A NOTE ON DISCRETENESS OF $F$-JUMPING NUMBERS

KARL SCHWEDE

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Abstract. Suppose that $R$ is a ring essentially of finite type over a perfect field of characteristic $p > 0$ and that $a \subseteq R$ is an ideal. We prove that the set of $F$-jumping numbers of $\tau_b(R; a^t)$ has no limit points under the assumption that $R$ is normal and $Q$-Gorenstein – we make no assumption as to whether the $Q$-Gorenstein index is divisible by $p$. Furthermore, we also show that the $F$-jumping numbers of $\tau_b(R; \Delta, a^t)$ are discrete under the more general assumption that $K_R + \Delta$ is $\mathbb{R}$-Cartier.

1. Introduction

The test ideal is an important and subtle object associated to ideals $a$ in positive characteristic rings $R$. It measures the singularities of both the ambient ring and the elements of the ideal; see [HY03]. While the test ideal was initially introduced in the celebrated theory of tight closure of Hochster and Huneke (see [HH90]), more recent interest in the test ideal has been in regards to its connection with the multiplier ideal – a fundamental invariant of higher dimensional algebraic geometry in characteristic zero; see for example [Tak06] or [MY09].

Given a normal ring $R$ essentially of finite type over a perfect field of characteristic $p > 0$, an ideal $a \subseteq R$ and a real number $t \geq 0$, one can form the (big) test ideal $\tau_b(R; a^t)$ – an object which measures both algebraic and arithmetic properties of $R$ and $a$. Inspired by the test ideal’s close relation with the multiplier ideal $J(R, a^t)$, people have studied the numbers $t_i$ where $\tau_b(R; a^{t_i})$ changes. That is, people have studied the $F$-jumping numbers (see [MTW05]), real numbers which are by definition the $t_i > 0$ such that for every $\varepsilon > 0$,

$$\tau_b(R; a^{t_i - \varepsilon}) \neq \tau_b(R; a^{t_i}).$$

One easy to observe fact about multiplier ideals is that their jumping numbers are discrete and rational, at least when $R$ is $Q$-Gorenstein and normal; see [ELSV04]. Here, by discrete we mean that the set of jumping numbers with respect to a fixed ideal have no limit points. Because of this, various groups have recently worked to show that the $F$-jumping numbers of the test ideal are also discrete and rational; see [Har06], [BMS08], [BMS09], [KLZ09], and [BSTZ10]. In the most recent-mentioned work, the author, along with M. Blickle, S. Takagi, and W. Zhang, showed that

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the \( F \)-jumping numbers of test ideals formed a discrete set of rational numbers when \( R \) is normal and \( \mathbb{Q} \)-Gorenstein with index not divisible by \( p > 0 \). Recall that the index of a \( \mathbb{Q} \)-Gorenstein ring \( R \) is the smallest natural number \( n \) where 
\[
\omega_R^{(n)} = \mathcal{O}_{\text{Spec} \ R} (nK_R)
\]
is locally free.

The most fundamental case left open is the case when \( R \) is \( \mathbb{Q} \)-Gorenstein but of arbitrary index; see \cite{BSTZ10} Question 6.1. We answer this question at least for discreteness.

**Theorem 3.5** Suppose that \( R \) is a normal domain essentially of finite type over an \( F \)-finite field. Further suppose that \( a \subseteq R \) is an ideal and \( \Delta \) is an \( \mathbb{R} \)-divisor on \( X = \text{Spec} \ R \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier (for example, this holds if \( \Delta = 0 \) and \( R \) is \( \mathbb{Q} \)-Gorenstein). Then, as \( t \) varies, the \( F \)-jumping numbers of \( \tau_b (R; \Delta, a^t) \) have no limit points – they are discrete.

We also point out why the existing proofs of rationality do not seem to work in the case that \( R \) is \( \mathbb{Q} \)-Gorenstein with index divisible by \( p \).

