ESSENTIALLY FINITE VECTOR BUNDLES ON VARIETIES WITH TRIVIAL TANGENT BUNDLE

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Abstract. Let $X$ be a smooth projective variety, defined over an algebraically closed field of positive characteristic, such that the tangent bundle $TX$ is trivial. Let $F_X : X \rightarrow X$ be the absolute Frobenius morphism of $X$. We prove that for any $n \geq 1$, the $n$–fold composition $F^n_X$ is a torsor over $X$ for a finite group–scheme that depends on $n$. For any vector bundle $E \rightarrow X$, we show that the direct image $(F^n_X)_*E$ is essentially finite (respectively, $F$–trivial) if and only if $E$ is essentially finite (respectively, $F$–trivial).

1. Introduction

For a smooth projective variety $X$ over a field of characteristic zero, the tangent bundle $TX$ is trivial if and only if $X$ is an abelian variety. This is not true for fields of characteristic $p > 0$. Examples of varieties, different from abelian varieties, with trivial tangent bundle can be found in [4], [5] (see also [6]).

Let $X$ be a smooth projective variety, defined over an algebraically closed field of positive characteristic, with the property that $TX$ is trivial. Let $F_X : X \rightarrow X$ be the absolute Frobenius morphism. For any $n \geq 1$, let $F^n_X$ be the $n$–fold composition of the self–map $F_X$.

Nori introduced the fundamental group–scheme [9], [10]. We recall that after introducing the essentially finite vector bundles and then showing that they form a neutral Tannakian category, Nori defined the fundamental group–scheme to be the one given by this neutral Tannakian category. We also recall that a vector bundle $V$ on a smooth projective variety is essentially finite if and only if the pullback of $V$ by some projective surjective morphism is trivial. A special class of essentially finite vector bundles is the $F$–trivial vector bundle; a vector bundle is called $F$–trivial if its pullback by some power of the Frobenius morphism is trivial. The $F$–trivial vector bundles also define a neutral Tannakian category. The corresponding group–scheme is called the local fundamental group–scheme.

We prove the following proposition (see Proposition 2.5):
Proposition 1.1. Let $E$ be a vector bundle on a smooth projective variety $X$ with trivial tangent bundle. If $E$ is essentially finite, then the direct image $(F^n_X)_*E$ is essentially finite for every $n$.

If $(F^n_X)_*E$ is essentially finite for some $n$, then $E$ is essentially finite.

We also prove the following theorem (see Theorem 2.6):

Theorem 1.2. Let $X$ and $E$ be as in Proposition 1.1. If $E$ is $F$–trivial, then $(F^n_X)_*E$ is $F$–trivial for every $n$.

If $(F^n_X)_*E$ is $F$–trivial for some $n$, then $E$ is $F$–trivial.

The condition that $TX$ is trivial is used in the following three ways: All nonzero vector fields on $X$ are nowhere vanishing; the cotangent bundle $\Omega^1_X$ is a subbundle of a trivial vector bundle (equivalently, $TX$ is globally generated), and for a vector bundle $E$ on $X$, the Chern class $c_1(E)$ is numerically trivial if and only if $c_1((F^n_X)_*E)$ is numerically trivial. More precisely, the crucial Lemma 2.1 is proved using the fact that all nonzero vector fields on $X$ are nowhere vanishing and that $TX$ is globally generated. In the proofs of Proposition 2.5 and Theorem 2.6, we use the fact that $c_1(E)$ is numerically trivial if and only if $c_1((F^n_X)_*E)$ is numerically trivial.

2. Varieties with trivial tangent bundle

Let $k$ be an algebraically closed field of characteristic $p$, with $p > 0$. Let $X$ be an irreducible smooth projective variety defined over $k$ such that the tangent bundle $TX$ is trivial. Let $d$ be the dimension of $X$.

The $d$–dimensional vector space $H^0(X, TX)$ is equipped with the Lie bracket operation. Note that $H^0(X, TX)$ is a $p$–Lie algebra. There is a natural bijective correspondence between the $p$–Lie algebras over $k$ and the local group–schemes over $k$ of height one [8, p. 139]. Let

\[(1)\quad G\]

be the local group–scheme of height one corresponding to the $p$–Lie algebra $H^0(X, TX)$.

