THE BRAUER GROUP OF MODULI SPACES OF VECTOR BUNDLES OVER A REAL CURVE

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Abstract. Let $X$ be a geometrically connected smooth projective curve of genus $g_X \geq 2$ over $\mathbb{R}$. Let $M(r, \xi)$ be the coarse moduli space of geometrically stable vector bundles $E$ over $X$ of rank $r$ and determinant $\xi$, where $\xi$ is a real point of the Picard variety $\text{Pic}^d(X)$. If $g_X = r = 2$, then let $d$ be odd. We compute the Brauer group of $M(r, \xi)$.

1. Introduction

Let $X_{\mathbb{C}}$ be a connected smooth projective curve of genus $g_X \geq 2$ over $\mathbb{C}$. Fix integers $r \geq 2$ and $d$. Given a line bundle $\xi_{\mathbb{C}}$ of degree $d$ over $X_{\mathbb{C}}$, we denote by $M(r, \xi_{\mathbb{C}})$ the coarse moduli space of stable vector bundles over $X_{\mathbb{C}}$ of rank $r$ and determinant $\xi_{\mathbb{C}}$.

The Picard group of such moduli spaces has been studied intensively; see for example [DN, KN, LS, Sc, BLS, Fa, Te, BHo1]. We view the Brauer group as a natural higher order analogue of the Picard group. It is related to the classical rationality problem [CS].

We assume that $d$ is odd if $g_X = r = 2$; otherwise $d$ is arbitrary. The Brauer group of $M(r, \xi_{\mathbb{C}})$ has been computed in [BBCN]; the result is a canonical isomorphism

$$\text{Br}(M(r, \xi_{\mathbb{C}})) \cong \mathbb{Z}/\gcd(r, d).$$

The corresponding generator $\beta_{\mathbb{C}} \in \text{Br}(M(r, \xi_{\mathbb{C}}))$ can be viewed as the obstruction against the existence of a Poincaré bundle, or universal vector bundle, over $M(r, \xi_{\mathbb{C}}) \times X_{\mathbb{C}}$.

Now suppose $X_{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$ for a smooth projective curve $X$ over $\mathbb{R}$. Then some of the above moduli spaces carry interesting real algebraic structures, and there has been a growing interest in understanding these structures [BbB, BHI, BHu, Sch]. In this paper we compute the Brauer group of such real algebraic moduli spaces.

More precisely, assume that the line bundle $\xi_{\mathbb{C}}$ comes from a real point $\xi$ of the Picard variety $\text{Pic}^d(X)$. Let $M(r, \xi)$ be the coarse moduli space of geometrically stable vector bundles $E$ over $X$ of rank $r$ and determinant $\xi$. It is a smooth quasi-projective variety over $\mathbb{R}$, with $M(r, \xi) \otimes_{\mathbb{R}} \mathbb{C} \cong M(r, \xi_{\mathbb{C}})$; see Section 2. Our main result, Theorem 3.3, describes the Brauer group of $M(r, \xi)$ as follows.

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Theorem 1.1. With $\chi := r(1 - g_X) + d$, there is a canonical isomorphism

$$\text{Br}(M(r, \xi)) \cong \begin{cases} \mathbb{Z}/\gcd(r, \chi) \oplus \mathbb{Z}/2 & \text{if } \xi \text{ comes from a line bundle defined over } \mathbb{R}, \\
\mathbb{Z}/\gcd(2r, \chi) & \text{otherwise.} \end{cases}$$

Note that $\gcd(r, \chi) = \gcd(r, d)$. The groups $\mathbb{Z}/\gcd(r, \chi)$ and $\mathbb{Z}/\gcd(2r, \chi)$ are generated by a canonical class $\beta \in \text{Br}(M(r, \xi))$, the obstruction against a Poincaré bundle over $M(r, \xi) \times X$. The order of this obstruction class $\beta$ is computed in Proposition 3.2. The remaining direct summand $\mathbb{Z}/2$ comes from the Brauer group of $\mathbb{R}$.