Recently, in \cite{DH09}, de Fernex and Hacon gave a definition of the multiplier ideal without the \( \mathbb{Q} \)-Gorenstein assumption and asked the question of whether discreteness and rationality of the \( F \)-jumping numbers still holds in this context. Following this, Urbinati showed that rationality need not hold but gave some evidence that discreteness may hold in general; see \cite{Urb10}. This suggests that one should not expect rationality to hold in positive characteristic either.

## 2. Definition of the Test Ideal

We only give a very brief description of the big test ideal in this paper. Please see \cite{BSTZ10} for a more detailed description of the test ideal.

First we fix some notation. Given a ring \( R \) of characteristic \( p > 0 \) and \( M \) an \( R \)-module, we set \( F^e_r M \) to be the \( R \)-module which agrees with \( M \) as an additive group but where the \( R \)-module structure is defined by the rule \( r.m = r^p.m \). Also recall that \( R \) is said to be \( F \)-finite if \( F^e_r R \) is a finitely generated \( R \)-module.

**Convention.** Throughout this paper, all rings will be assumed to be \( F \)-finite.

Recall that an \( \mathbb{R} \)-divisor on a normal scheme \( X \) is a formal linear combination of prime Weil divisors \( D_i \) with real coefficients. An \( \mathbb{R} \)-divisor \( D \) is called \( \mathbb{R} \)-Cartier if it is equal to an \( \mathbb{R} \)-linear combination of Cartier divisors.

We now define the test ideal \( \tau_b (R; \Delta, a^t b_1^{s_1} \ldots b_m^{s_m}) \). We work in this greater generality because when proving our main theorem, we perturb our initial triple \( (R, \Delta, a^t) \) to a new triple \( (R, \Delta', a^t b_1^{s_1} \ldots b_m^{s_m}) \) which has the same test ideal.

**Definition 2.1** \cite{HH90, Hoc07, Sch10}. Suppose that \( R \) is an \( F \)-finite normal domain, \( \Delta \geq 0 \) is an \( \mathbb{R} \)-divisor on \( X = \text{Spec} \ R \), \( a, b_1, \ldots, b_m \subseteq R \) are non-zero ideals and \( t, s_1, \ldots, s_m \geq 0 \) are real numbers. Then the **big test ideal** \( \tau_b (R; \Delta, a^t b_1^{s_1} \ldots b_m^{s_m}) \) is defined to be the unique smallest non-zero ideal \( J \subseteq R \) such that
\[
(1) \quad \phi \left( F^e_r (a(t(p^e - 1))^{[s_1(p^e - 1)]} \ldots b_m^{[s_m(p^e - 1)]}, J) \right) \subseteq J
\]
for every \( e \geq 0 \) and every \( \phi \in \text{Hom}_R (F^e_r R ([ (p^e - 1)\Delta ]), R) \). This ideal always exists in the context described.

**Remark 2.2.** In the case that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, the **big test ideal** is known to equal the (finitistic) **test ideal** (which we will not define here); see \cite{Tak04} and \cite{BSTZ10} for details.
If all $b_i = R$, then we denote the associated big test ideal by $\tau_b(R; \Delta, a')$. Likewise if $\Delta = 0$, then we denote the associated big test ideal using the notation $\tau_b(R; a' b_1^{s_1} \ldots b_m^{s_m})$. Finally, if the $b_i = (f_i)$ are principal, we denote the associated big test ideal by $\tau_b(R; \Delta, a' f_1^{s_1} \ldots f_m^{s_m})$.