Let

\[(2)\quad F_X : X \rightarrow X\]

be the absolute Frobenius morphism. For any integer $n \geq 1$, let

\[F^n_X := \underbrace{F_X \circ \cdots \circ F_X}_{n\text{-times}} : X \rightarrow X\]

be the $n$–fold iteration of $F_X$. By $F^n_X$ we will denote the identity morphism of $X$.

Lemma 2.1. The absolute Frobenius morphism $F_X$ defines a $G$–torsor over $X$, where $G$ is defined in \[(1)\].

Proof. Since $G$ corresponds to a Lie algebra of derivations on $X$, the group–scheme $G$ has a tautological action on $X$. Let

\[q : X \rightarrow Z := X/G\]

be the quotient morphism. We note that the action of $G$ on $X$ is free, because all the nonzero vector fields on $X$ are nowhere vanishing. Therefore, the above morphism $q$ defines a $G$–torsor on $Z$. 

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It can be shown that the Frobenius morphism $F_X$ factors as

$$
\begin{array}{ccc}
X & \xrightarrow{q} & Z = X/G \\
\| & \Downarrow \phi & \\
X & \xrightarrow{F_X} & X
\end{array}
$$

To prove this, let $E$ denote the above principal $G$–bundle $q : X \to Z$. Let $F_Z : Z \to Z$ be the Frobenius morphism of $Z$. The pullback $F_Z^*E$ is identified with the principal $G$–bundle obtained by extending the structure group of $E$ using the Frobenius homomorphism $F_G : G \to G$. We recall that $G$ is of height one, which means that the image of $F_G$ is the identity. Hence $F_Z^*E$ is the trivial principal $G$–bundle $Z \times G \to Z$. Therefore, we have a commutative diagram:

$$
\begin{array}{ccc}
Z \times G & \xrightarrow{\psi} & X \\
\| & \Downarrow & \\
Z & \xrightarrow{F_Z} & Z
\end{array}
$$

The morphism $\phi$ in (3) is the restriction of $\psi$ to the Cartesian product of $Z$ with the identity of $G$.

Now considering the scheme theoretic fibers of $q$ and $F_X$ we conclude that $\phi$ is an isomorphism.

Essentially finite vector bundles were defined in [9, 10]. A vector bundle $E$ on a smooth projective variety $Y$ is essentially finite if and only if there is a projective variety $Z$ and a surjective morphism $\psi : Z \to Y$ such that the pullback $\psi^*E$ is trivial [2, Theorem 1.1].

**Corollary 2.2.** The vector bundle $(F_X)^*(F_X)_*O_X \to X$ is trivial. In particular, the direct image $(F_X)_*O_X$ is essentially finite.

**Proof.** From Lemma 2.1 it follows that $(F_X)^*(F_X)_*O_X$ is the trivial vector bundle over $X$ with fiber $k[G]$ (see [8, p. 120, Corollary 2]). Hence $(F_X)_*O_X$ is essentially finite by the above criterion.

**Lemma 2.3.** Let $\gamma : Y \to X$ be an étale cover. Then the tangent bundle of $Y$ is trivial.

**Proof.** Let $d\gamma : TY \to \gamma^*TX$ be the differential of the morphism $\gamma$. Since $\gamma$ is étale, we know that $d\gamma$ is an isomorphism. The vector bundle $\gamma^*TX$ is trivial because $TX$ is trivial. Hence the isomorphic vector bundle $TY$ is trivial.

**Lemma 2.4.** Let $E$ be an essentially finite vector bundle over $X$. Then the direct image $(F_X)_*E$ is also essentially finite.

**Proof.** There is an étale Galois covering

$$
\gamma : Y \to X
$$

and a positive integer $m$ such that $(F_Y^m)^*\gamma^*E$ is trivial, where

$$
F_Y : Y \to Y
$$
is the absolute Frobenius morphism of $Y$ [10, Chapter II, Proposition 7] (see also [11, p. 557]). Since $\gamma$ is étale, the following diagram is Cartesian:

\[
\begin{array}{ccc}
Y & \xrightarrow{\gamma \circ F^n} & X \\
\downarrow F_Y & & \downarrow F_X \\
Y & \xrightarrow{F^n \circ \gamma} & X
\end{array}
\]

Hence

\[
F_Y (F_Y^n)^* \gamma^* E = F_Y (\gamma \circ F^n)_* E = (F_X^n \circ \gamma)^* ((F_X)_* E)
\]

(see [3, p. 255, Proposition 9.3]).