2. Moduli of vector bundles over a real curve

Let $X$ be a geometrically connected smooth projective algebraic curve of genus $g_X \geq 2$ defined over $\mathbb{R}$. We will denote the base change from $\mathbb{R}$ to $\mathbb{C}$ by a subscript $\mathbb{C}$. In particular, $X_\mathbb{C} := X \otimes_\mathbb{R} \mathbb{C}$ is the associated algebraic curve over $\mathbb{C}$.

Let $\sigma : \mathbb{C} \to \mathbb{C}$ denote complex conjugation. The involutive morphism of schemes

$$\sigma_X := \text{id}_X \otimes \sigma : X_\mathbb{C} \to X_\mathbb{C}$$

lies over $\sigma : \mathbb{C} \to \mathbb{C}$. The closed points of $X_\mathbb{C}$ fixed by $\sigma_X$ are the real points of $X$.

Let $\xi$ be a real point of the Picard variety $\text{Pic}(X)$. Viewing the associated complex point $\xi_\mathbb{C}$ of $\text{Pic}(X)_\mathbb{C}$ as a line bundle over $X_\mathbb{C}$, we have $\xi_\mathbb{C} \cong \sigma_X^*(\xi_\mathbb{C})$.

A real (respectively, quaternionic) structure on $\xi_\mathbb{C}$ is by definition an isomorphism

$$\eta : \xi_\mathbb{C} \to \sigma_X^*(\xi_\mathbb{C})$$

of line bundles over $X_\mathbb{C}$ with $\sigma_X^* \eta \circ \eta = \text{id}_{\xi_\mathbb{C}}$ (respectively, $\sigma_X^* \eta \circ \eta = -\text{id}_{\xi_\mathbb{C}}$). The line bundle $\xi_\mathbb{C}$ admits either a real structure $\eta$ or a quaternionic structure $\eta$, and in both cases the resulting pair $(\xi_\mathbb{C}, \eta)$ is uniquely determined up to an isomorphism; cf. for example [Ve, Proposition 2.5] or [BHH, Proposition 3.1].

The real point $\xi$ of $\text{Pic}(X)$ is called quaternionic if $\xi_\mathbb{C}$ admits a quaternionic structure. Otherwise, $\xi_\mathbb{C}$ admits a real structure, so we can view $\xi$ as a real line bundle over $X$.

A vector bundle $E$ over $X$ is called geometrically stable if the vector bundle $E_\mathbb{C}$ over $X_\mathbb{C}$ is stable. Not every stable vector bundle $E$ over $X$ is geometrically stable, but it is always geometrically polystable. Fix integers $r \geq 2$ and $d$. We denote by

$$\mathcal{M}(r, d) \supset \mathcal{M}(r, d)^s \to M(r, d)$$

the moduli stack of vector bundles $E$ over $X$ of rank $r$ and degree $d$, the open substack of geometrically stable $E$, and their coarse moduli scheme, respectively. Since geometrically stable $E$ have only scalar automorphisms, $\mathcal{M}(r, d)^s$ is a gerbe with band $\mathbb{G}_m$ over $M(r, d)$.

Let $\mathcal{L}(\det)$ denote the determinant of the cohomology line bundle over $\mathcal{M}(r, d)$. Its fiber over the moduli point of a vector bundle $E$ is by definition $\det \text{det} H^0(E) \otimes \text{det}^{-1} H^1(E)$.

All three moduli spaces or stacks in (2.1) come with a determinant map to the Picard variety $\text{Pic}^d(X)$. Given a real point $\xi$ of $\text{Pic}^d(X)$, we denote by

$$\mathcal{M}(r, \xi) \supset \mathcal{M}(r, \xi)^s \to M(r, \xi)$$
the corresponding fibers over \( \xi \). So \( M(r, \xi) \) is a smooth quasiprojective variety over \( \mathbb{R} \), whose base change \( M(r, \xi)_C \) is the moduli space of stable vector bundles over \( X_C \) of rank \( r \) and determinant \( \xi_C \). By restriction, \( M(r, \xi)^s \) is a gerbe with band \( \mathbb{G}_m \) over \( M(r, \xi) \).

