where the function $g(t, y, z)$: $[0, T] \times R \times R \to R$ is $\mathcal{F}_t$-progressively measurable for each $(y, z) \in R \times R$. It is by now well known that equation (2.1) has a unique adapted solution

$$(y_t^\xi, z_t^\xi)_{t\in[0,T]} \in \mathcal{S}^2 \times H^2$$

if ξ and g satisfy the following assumptions:

(A$_1$) $g(\cdot, y, z) \in L^2(\mathcal{F}_T), (y, z) \in R^2, \xi \in L^2(\mathcal{F}_T)$;

(A$_2$) $|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|), t \in R, y_1, y_2, z_1, z_2 \in R$.

We denote by $\{(y_t^\xi, z_t^\xi), t \in [0, T]\}$ the unique solution of BSDE (2.1). In addition, if g satisfies the assumption

(A_3) $g(t, y, 0) \equiv 0, (t, y) \in [0, T] \times R$,

y_0^ξ is called the g-expectation of ξ in [9]. The nonlinear g-expectation remains constant-preserving and monotone. The notion of g-expectation can be considered as a nonlinear extension of the well-known Girsanov transformations.

We now introduce the following assumption, which plays a key role in introducing the nonlinear expectation not preserving constants:

$(A_3^{'})$ $g(t, 0, 0) \equiv 0, t \in [0, T]$.

Remark 2.1. An explanation of $(A_3^{'})$ in the financial market is that a contingent claim in the future is zero. Then, under the pricing mechanism driven by $g(\cdot, \cdot, \cdot)$, the pricing process of the contingent claim is also zero.

We denote the solution y_t^ξ of BSDE (2.1) by $E_t^g(\xi)$ and y_0^ξ by $E^g(\xi)$ when g satisfies the conditions $(A_1), (A_2)$ and $(A_3^{'})$.

A solution of RBSDE (see El Karoui et al. [4]) is a triple $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$ of $\mathcal{F}_t$-progressively measurable processes taking values in $R \times R \times R_+$, respectively,

and satisfying:

$$
(2.2)\quad \begin{cases} (i)\ Z \in H_2, E\int_0^T |Z_t|^2 < +\infty; \\ (ii)\ Y \in \mathcal{S}_2 \text{ and } K_T \in L^2; \\ (iii)\ Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, 0 \le t \le T; \\ (iv)\ Y_t \ge S_t, 0 \le t \le T; \\ (v)\ K_t \text{ is continuous and increasing}, K_0 = 0 \text{ and } \int_0^T (Y_t - S_t)dK_t = 0, \end{cases}
$$

where $S_t, 0 \le t \le T$, is a continuous progressively measurable process satisfying $E(\sup_{0\le t\le T} |S_t|^2) < +\infty$. There exists a unique solution $\left(Y_t^\xi, Z_t{}^\xi, K_t^\xi\right)$ to the RBSDE (2.2), and we denote the solution Y_t^ξ of RBSDE by $\mathcal{E}_t^g(\xi)$ and Y_0^ξ by $\mathcal{E}^g(\xi)$ if g satisfies $(A_1), (A_2)$ and $(A_3^{'})$ (see [4]).

Hereinafter, we always assume that g satisfies $(A_1), (A_2)$ and $(A_3^{'})$. We call $E^g(\xi)$ and $\mathcal{E}^g(\xi)$ the corresponding nonlinear expectation of ξ. It is clear that the property of constant preservation does not hold for $E^g(\cdot)$ and $\mathcal{E}^g(\cdot)$.