Since $(F_Y^n)^* \gamma^* E$ is trivial, from Corollary [2,2] we know that $F_Y (F_Y^n)^* \gamma^* E$ is essentially finite (note that Lemma [2,3] implies that Corollary [2,2] applies). Hence $(F_X^n \circ \gamma)^* ((F_X)_* E)$ is essentially finite by (5). This implies that the direct image $(F_X)_* E$ is essentially finite.

\begin{proposition}
For any essentially finite vector bundle $E \to X$, and any $n \geq 1$, the direct image $(F_X^n)_* E$ is essentially finite. In particular, $(F_X^n)_* \mathcal{O}_X$ is essentially finite.
\end{proposition}

\begin{proof}
Since $(F_X^{n+1})_* E = (F_X^n)_* (F_X^n)_* E$, from Lemma [2,4] we conclude that $(F_X^n)_* E$ is essentially finite if $E$ is essentially finite.

To prove the second part, let $E \to X$ be a vector bundle, and let $n$ be a positive integer such that the direct image $(F_X^n)_* E$ is essentially finite. Since $TX$ is trivial and $(F_X^n)_* E$ is essentially finite, it follows that $c_1(E)$ is numerically equivalent to zero.

The pullback $(F_X^n)^* (F_X^n)_* E$ is essentially finite because $(F_X^n)_* E$ is as well. Since $F_X^n$ is a finite morphism, the natural homomorphism

\[
(F_X^n)^* (F_X^n)_* E \to E
\]

is surjective; hence $E$ is a quotient of the vector bundle $(F_X^n)^* (F_X^n)_* E$. As $E$ is a quotient of an essentially finite vector bundle and $c_1(E)$ is numerically equivalent to zero, we conclude that $E$ is essentially finite; see Remark [2,7].

We recall that a vector bundle $E$ over $X$ is called $F$-trivial if the vector bundle $(F_X^n)^* E$ is trivial for some $n_0$.

\begin{theorem}
Let $E \to X$ be an $F$-trivial vector bundle. Then for every $n$, the direct image $(F_X)_* E$ is $F$-trivial.

If $E$ is a vector bundle on $X$ such that $(F_X^n)_* E$ is $F$-trivial for some $n$, then $E$ is $F$-trivial.
\end{theorem}

\begin{proof}
To prove the first part, it suffices to show that $(F_X)_* E$ is $F$-trivial.

Let $n_0$ be an integer such that the vector bundle $(F_X^{n_0})^* E$ is trivial. In [1], take $Y = X$ and set $\gamma$ to be the identity morphism of $X$. Now, from (6),

\[
F_X (F_X^{n_0})^* E = (F_X^{n_0})^* ((F_X)_* E).
\]

To prove that $(F_X)_* E$ is $F$-trivial, it is enough to show that $(F_X^{n_0})^* ((F_X)_* E)$ is $F$-trivial.
On the other hand, $F_X \times (F_X^m)^* E$ is $F$–trivial if $F_X \mathcal{O}_X$ is $F$–trivial, because $(F_X^m)^* E$ is a trivial vector bundle. Hence from (4) we conclude that $(F_X^m)^* ((F_X)_E)$ is $F$–trivial if $F_X \mathcal{O}_X$ is $F$–trivial. But $F_X \mathcal{O}_X$ is $F$–trivial by Corollary 2.2. This completes the proof of the first part.

To prove the second part, let $E \longrightarrow X$ be a vector bundle of rank $r$, and let $n$ be a positive integer such that the direct image $(F_X^n)_E$ is $F$–trivial. So there is a positive integer $m$ such that the pullback $(F_X^m)^*(F_X^n)_E$ is a trivial vector bundle. Any pullback of a trivial vector bundle is trivial. Hence we may – and we will – assume that $m \geq n$. We saw in the proof of Proposition 2.5 that $E$ is a quotient of $(F_X^m)^*(F_X^n)_E$. Hence the vector bundle $(F_X^{m-n})^*E$ is a quotient of the trivial vector bundle