Suppose for the moment that \( \xi \) is a real line bundle. Then we can define a line bundle \( L(\xi) \) over \( M(r, \xi) \) whose fiber over the moduli point of a vector bundle \( E \) is \( \text{Hom}(\xi, \det E) \). To state this more precisely, let \( S \) be a scheme over \( \mathbb{R} \). Then the pullback of \( L(\xi) \) along the classifying morphism \( S \to M(r, \xi) \) of a vector bundle \( E \) over \( X \times S \) is by definition the line bundle \( \text{pr}_2^*(\text{pr}_1^*\xi^{-1} \otimes \det E) \) over \( S \). This defines a line bundle \( L(\xi) \) over \( M(r, \xi) \).

Now suppose that \( \xi \) is quaternionic. Then the same recipe defines a line bundle over \( M(r, \xi)_C \) endowed with a quaternionic structure. We denote this pair again by \( L(\xi) \).

In both cases, \( L(\xi) \) gives us a line bundle \( L(\xi)_C \) over \( M(r, \xi)_C \). If we trivialize the fiber of \( \xi_C \) over one closed point \( x_0 \in X_C \), we can identify \( L(\xi)_C \) with the line bundle whose fiber at the moduli point of a vector bundle \( E_C \) over \( X_C \) is the fiber of \( \det E_C \) over \( x_0 \).

**Proposition 2.1.** The Picard group \( \text{Pic}(M(r, \xi)) \) is generated by

i) \( L(\det) \) and \( L(\xi) \) if \( \xi \) is a real line bundle,

ii) \( L(\det) \) and \( L(\xi)^{\otimes 2} \) if \( \xi \) is quaternionic.

The restrictions of these line bundles also generate \( \text{Pic}(M(r, \xi)^s) \).

**Proof.** Let \( \tilde{M}(r, \xi_C) \) denote the moduli stack of vector bundles \( E \) of rank \( r \) over \( X_C \) together with an isomorphism \( \xi_C \cong \det E \). The forgetful map

\[ \pi : \tilde{M}(r, \xi_C) \to M(r, \xi)_C \]

is the \( \mathbb{G}_m \)-torsor given by the line bundle \( L(\xi)_C \). It is easy to check that the kernel of

\[ \pi^* : \text{Pic}(M(r, \xi)_C) \to \text{Pic}(\tilde{M}(r, \xi)_C) \]

is generated by \( L(\xi)_C \); cf. the proof of [BL] Lemma 7.8]. The Picard group of \( \tilde{M}(r, \xi_C) \) is generated by \( \pi^*(L(\det)_C) \), according to [BL] Remark 7.11 and Proposition 9.2.

This shows that \( \text{Pic}(M(r, \xi)_C) \) is generated by \( L(\det)_C \) and \( L(\xi)_C \). We have just seen that all these line bundles admit a real or quaternionic structure. This real or quaternionic structure is unique, since \( \Gamma(M(r, \xi)_C, \mathcal{O}^*) = \mathbb{C}^* \). It follows that \( \text{Pic}(M(r, \xi)) \) is the subgroup of line bundles in \( \text{Pic}(M(r, \xi)_C) \) which are real, not quaternionic. Hence \( \text{Pic}(M(r, \xi)) \) is generated by the line bundles as claimed.

As \( M(r, \xi) \) is smooth, the restriction map \( \text{Pic}(M(r, \xi)) \to \text{Pic}(M(r, \xi)^s) \) is surjective; cf. for example [BHo2] Lemma 7.3]. So these line bundles also generate \( \text{Pic}(M(r, \xi)^s) \).

Now let \( \mathcal{M} \to M \) be a gerbe with band \( \mathbb{G}_m \) over an irreducible Noetherian scheme \( M \). As a basic example, we have the gerbe \( \mathcal{M}(r, d)^s \to M(r, d) \) in mind.