Let $\xi_1, \xi_2, ...$ be a sequence of independent identically distributed random variables with finite second moment. Then Kolomogorov's strong law of large numbers holds; i.e.,

$$
\lim_{n\to\infty} (\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n}) = E(\xi_1), \quad \mathcal{P} - \text{a.s.},
$$

which can be viewed as the behavior of the random walk in the physical world $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$. But, for a special financial market $(\Omega, \mathcal{F}, \mathcal{F}_t, E^g)$ or $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{E}^g)$, we are more concerned with the question of whether

$$
\lim_{n\to\infty} E^g(\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n}) = E^g(E(\xi_1))
$$

or

$$
\lim_{n\to\infty} \mathcal{E}^g(\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n}) = \mathcal{E}^g(E(\xi_1))
$$

holds.

In this paper, we prove that the nonlinear expectation of the random variable $(\frac{\xi_1+\xi_2+\cdots+\xi_n}{n})$ converges to the nonlinear expectation of $E(\xi_1)$. We call it the law of large numbers under the nonlinear expectation.

3. Some preliminaries and lemmas

Let $L^p(\mathcal{F}_T)$ be the space of $\mathcal{F}_T$-measurable random variables ξ satisfying $E|\xi|^P = \int_\Omega |\xi|^p d\mathcal{P} < \infty$. Let $\xi_1, \xi_2, ...$ be a sequence of independent identically distributed random variables with finite second moments and $S_n = \xi_1+\xi_2+\cdots+\xi_n, n = 1, 2, \cdots$.

We first introduce several lemmas.

Lemma 3.1 (see [10], p. 389)**.** *If there are positive numbers b_n such that $b_n \uparrow \infty$ and $\sum \frac{V\xi_n}{b_n^2} < \infty$, then $\frac{S_n - ES_n}{b_n} \to 0, \mathcal{P} - a.s.$ In particular, if $\sum \frac{V\xi_n}{n^2} < \infty$, then $\frac{S_n - ES_n}{n} \to 0, \mathcal{P} - a.s.$*

The following assumption has a very good interpretation in finance.

(A_4) For each constant K, $\{E_t^g(K); t \in [0,T]\}$ $\Big(\{\mathcal{E}_t^g(K); t \in [0,T]\}\Big)$ is a deterministic function in t.

Remark 3.2. In the financial market $(\Omega, \mathcal{F}, \mathcal{F}_t, E^g)$, for each constant K, $\{E_t^g(K); t \in [0,T]\}$ may be viewed as the pricing process of the future risk-free deterministic asset K. The assumption (A_4) means that the pricing process of the future risk-free asset K is a deterministic function in t and that if $K \neq 0$, then $E_t^g(K) \neq 0$. A similar interpretation can be made for $\mathcal{E}_t^g(K)$.

We write $g_0(y) = g(y,0)$ and $g_1(y,z) = g(y,z) - g_0(y)$. It is easy to show that $g_0(0) = 0, g_1(y,0) = 0$, and both $g_0(y)$ and $g_1(y,z)$ satisfy the conditions $(A_1), (A_2)$ and $(A_3^{'})$.

Lemma 3.3. *If g_0 satisfies (A_4), then g also satisfies (A_4) and $E_t^{g_0}(K) = E_t^g(K)$.*

Proof. For a given constant $K \in R, 0 \leq t \leq T$, we have

$$\begin{aligned} E_t^{g_0}(K) &= K + \int_t^T g_0(E_s^{g_0}(K))ds - \int_t^T 0dB_s \\ &= K + \int_t^T (g_0(E_s^{g_0}(K)) + g_1(0,0))ds - \int_t^T 0dB_s \\ &= K + \int_t^T (g_0(E_s^{g_0}(K)) + g_1(E_s^{g_0}(K),0))ds - \int_t^T 0dB_s \\ &= K + \int_t^T g(E_s^{g_0}(K),0)ds - \int_t^T 0dB_s. \end{aligned}$$

By the uniqueness of the solution of BSDE (2.1), we obtain $E_t^{g_0}(K) = E_t^g(K)$, which is also a deterministic function. □

4. Main results

Let $\xi_1, \xi_2, \cdots$ be a sequence of independent identically distributed random variables with finite second moments. Set $C{=}E(\xi_1)$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, $n = 1, 2, \cdots$.