**Remark 2.3.** Given a non-zero element $c \in \tau_b(R; \Delta, a' b_1^{s_1} \ldots b_m^{s_m})$ (such an element is called a *big sharp test element*), we note that

$$\tau_b(R; \Delta, a' b_1^{s_1} \ldots b_m^{s_m}) = \sum_{c > 0} \sum \phi \left( F_c^e(cq^{[s_1(p_e - 1)]} b_1^{[s_1(p_e - 1)]} \ldots b_m^{[s_m(p_e - 1)]}) \right),$$

where the inner sum is over $\phi \in \text{Hom}_R(F_c^e(R((p_e - 1)\Delta)), R)$. To see this, simply note that the right side satisfies equation (1), and it is by definition the smallest ideal containing $c$ satisfying equation (1); note $a^0 = b^0 = R$.

Suppose that $X = \text{Spec } R$ is normal. Then given $\phi \in \text{Hom}_R(F_c^e(R, R) \cong F_c^e\mathcal{O}_X((1-p_e)K_X)$, we may view $\phi$ as determining an effective Weil divisor linearly equivalent to $(1-p_e)K_X$.

**Definition 2.4.** We use $D_\phi$ to denote the Weil divisor associated to $\phi$ in this way.

Given an $\mathbb{R}$-divisor $\Delta \geq 0$ on $X$, one has an inclusion

$$(3) \quad \text{Hom}_R(F_c^e(R((p_e - 1)\Delta)), R) \subseteq \text{Hom}_R(F_c^e(R, R)).$$

The following lemma gives a nice interpretation of this submodule.

**Lemma 2.5.** An element $\phi \in \text{Hom}_R(F_c^e(R, R))$ is contained in the submodule $\text{Hom}_R(F_c^e(R((p_e - 1)\Delta)), R)$ if and only if $D_\phi \geq (p_e - 1)\Delta$.

**Proof.** Because all the modules are reflexive, the statement can be reduced to the case when $R$ is a discrete valuation ring and $\Delta = s \text{ div}(x)$, where $x$ is the parameter for the DVR $R$ and $s \geq 0$ is a real number. In this case, the inclusion from equation (3) can be identified with the multiplication map $R \rightarrow R$ which sends $1$ to $x^{[s(p_e - 1)]}$. Thus, $\phi \in \text{Hom}_R(F_c^e(R, R) \cong R$ is contained inside $\text{Hom}_R(F_c^e(R((p_e - 1)\Delta)), R) \cong x^{[s(p_e - 1)]} R$ if and only if $D_\phi \geq [s(p_e - 1)] \text{ div}(x) = [(p_e - 1)\Delta]$. However, since $D_\phi$ is integral, it is harmless to remove the round-up $\lceil \cdots \rceil$.

### 3. Discreteness of $F$-Jumping Numbers

In this section we prove our main result. We accomplish this by perturbing our triples $(R, \Delta, a')$ in order to reduce the discreteness statement to the case where the (log) $\mathbb{Q}$-Gorenstein index is not divisible by $p > 0$. First we need a lemma.

**Lemma 3.1.** Suppose that $(X = \text{Spec } R, \Delta, a' b_1^{s_1} \ldots b_m^{s_m})$ is a triple and that $\Delta = \Gamma + b \text{ div}(f)$ for some $f \in R \setminus \{0\}$ and non-negative number $b \in \mathbb{R}$. Then

$$\tau_b(R; \Delta, a' b_1^{s_1} \ldots b_m^{s_m}) = \tau_b(R; \Gamma, f^b a' b_1^{s_1} \ldots b_m^{s_m}).$$

This type of statement is essentially obvious for multiplier ideals, but because of certain issues surrounding the construction of test ideals we have thus-far presented, it is somewhat less obvious in this context. However, it is still quite straightforward, especially from the definition of the generalized test ideal by Harada-Yoshida-Takagi (the proof in that case uses the theory tight closure); see [HY03] and [Tak04]. We provide a short proof here, certainly acknowledging that this statement is obvious to experts.
Thus, since $\geq t$