$$(F_X^{m-n})^*(F_X^m)^*(F_X^n)_E = (F_X^m)^*(F_X^n)_E.$$ 

Define $V_0 := H^0(X, (F_X^m)^*(F_X^n)_E)$, and let $\text{Gr}(V_0, r)$ be the Grassmannian parametrizing quotients of $V_0$ of dimension $r$. Note that the evaluation of sections identifies $(F_X^m)^*(F_X^n)_E$ with the trivial vector bundle $X \times V_0$. The quotient map

$$(F_X^m)^*(F_X^n)_E \longrightarrow (F_X^{m-n})^* E$$ 

produces a quotient map $X \times V_0 \longrightarrow (F_X^{m-n})^* E$. Therefore, there is a morphism

$$f_E : X \longrightarrow \text{Gr}(V_0, r)$$ 

such that $(F_X^{m-n})^* E$ is the pullback $f_E^* Q$, where $Q \longrightarrow \text{Gr}(V_0, r)$ is the tautological quotient bundle.

Since $TX$ is trivial and $(F_X^n)_E$ is $F$–trivial, it follows that $c_1(E)$ is numerically trivial. Hence $c_1((F_X^{m-n})^* E)$ is numerically trivial (this also uses the assumption that $TX$ is trivial). Since the line bundle $\det Q \longrightarrow \text{Gr}(V_0, r)$ is ample and $c_1((F_X^{m-n})^* E) = c_1(f_E^* Q)$ is numerically trivial, it follows that $f_E$ is a constant morphism. Consequently, the vector bundle $(F_X^{m-n})^* E = f_E^* Q$ is trivial. In particular, $E$ is $F$–trivial.

\[ \square \]

Remark 2.7. Let $E$ be an essentially finite vector bundle over a smooth projective variety $M$, and let $Q$ be a quotient bundle of $E$ such that $c_1(Q)$ is numerically trivial. Then $Q$ is also essentially finite. This can be derived from the definition of semistable bundles given in [10, p. 81] (note that this definition differs from the usual definition of semistability) and the definition of an essentially finite vector bundle given in [10, p. 82]. Here we have used the following characterization of an essentially finite vector bundle: There is an étale Galois covering

$$\gamma : \tilde{M} \longrightarrow M$$ 

such that $(F^{n}_{\tilde{M}})^* \gamma^* E$ is trivial, where $n$ is some positive integer and $F_{\tilde{M}}$ is the absolute Frobenius morphism of $\tilde{M}$. The Chern class $c_1((F^{n}_{\tilde{M}})^* \gamma^* Q) = (F^{n}_{\tilde{M}})^* \gamma^* c_1(Q)$ is numerically trivial because $c_1(Q)$ is as well. Since $(F^{n}_{\tilde{M}})^* \gamma^* Q$ is a quotient of the trivial vector bundle $(F^{n}_{\tilde{M}})^* \gamma^* E$ and $c_1((F^{n}_{\tilde{M}})^* \gamma^* Q)$ is numerically trivial, we know that the vector bundle $(F^{n}_{\tilde{M}})^* \gamma^* Q$ is trivial; see the proof of the second part of Theorem 2.6. Consequently, $Q$ is essentially finite [10], [2, Theorem 1.1].
3. THE LOCAL FUNDAMENTAL GROUP–SCHEME

We continue with the set-up of the previous section. Let $X$ be ordinary.
Let \( \{ e_1, \ldots, e_d \} \) be a basis of the \( k \)-vector space \( H^0(X, TX) \). Recall that \( H^0(X, TX) \) is the Lie algebra of \( G \).

Given a vector field \( \theta \in H^0(X, TX) \), we have a vector field \( \theta^p \) defined by

\[
\theta^p(f) := \theta \circ \cdots \circ \theta(f)
\]

for all locally defined functions on \( X \). Let

\[
\mu : H^0(X, TX) \rightarrow H^0(X, TX)
\]

be the additive group homomorphism defined by \( \theta \mapsto \theta^p \). Note that

\[
\mu(c \theta) = c^p \mu(\theta)
\]

for all \( c \in k \). Since \( \mu \) is an additive group homomorphism and \( \mu^{-1}(0) = 0 \), we conclude that \( \mu \) is injective. Since

\[
\mu(\sum_{i=1}^d c_i \cdot e_i) = \sum_{i=1}^d c_i^p \cdot \mu(e_i),
\]

it follows that \( \{ \mu(e_1), \ldots, \mu(e_d) \} \) is a basis for the \( k \)-vector space \( H^0(X, TX) \).