4.1. LLN under the nonlinear expectation induced by BSDE.

We are in a position to prove that LLN holds under the nonlinear expectation induced by BSDE .

Theorem 4.1. *If g satisfies (A_1), (A_2) and $(A_3^{'})$, $g_0(y)$ satisfies (A_4), and there exists a random variable $\eta \in L^2(\mathcal{F}_T)$ such that $|\xi_i| \leq |\eta|, i = 1, 2, \cdots$, then $\lim_{n\to\infty} E^g(\frac{S_n}{n}) = E^g(C)$.*

Proof. Set $\frac{S_n}{n} = C + \varepsilon_n$. By Lemma 3.1, $\lim_{n\to\infty} \frac{S_n}{n} = C, \mathcal{P}-$a.s., i.e., $\lim_{n\to\infty} \varepsilon_n = 0, \mathcal{P}-$a.s. For $0 \leq t \leq T$, we have

$$\begin{aligned} y_t &= C + \int_t^T g_0(y_s)ds - \int_t^T 0dB_s, \\ y_t^n &= \varepsilon_n + \int_t^T g(y_s^n, z_s^n)ds - \int_t^T z_s^n dB_s, \\ \overline{y_t^n} &= (C + \varepsilon_n) + \int_t^T g(\overline{y_s^n}, \overline{z_s^n})ds - \int_t^T \overline{z_s^n} dB_s; \end{aligned}$$

that is,

$$dy_t = -g_0(y_t)dt, \qquad y_T = C, \tag{4.1a}$$

$$dy_t^n = -g(y_t^n, z_t^n)dt + z_t^n dB_t, \qquad y_T^n = \varepsilon_n, \tag{4.1b}$$

$$d\overline{y_t^n} = -g(\overline{y_t^n}, \overline{z_t^n})dt + \overline{z_t^n}dB_t, \qquad \overline{y_T^n} = C + \varepsilon_n. \tag{4.1c}$$

From (4.1), we obtain

$$d(\overline{y_t^n} - y_t) = -\big(g(\overline{y_t^n}, \overline{z_t^n}) - g_0(y_t)\big)dt + \overline{z_t^n}dB_t,\ 0 \le t \le T.$$

Denote $\widetilde{y_t^n} = \overline{y_t^n} - y_t$ and $\widetilde{z_t^n} = \overline{z_t^n}$. Applying Itô's formula to $e^{\beta t}(\widetilde{y_t^n})^2$, we have

$$\begin{cases} d(e^{\beta t}(\widetilde{y_t^n})^2) = e^{\beta t}[\beta(\widetilde{y_t^n})^2 dt - \widetilde{y_t^n}(a_t\widetilde{y_t^n} + b_t\widetilde{z_t^n})dt + 2\widetilde{y_t^n}\widetilde{z_t^n}dB_t + \widetilde{z_t^n}^2 dt], \\ \quad e^{\beta T}\widetilde{y_T^n}^2 = {\varepsilon_n}^2 e^{\beta T}, \end{cases} \tag{4.2}$$

where

$$a_t = \begin{cases} \dfrac{(g(\overline{y_t^n}, \overline{z_t^n}) - g(y_t^n, \overline{z_t^n}))}{\overline{y_t^n} - y_t}, & \text{if} \quad \overline{y_t^n} \neq y_t, \\ 0, & \text{else}, \end{cases}$$

$$b_t = \begin{cases} \dfrac{(g(y_t^n, \overline{z_t^n}) - g_0(y_t))}{\overline{z}_t^n}, & \text{if} \quad \overline{z}_t^n \neq 0, \\ 0, & \text{else}. \end{cases}$$