**Proof.** Choose $c$ to be a big sharp test element for both $(X, \Delta, a^i b^i_1 \ldots b^i_m)$ and $(X, \Gamma, f^i a^i b^i_1 \ldots b^i_m)$. Then we know that the test ideal $\tau_b(R; \Gamma, f^i a^i b^i_1 \ldots b^i_m)$ equals

$$
\sum_{\phi, D_{\phi} \geq (p^e-1)\Gamma} \phi \left(F^e_c \phi f [b(p^e-1)] a^{[t(p^e-1)]} b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

$$
= \sum_{\phi, D_{\phi} \geq (p^e-1)\Gamma + \text{div } f[b(p^e-1)]} \phi \left(F^e_c \phi f [t(p^e-1)] b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

Consequently,

$$
\sum_{\phi, D_{\phi} \geq (p^e-1)\Gamma + [b(p^e-1) \text{div } f]} \phi \left(F^e_c \phi f [t(p^e-1)] b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

$$
\subseteq \sum_{\phi, D_{\phi} \geq (p^e-1)\Gamma + \text{div } f} \phi \left(F^e_c \phi f [t(p^e-1)] b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

$$
= \tau_b(R; \Delta, a^i b^i_1 \ldots b^i_m),
$$

and so $\tau_b(R; \Gamma, f^i a^i b^i_1 \ldots b^i_m) \subseteq \tau_b(R; \Delta, a^i b^i_1 \ldots b^i_m)$. For the converse inclusion, observe first that

$$
\text{div } f[b(p^e-1)] = b(p^e - 1) \text{div } f \leq \text{div } f).
$$

Thus, since $cf$ is also a test element,

$$
\tau_b(R, \Delta; a^i b^i_1 \ldots b^i_m) = \sum_{\phi, D_{\phi} \geq (p^e-1)\Delta} \phi \left(F^e_c \phi f a^{[t(p^e-1)]} b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

$$
= \sum_{\phi, D_{\phi} \geq (p^e-1)\Gamma + \text{div } f} \phi \left(F^e_c \phi f a^{[t(p^e-1)]} b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

Consequently,

$$
\sum_{\phi, D_{\phi} \geq (p^e-1)\Gamma + \text{div } f} \phi \left(F^e_c \phi f [b(p^e-1)] a^{[t(p^e-1)]} b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

$$
\subseteq \sum_{\phi, D_{\phi} \geq (p^e-1)\Gamma + \text{div } f} \phi \left(F^e_c \phi f [t(p^e-1)] b_1^{[s_1(p^e-1)]} \ldots b_m^{[s_m(p^e-1)]} \right)
$$

$$
= \tau_b(R, \Delta; a^i b^i_1 \ldots b^i_m),
$$

and so $\tau_b(R; \Delta, a^i b^i_1 \ldots b^i_m) \subseteq \tau_b(R; \Gamma, f^i a^i b^i_1 \ldots b^i_m)$, as desired. \hfill \Box

We also need a very special case of Skoda’s theorem.

**Lemma 3.2** ([HT04 Theorem 4.1]). Suppose that $X = \text{Spec } R$, $\Delta > 0$, $a \subseteq R$ and $t \geq 0$ is as above. Further suppose that $f \in R$ is a non-zero element. Then

$$
\tau_b(X; \Delta + \text{div } f, a^i b^i_1 \ldots b^i_m) = f \tau_b(X; \Delta, a^i b^i_1 \ldots b^i_m).
$$

**Proof.** We leave the proof to reader; see [HT04 Theorem 4.2] and [BSTZ10 Lemma 3.26]. \hfill \Box

Now we can prove the following result.