Now from (9) it follows that \( \mu \) is surjective. Therefore, \( \mu \) is an isomorphism.

Recall the above basis \( \{ e_1, \ldots, e_d \} \) of \( H^0(X, TX) \). Let \( \{ e_1^*, \ldots, e_d^* \} \) be the dual basis of \( H^0(X, TX)^* \). For any \( i \in [1, d] \), the element of the coordinate ring \( k[G] \) of \( G \) given by \( e_i^* \) will be denoted by \( x_i \). So

\[
k[G] = \frac{k[x_1, \ldots, x_d]}{(x_1^p, \ldots, x_d^p)}.
\]

Let

\[
x_i \mapsto \sum_{j} f_j^i \otimes g_j^i
\]

be the co–multiplication structure of \( k[G] \).

Define \( G_1 := G \). Let \( G_2 \) be the group–scheme defined by the rule

\[
k[G_2] := \frac{k[x_1^{1/p}, \ldots, x_d^{1/p}]}{(x_1^p, \ldots, x_d^p)}
\]

with the co–multiplication rule

\[
x_i^{1/p} \mapsto \sum_{j} (f_j^i \otimes g_j^i)^{1/p},
\]

where \( f_j^i \) and \( g_j^i \) are defined in (10). We note that \( G_2 \) is a local group–scheme of height two.

Let \( F_{G_2} : G_2 \rightarrow G_2 \) be the absolute Frobenius morphism. Note that the image \( F_{G_2}(G_2) \) coincides with the image of the natural homomorphism \( G_1 := G \rightarrow G_2 \).

More precisely, we have a short exact sequence of group–schemes:

\[
e \rightarrow G_1 \rightarrow G_2 \xrightarrow{F_{G_2}} G_1 \rightarrow e.
\]
More generally, for any positive integer \( n \), define the group–scheme \( G_{n+1} \) as follows:

\[
k[G_{n+1}] := \frac{k[x_1^{1/p^n}, \ldots, x_d^{1/p^n}]}{(x_1, \ldots, x_d)}
\]

with co–multiplication rule

\[
x_i^{1/p^n} \mapsto \sum_j (f_j \otimes g_j)^{1/p^n}.
\]

Note that \( G_n \) is a local group–scheme of height \( n \).

Let \( F_{G_n} : G_n \to G_n \) be the absolute Frobenius morphism. The image \( F_{G_n}(G_n) \) coincides with the image of the natural homomorphism \( G_{n-1} \to G_n \). As in (11), we have a short exact sequence of group–schemes:

\[
(12)
\]

We will show that \( G_{n+1} \) acts on \( X \).

Given a vector field \( D \) on \( X \) and any positive integer \( n \), let \( D^{1/p^n} \) be the vector field on \( X \) defined by

\[
(13)
\]

for all locally defined functions \( f \) on \( X \). So \( D^{1/p^n} = (\mu^n)^{-1}(D) \), where \( \mu \) is the isomorphism in (4).

Let \( S = \text{Spec} k[t]/t^2 \) be the Artin local \( k \)–algebra. Consider \( G_{n+1}(S) \). We will construct an action of \( G_{n+1}(S) \) on \( X(S) \). A point of \( G_{n+1}(S) \) is a \( k \)–algebra homomorphism:

\[
k[x_1^{1/p^n}, \ldots, x_d^{1/p^n}] \to k[t]/t^2.
\]

Consider the homomorphism

\[
k[x_1^{1/p^n}, \ldots, x_d^{1/p^n}] \to k[t]/t^2
\]

defined by \( x_i^{1/p^n} \mapsto t \). The action of this point of \( G_{n+1}(S) \) on \( X(S) \) is constructed as follows: The action of \( (e_i)^{1/p^n} \) sends a tangent vector \( v \) to \( v + (e_i)^{1/p^n} \). (The tangent vector \( (e_i)^{1/p^n} \) is defined in (1); the tangent vectors \( \{e_i\} \) are dual to \( x_i \).)

The following diagram is commutative:

\[
(14)
\]

where the morphisms \( G_n \times X \to X \) and \( G_{n-1} \times X \to X \) are the actions on \( X \) of \( G_n \) and \( G_{n-1} \) respectively.