Integrating the first equality in (4.2) from t to T, we arrive at

$$\begin{aligned} &{\varepsilon_n}^2 e^{\beta T} - (e^{\beta t}(\widetilde{y_t^n})^2) \\ &= \int_t^T e^{\beta s}(\beta\widetilde{y_s^n}^2 + \widetilde{z_s^n}^2 - a_s\widetilde{y_s^n}^2 - b_s\widetilde{y_s^n}\widetilde{z_t^n})ds + 2\int_t^T e^{\beta s}\widetilde{y_s^n}\widetilde{z^n}dB_s. \end{aligned} \tag{4.3}$$

Then we get

$$\begin{aligned} &E({\varepsilon_n}^2 e^{\beta T}|\mathcal{F}_t) + E(\int_t^T (a_s\widetilde{y_s^n}^2 + b_s\widetilde{y_s^n}\widetilde{z_s^n})e^{\beta s}ds|\mathcal{F}_t) \\ &= (\widetilde{y_t^n})^2 e^{\beta t} + E(\int_t^T (\beta\widetilde{y_s^n}^2 + \widetilde{z_s^n}^2)e^{\beta s}ds|\mathcal{F}_t). \end{aligned} \tag{4.4}$$

In terms of (4.4), it follows that

$$\begin{aligned} &\widetilde{y_t^n}^2 + E(\int_t^T e^{\beta(s-t)}(\beta\widetilde{y_s^n}^2 + \widetilde{z_s^n}^2)ds|\mathcal{F}_t) \\ &= E({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t) + E(\int_t^T (a_s\widetilde{y_s^n}^2 + b_s\widetilde{y_s^n}\widetilde{z_s^n})e^{\beta(s-t)}ds|\mathcal{F}_t). \end{aligned} \tag{4.5}$$

Because g satisfies (A_1), (A_2), $(A_3^{'})$ and (A_4), there exists a constant $K > 0$ such that

$$|a_t| \le K, |b_t| \le K, 0 \le t \le T.$$

Therefore,

$$
\begin{aligned}
&\widetilde{y_t^n}^2 + E\left(\int_t^T e^{\beta(s-t)}(\beta\widetilde{y_s^n}^2 + \widetilde{z_s^n}^2)ds|\mathcal{F}_t\right) \\
&\le E({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t) + KE(\int_t^T \widetilde{y_s^n}^2 e^{\beta(s-t)}ds|\mathcal{F}_t) \\
&\quad + KE(\int_t^T e^{\beta(s-t)}|\widetilde{y_s^n}\widetilde{z_s^n}|ds|\mathcal{F}_t).
\end{aligned} \tag{4.6}
$$

From

$$
|2K\widetilde{y_s^n}\widetilde{z_s^n}| \le \frac{\beta}{2}\widetilde{y_s^n}^2 + \frac{2K^2}{\beta}\widetilde{z_s^n}^2,
$$

we have

$$
\begin{aligned}
&(\widetilde{y_t^n})^2 + E\left(\int_t^T e^{\beta(s-t)}(\beta\widetilde{y_s^n}^2 + \widetilde{z_s^n}^2)ds|\mathcal{F}_t\right) \\
&\le E({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t) \\
&\quad + \ (K + \frac{\beta}{4})E\left(\int_t^T e^{\beta(s-t)}\widetilde{y_s^n}^2 ds|\mathcal{F}_t + \frac{K^2}{\beta}e^{\beta(s-t)}\widetilde{z_s^n}^2 ds|\mathcal{F}_t\right).
\end{aligned}
$$

For a sufficiently large positive number β, it follows that there exist two positive numbers K_1 and K_2 such that

$$
\begin{aligned}
&(\widetilde{y_t^n})^2 + K_1E(\int_t^T e^{(s-t)}\beta\widetilde{y_s^n}^2 ds|\mathcal{F}_t) + K_2E(\int_t^T e^{(s-t)}\widetilde{z_s^n}^2 ds|\mathcal{F}_t) \\
&\le E({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t).
\end{aligned} \tag{4.7}
$$