**Theorem 3.3.** Suppose that $R$ is an $F$-finite normal domain and further suppose that $(X = \text{Spec } R, \Delta, a^i)$ is a triple where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then for each point $x \in X$, there exists an open set $U = \text{Spec } R' = \text{Spec } R[\text{h}^{-1}]$ containing $x \in X$
with the following properties: There exists an effective \( \mathbb{Q} \)-divisor \( \Gamma \) on \( U \), elements \( f_1, \ldots, f_m \in R' \setminus \{0\} \) and non-negative real numbers \( b_1, \ldots, b_m \) such that

1. \( K_U + \Gamma \) is \( \mathbb{Q} \)-Cartier with index not divisible by \( p > 0 \), and, furthermore, \( (p^e - 1)(K_U + \Gamma) \sim 0 \) for some integer \( e > 0 \).
2. The \( F \)-jumping numbers of \( \tau_b(U; \Delta|_U, (aR')^t) \) are the same as the \( F \)-jumping numbers of \( \tau_b(U; \Delta + \Delta|_U, (aR')^t) \) (both sets of jumping numbers are with respect to \( t \)).

Proof. Choose a non-zero section \( \phi \) of \( \text{Hom}_R(F^*_x R, R) \) and set \( \Gamma := \Gamma_{\Phi} \). It follows that \( K_X + \Gamma \) satisfies condition (1) on \( X \). Therefore, \( (K_X + \Delta) - (K_X + \Gamma) = \Delta - \Gamma \) is \( \mathbb{R} \)-Cartier, and so we may write \( \Delta - \Gamma = d_1 D_1 + \cdots + d_m D_m \) for some integral effective Cartier divisors \( D_i \) and real numbers \( d_i \in \mathbb{R} \). We choose our open set \( U = \text{Spec} R[h^{-1}] = \text{Spec} R' \) to be any such set containing \( x \in X \) where all of the \( D_i|_U \) are principal divisors.

Now write \( D_i|_U = \text{div}(f_i) \) for some \( f_i \in R' \setminus \{0\} \) and also by abuse of notation denote \( \Gamma := \Gamma_{\Phi} \). Choose natural numbers \( l_i \) such that \( b_i := l_i + d_i > 0 \) for all \( i \) and set \( g := f_1^{l_1} \cdots f_m^{l_m} \in R' \). Notice that \( (|_U + \text{div}(g)) - \Gamma = b_1 \text{div}(f_1) + \cdots + b_m \text{div}(f_m) \).

By Lemma 3.2 the \( F \)-jumping numbers of \( \tau_b(U; \Delta|_U, (aR')^t) \) and the \( F \)-jumping numbers of \( \tau_b(U; \Delta + \text{div}(g), (aR')^t) \) coincide. Now using Lemma 3.1 we have

\[
\begin{align*}
\tau_b(U; \Delta|_U + \text{div}(g), (aR')^t) &= \tau_b(U; \Gamma + b_1 \text{div}(f_1) + \cdots + b_m \text{div}(f_m), (aR')^t) \\
&= \tau_b(U; \Gamma, f_1^{b_1} \cdots f_m^{b_m} (aR')^t),
\end{align*}
\]

which proves the theorem. \( \square \)

Remark 3.4. If, in Theorem 3.3, \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, then one needs only a single \( f_1^{b_1} \) (and no other \( f_i^{b_i} \)). However, if the index of \( K_X + \Delta \) is divisible by \( p > 0 \), then it follows by construction that \( b_1 \) will be a rational number with its denominator divisible by \( p > 0 \).

We are now in a position to prove the discreteness of the \( F \)-jumping numbers in the case that \( X \) is essentially of finite type over a field. The proof idea follows the usual lines.

Theorem 3.5. Suppose that \( R \) is a normal domain essentially of finite type over an \( F \)-finite field. Further suppose that \( a \subseteq R \) is an ideal and \( \Delta \) is an \( \mathbb{R} \)-divisor on \( X = \text{Spec} R \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier (for example, this holds if \( \Delta = 0 \) and \( R \) is \( \mathbb{Q} \)-Gorenstein). Then, as \( t \) varies, the \( F \)-jumping numbers of \( \tau_b(R; \Delta, a^t) \) have no limit points -- they are discrete.