The quotient \( X/G_n \) is identified with \( X \), and the quotient morphism

\[
X \to X/G_n
\]

coincides with the morphism

\[
F^n_X : X \to X.
\]
This is already proved for \( n = 1 \) (see the proof of Lemma 2.1). To prove the general case, use induction on \( n \). The quotient by the subgroup–scheme \( G = G_1 \) in (12) is the Frobenius morphism \( F_X \) (the case of \( n = 1 \)); hence from (12) it follows that the quotient by \( G_n \) is the composition \( F_X^n \).

Thus the morphism \( F_X^n \) defines a principal \( G_n \)-bundle over \( X \).

Fix a \( k \)-rational point \( x_0 \in X \). The local fundamental group–scheme \( \varpi_1^{\text{loc}}(X,x_0) \) is defined to be the group–scheme associated to the neutral Tannakian category given by the \( F \)-trivial vector bundles on \( X \) (see [7]); the definition of \( F \)-trivial vector bundles is recalled in Section 2. There is a tautological universal principal \( \varpi_1^{\text{loc}}(X,x_0) \)-bundle

\[
\hat{X} \longrightarrow X.
\] (15)

**Theorem 3.1.** The local fundamental group–scheme \( \varpi_1^{\text{loc}}(X,x_0) \) is the inverse limit of the group–schemes \( \{G_n\}_{n \geq 1} \) constructed using the homomorphisms in (12). The tautological principal \( \varpi_1^{\text{loc}}(X,x_0) \)-bundle \( \hat{X} \) in (15) is the inverse limit of the morphisms \( F_X^n : X \longrightarrow X \).

**Proof.** Let \( \mathcal{G} \) denote the inverse limit of the group–schemes \( \{G_n\}_{n \geq 1} \) constructed using the homomorphisms in (12). Let

\[
E_{\mathcal{G}} \longrightarrow X
\]

be the principal \( \mathcal{G} \)-bundle defined by the inverse limit of the morphisms \( F_X^n : X \longrightarrow X \). From the commutativity of the diagram in (14) it follows that the inverse limit of the morphisms \( F_X^n \) is a principal \( \mathcal{G} \)-bundle. Take any rational representation \( V \) of \( \mathcal{G} \). Therefore \( V \) is a rational representation of \( G_{n_0} \) for some \( n_0 \). Let

\[
E_{V}^{n_0} \longrightarrow X
\]

be the vector bundle associated to the principal \( G_{n_0} \)-bundle

\[
F_{X}^{n_0} : X \longrightarrow X
\]

for the \( G_{n_0} \)-module \( V \). So the vector bundle \( (F_X^{n_0})^*E_{V}^{n_0} \) is trivializable. Thus \( E_{V}^{n_0} \) is \( F \)-trivial. Consequently, we obtain a homomorphism,

\[
\rho : \varpi_1^{\text{loc}}(X,x_0) \longrightarrow \mathcal{G}.
\] (16)

This also produces an isomorphism of the principal \( \mathcal{G} \)-bundle \( E_{\mathcal{G}} \) with the principal \( \mathcal{G} \)-bundle \( \hat{X} \times \varpi_1^{\text{loc}}(X,x_0) \mathcal{G} \) obtained by extending the structure group of the principal \( \varpi_1^{\text{loc}}(X,x_0) \)-bundle \( \hat{X} \) using the homomorphism \( \rho \) in (16).

For the converse direction, let \( E \longrightarrow X \) be a \( F \)-trivial vector bundle of rank \( r \). Let \( n_0 \) be an integer such that the pullback \( (F_X^{n_0})^*E \) is trivializable. Fix an isomorphism of \( (F_X^{n_0})^*E \) with the trivial vector bundle \( X \times k^{[\bar{r}]} \). Using this trivialization, the natural action of \( G_{n_0} \) on the fiber \( ((F_X^{n_0})^*E)_{(F_X^{n_0})^{-1}(x_0)} \) defines a linear action of \( G_{n_0} \) on \( k^{[\bar{r}]} \). Therefore, we obtain a homomorphism,

\[
\eta : \mathcal{G} \longrightarrow \varpi_1^{\text{loc}}(X,x_0),
\]

which is the inverse of \( \rho \). \qed
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