Letting $t = 0$ in (4.7), we conclude that

$$
(\widetilde{y_t^n})^2 + K_1E\int_t^T e^s\beta\widetilde{y_s^n}^2 ds + K_2E\int_t^T e^s\widetilde{z_s^n}^2 ds \le E({\varepsilon_n}^2 e^{\beta T}). \tag{4.8}
$$

Because $|\xi_i| \le |\eta|, i = 1, 2, \cdots, \eta \in L^2(\mathcal{F}_T)$, by the Lebesgue dominated convergence theorem, we have

$$
\lim_{n\to\infty} E(\varepsilon_n^2) = \lim_{n\to\infty} E(\frac{s_n}{n} - C)^2 = 0.
$$

Therefore,

$$
\lim_{n\to\infty} \widetilde{y_0^n} = 0;
$$

i.e.,

$$
\lim_{n\to\infty} E^g(\frac{s_n}{n}) = E^g(C).
$$

□

4.2. **LLN under nonlinear expectation induced by RBSDE.** Let $\{\mathcal{B}_t, 0 \le t \le T\}$ be a 1-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. We now prove the LLN under nonlinear expectation induced by RBSDE.

Theorem 4.2. *If g satisfies (A_1),(A_2) and $(A_3^{'})$, $g_0(y)$ satisfies (A_4), and there exists a random variable $\eta \in L^2(\mathcal{F}_T)$ such that $|\xi_i| \le |\eta|, i = 1, 2, \cdots$, then*

$$
\lim_{n\to\infty} \mathcal{E}^g(\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n}) = \mathcal{E}^g(E(\xi_1)).
$$

Proof. As above, for $0 \le t \le T$, let

$$\begin{cases} y_t = C + \int_t^T g_0(y_s)ds + K_T - K_t - \int_t^T 0dB_s, \\ y_t \ge S_t, \\ \int_0^T (y_t - S_t)dK_t = 0 \end{cases} \tag{4.9}$$

and

$$\begin{cases} \overline{y_t^n} = (C + \varepsilon_n) + \int_t^T g(\overline{y_s^n}, \overline{z_s^n})ds + \overline{K_T} - \overline{K_t} - \int_t^T \overline{z_s^n}dB_s, \\ \overline{y_t^n} \ge S_t, \\ \int_0^T (\overline{y_t^n} - S_t)dK_t = 0. \end{cases} \tag{4.10}$$

From (4.9) and (4.10), we obtain

$$\begin{cases} d(\overline{y_t^n} - y_t) = -(g(\overline{y_t^n}, \overline{z_t^n}) - g_0(y_t))dt + \overline{z_t^n}dB_t - d(\overline{K_t^n} - K_t), \\ \overline{y_T^n} - y_T = \varepsilon_n. \end{cases}$$

Set $\widetilde{y_t^n} = \overline{y_t^n} - y_t$, $\widetilde{z_t^n} = \overline{z_t^n}$, $\widetilde{K_t^n} = \overline{K_t^n} - K_t$ and apply Itô's formula to $e^{\beta t}(\widetilde{y_t^n})^2$. Then

$$\begin{cases} d(e^{\beta t}(\widetilde{y_t^n})^2) = e^{\beta t}\big[\beta(\widetilde{y_t^n})^2dt - \widetilde{y_t^n}(a_t\widetilde{y_t^n} + b_t\widetilde{z_t^n})dt \\ \qquad\qquad + 2\widetilde{y_t^n}\widetilde{z_t^n}dB_t + \widetilde{z_t^n}^2dt - 2\widetilde{y_t^n}d\widetilde{K_t^n}\big], \\ e^{\beta T}\widetilde{y_T^n}^2 = {\varepsilon_n}^2 e^{\beta T}, \end{cases} \tag{4.11}$$

where

$$a_t = \begin{cases} \dfrac{(g(\overline{y_t^n}, \overline{z_t^n}) - g(y_t^n, \overline{z_t^n}))}{\overline{y_t^n} - y_t}, & \text{if} \quad \overline{y_t^n} \ne y_t, \\ 0, & \text{else} \end{cases}$$

and

$$b_t = \begin{cases} \dfrac{(g(y_t^n, \overline{z_t^n}) - g_0(y_t))}{\overline{z}_t^n}, & \text{if} \quad \overline{z}_t^n \ne 0, \\ 0, & \text{else.} \end{cases}$$