Proof. By [BSTZ10] Proposition 3.28, it is sufficient to answer this question on a finite affine cover of \( X \). Therefore, we reduce to the case that \( X \) is one of the charts from Theorem 3.3. More precisely, it is sufficient to prove our result for triples of the form \( \tau_b(R; \Gamma, f_1^{b_1} \cdots f_m^{b_m} a^t) \), where \( (p^e - 1)(K_X + \Gamma) \sim 0 \) for some \( e > 0 \). Using [BSTZ10] Lemma 4.2, Proposition 3.28, one can further assume that \( R \) is of finite type over an \( F \)-finite field of characteristic \( p > 0 \). One then has two possibilities:

(a) Mimic the proof of the main result of [BSTZ10] Section 4. In other words, use the methods of \( F \)-adjunction (as worked out in [Sch09a] and [BSTZ10]) to reduce to the case where \( R \) is a polynomial ring and then use
degree bounding methods similar to those found in [BMS08]. Note that in
[BSTZ10] one worked with triples \((R, \Delta, a^t)\) and not with the more complicated objects \((R, \Gamma, f_{b_1}^{b_{b_1}} \ldots f_{b_m}^{b_m} a^t)\), but the methods are easily generalized to our setting.

(b) Use the new language of [Bli09, Section 4]. We claim that the algebra of \(p^{-e}\)-linear maps associated to the triple \((R, \Gamma, f_{b_1}^{b_{b_1}} \ldots f_{b_m}^{b_m})\), as in [Sch09a, Remark 3.10], is “gauge bounded” (see [Bli09, Definition 4.7]). To see this claim, note that by [Sch09a Lemma 3.9] or [Sch09b, Remark 4.4] the Cartier-algebra associated to \((R, \Gamma, f_{b_1}^{b_{b_1}} \ldots f_{b_m}^{b_m})\) is finitely generated and thus gauge bounded by [Bli09, Proposition 4.8]. It then follows from [Bli09, Proposition 4.13] that the Cartier-algebra associated to \((R, \Gamma, f_{b_1}^{b_{b_1}} \ldots f_{b_m}^{b_m})\) is also gauge bounded, as claimed. To finish the proof, apply [Bli09 Theorem 4.14].

In either case, the result follows easily from the theories previously developed. □

4. ON THE QUESTION OF RATIONALITY

Note that the usual way to prove the rationality of the \(F\)-jumping numbers employs the following theorem. First recall that a pair \((X, \Delta)\) is called log \(\mathbb{Q}\)-Gorenstein with index \(n\) if \(n(K_X + \Delta)\) is Cartier and \(n > 0\) is the smallest integer with this property.

**Theorem 4.1 (BMS08, BSTZ10).** Suppose \((X, \Delta)\) is log \(\mathbb{Q}\)-Gorenstein with index \(n\) such that \(n\) divides \((p^e - 1)\) for some fixed \(e > 0\). Further suppose that \(a\) is an ideal sheaf of \(X\). Then if \(t_0\) is an \(F\)-jumping number of \(\tau(X; \Delta, a^t)\), then \(p^e t_0\) is also an \(F\)-jumping number.

However, without the “index not divisible by \(p\)” assumption, this theorem is false. Consider the following example (which in some sense is typical by Remark 3.4).

**Example 4.2.** Set \(X = \mathbb{A}^1_k = \text{Spec } k[x]\), \(\Delta = \frac{1}{p} \text{div}(x)\) and \(a = (x)\). Then the \(F\)-jumping numbers of \((X, \Delta, a^t) = (X, (x)^{1/p} a^t)\) with respect to \(t\) are

\[
p - 1, \frac{2p - 1}{p}, \frac{3p - 1}{p}, \ldots.
\]

In particular, \(p\) (or \(p^e\)) times any of them is not an \(F\)-jumping number.

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**References**


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DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

E-mail address: schwede@math.psu.edu

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