Integrating both sides of (4.11) from t to T yields

$$\begin{aligned} &{\varepsilon_n}^2 e^{\beta T} - (e^{\beta t}(\widetilde{y_t^n})^2) \\ &= \int_t^T e^{\beta s}(\beta\widetilde{y_s^n}^2 + \widetilde{z_t^n}^2 - a_s\widetilde{y_s^n}^2 - b_s\widetilde{y_s^n}\widetilde{z_t^n})ds \\ &\quad + 2\int_t^T e^{\beta s}\widetilde{y_s^n}\widetilde{z^n}ds - 2\int_t^T e^{\beta s}\widetilde{y_s^n}d\widetilde{K_t^n}. \end{aligned} \tag{4.12}$$

Taking conditional expectation with respect to $\mathcal{F}_t$ in (4.12), we have

$$\begin{aligned}
&\widetilde{y_t^n}^2 + \mathcal{E}^g\Big(\int_t^T e^{\beta(s-t)}(\beta\widetilde{y_t^n}^2 + \widetilde{z_s^n}^2)ds|\mathcal{F}_t\Big) \\
&= \mathcal{E}^g({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t) + \mathcal{E}^g\Big(\int_t^T (a_s\widetilde{y_s^n}^2 + b_s\widetilde{y_s^n}\widetilde{z_s^n})e^{\beta(s-t)}ds|\mathcal{F}_t\Big) \\
&\quad + 2\mathcal{E}^g\Big[\int_t^T e^{\beta(s-t)}\widetilde{y_s^n}d\widetilde{K_s^n}|\mathcal{F}_t\Big].
\end{aligned} \tag{4.13}$$

Because g satisfies (A_1)-(A_4), there exists a constant K such that

$$|a_t| \le K, |b_t| \le K,\ 0 \le t \le T. \tag{4.14}$$

Combining (4.13) with (4.14), we have

$$\begin{aligned}
&\widetilde{y_t^n}^2 + \mathcal{E}^g\Big(\int_t^T e^{\beta(s-t)}(\beta\widetilde{y_t^n}^2 + \widetilde{z_s^n}^2)ds|\mathcal{F}_t\Big) \\
&\le \mathcal{E}^g({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t) + K\mathcal{E}^g\Big(\int_t^T \widetilde{y_s^n}^2 e^{\beta(s-t)}ds|\mathcal{F}_t\Big) \\
&\quad + K\mathcal{E}^g\Big(\int_t^T e^{\beta(s-t)}\widetilde{y_s^n}\widetilde{z_s^n}ds|\mathcal{F}_t\Big) + 2\mathcal{E}^g\Big[\int_t^T e^{\beta(s-t)}\widetilde{y_s^n}d\widetilde{K_s^n}|\mathcal{F}_t\Big].
\end{aligned} \tag{4.15}$$

Since

$$|2K\widetilde{y_s^n}\widetilde{z_s^n}| \le \frac{\beta}{2}\widetilde{y_s^n}^2 + \frac{2K^2}{\beta}\widetilde{z_s^n}^2,$$

we have

$$\begin{aligned}
&(\widetilde{y_t^n})^2 + \mathcal{E}^g\Big[\int_t^T e^{\beta(s-t)}(\beta\widetilde{y_t^n}^2 + \widetilde{z_s^n}^2)ds|\mathcal{F}_t\Big] \\
&\le \mathcal{E}^g({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t) + (K + \frac{\beta}{4})\mathcal{E}^g\Big(\int_t^T e^{\beta(s-t)}\widetilde{y_s^n}^2 ds|\mathcal{F}_t\Big) \\
&\quad + \frac{K^2}{\beta}\mathcal{E}^g\Big(\int_t^T e^{\beta(s-t)}\widetilde{z_s^n}^2 ds|\mathcal{F}_t\Big) + 2\mathcal{E}^g\Big[\int_t^T e^{\beta(s-t)}\widetilde{y_s^n}d\widetilde{K_s^n}|\mathcal{F}_t\Big].
\end{aligned}$$

For a sufficiently large positive number β, it follows that there exist two positive numbers K_1 and K_2 such that

$$\begin{aligned}
&(\widetilde{y_t^n})^2 + K_1\mathcal{E}^g\Big(\int_t^T e^{(s-t)}\beta\widetilde{y_s^n}^2 ds|\mathcal{F}_t\Big) + K_2\mathcal{E}^g\Big(\int_t^T e^{(s-t)}\widetilde{z_s^n}^2 ds|\mathcal{F}_t\Big) \\
&\le \mathcal{E}^g({\varepsilon_n}^2 e^{\beta(T-t)}|\mathcal{F}_t) + 2\mathcal{E}^g\Big[\int_t^T e^{\beta(s-t)}\widetilde{y_s^n}d\widetilde{K_s^n}|\mathcal{F}_t\Big].
\end{aligned} \tag{4.16}$$

Letting $t = 0$ in (4.16), we conclude that

$$\begin{aligned}
&(\widetilde{y_0^n})^2 + K_1\mathcal{E}^g\Big(\int_0^T e^s\beta\widetilde{y_s^n}^2 ds\Big) + K_2\mathcal{E}^g\Big(\int_0^T e^s\widetilde{z_s^n}^2 ds\Big) \\
&\le e^{\beta T}\mathcal{E}^g({\varepsilon_n}^2) + 2\mathcal{E}^g\int_0^T e^{\beta s}\widetilde{y_s^n}d\widetilde{K_s^n}.
\end{aligned} \tag{4.17}$$

Since

$$\begin{aligned}\int_0^T \widetilde{y^n} d\widetilde{K_t^n} &= \int_0^T (\overline{y_t^n} - y_t) dK_t^n - \int_0^T (\overline{y_t^n} - y_t) dK_t \\ &= \int_0^T (\overline{y_t^n} - S_t^n) dK_t^n + \int_0^T (S_t^n - \overline{y_t^n}) dK_t^n \\ &\quad - \int_0^T (\overline{y_t^n} - y_t) dK_t + \int_0^T (S_t^n - \overline{y_t^n}) dK_t \leq 0,\end{aligned}$$

we have

$$2\mathcal{E}^g[\int_0^T e^{\beta(s-t)} \widetilde{y_s^n} d\widetilde{K_s^n} | \mathcal{F}_t] \leq 2e^{\beta(-t)} \mathcal{E}^g[\int_0^T \widetilde{y_s^n} d\widetilde{K_s^n} | \mathcal{F}_t] \leq 0.$$

Furthermore, we have

$$(\widetilde{y_0^n})^2 + K_1 \mathcal{E}^g(\int_0^T e^{(s-t)} \beta \widetilde{y_t^n}^2 ds) + K_2 \mathcal{E}^g(\int_0^T e^{(s-t)} \widetilde{z_s^n}^2 ds) \leq e^{\beta(T-t)} \mathcal{E}^g({\varepsilon_n}^2).$$

For $|\xi_i| \leq |\eta|, i = 1, 2, \cdots, \eta \in L^2(\mathcal{F}_T)$, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \mathcal{E}^g(\varepsilon_n^2) = \lim_{n\to\infty} \mathcal{E}^g(\frac{s_n}{n} - C)^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} \widetilde{y_0^n} = 0;$$

i.e.,

$$\lim_{n\to\infty} \mathcal{E}^g(\frac{s_n}{n}) = \mathcal{E}^g(C). \qquad \square$$

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