**Definition 2.2.** Let \( \mathcal{L} \) be a line bundle over \( \mathcal{M} \). Then the automorphism groups \( \mathbb{G}_m \) in \( \mathcal{M} \) act on the fibers of \( \mathcal{L} \). These \( \mathbb{G}_m \) act by the same power \( w \in \mathbb{Z} \) on every fiber of \( \mathcal{L} \), since \( \mathcal{M} \) is connected. The integer \( w \) is called the weight of \( \mathcal{L} \).
The weight of a quaternionic line bundle $\mathcal{L}$ is by definition the weight of the associated complex line bundle $\mathcal{L}_C$. For example, the real or quaternionic line bundle $\mathcal{L}(\xi)$ over $\mathcal{M}(r,\xi)^s$ has weight $r$. The real line bundle $\mathcal{L}(\det)$ over $\mathcal{M}(r,d)^s$ has weight

$$\chi := r(1-g\chi) + d$$

according to Riemann-Roch. Consider the integers

$$\chi' := \chi/g(\chi) \quad \text{and} \quad r' := r/g(\chi).$$

The real or quaternionic line bundle

$$\mathcal{L}(\Theta) := \mathcal{L}(\det)^\otimes r' \otimes \mathcal{L}(\xi)^\otimes \chi'$$

over $\mathcal{M}(r,\xi)^s$ has weight 0. Hence it descends to a real or quaternionic line bundle over $\mathcal{M}(r,\xi)$, which we again denote by $\mathcal{L}(\Theta)$. The line bundle $\mathcal{L}(\Theta)_C$ is ample on $\mathcal{M}(r,\xi)_C$, and it generates the Picard group $\text{Pic}(\mathcal{M}(r,\xi)_C)$ according to [DN Théorèmes A & B].

**Proposition 2.3.** The Picard group $\text{Pic}(\mathcal{M}(r,\xi))$ is generated by

i) $\mathcal{L}(\Theta)$ if $\xi$ is a real line bundle or $\chi'$ is even,

ii) $\mathcal{L}(\Theta)^\otimes 2$ if $\xi$ is quaternionic and $\chi'$ is odd.

**Proof.** The line bundles over $\mathcal{M}(r,\xi)$ are the line bundles of weight 0 over $\mathcal{M}(r,\xi)^s$. According to Proposition [2], these are of the form $\mathcal{L}(\det)^\otimes a \otimes \mathcal{L}(\xi)^\otimes b$ with $a\chi + br = 0$, where moreover $b$ has to be even if $\xi$ is quaternionic. \qed

### 3. The Brauer Group

The Brauer group $\text{Br}(S)$ of a Noetherian scheme $S$ is by definition the abelian group of Azumaya algebras over $S$ up to Morita equivalence. It is a torsion group, and it embeds canonically into the étale cohomology group $H^2_{\text{ét}}(S, \mathbb{G}_m)$.

If $S$ is smooth and quasiprojective over a field, then $H^2_{\text{ét}}(S, \mathbb{G}_m)$ is also a torsion group [Gi Proposition 1.4], and the embedding of $\text{Br}(S)$ into $H^2_{\text{ét}}(S, \mathbb{G}_m)$ is an isomorphism [G].

Our aim is to compute the Brauer group of the real moduli space $\mathcal{M}(r,\xi)$. Let

$$\beta \in H^2_{\text{ét}}(\mathcal{M}(r,\xi), \mathbb{G}_m) = \text{Br}(\mathcal{M}(r,\xi))$$

(3.1)

denote the class given by the gerbe $\mathcal{M}(r,\xi)^s \to \mathcal{M}(r,\xi)$ with band $\mathbb{G}_m$. Since a section of this gerbe would yield a Poincaré bundle over $\mathcal{M}(r,\xi) \times X$, we can view the class $\beta$ as the obstruction against the existence of such a Poincaré bundle.

**Remark 3.1.** Choose an effective divisor $D \subset X$ defined over $\mathbb{R}$, for example a closed point in $X$. The Brauer class $\beta$ over $\mathcal{M}(r,\xi)$ can also be described by the Azumaya algebra with fibers $\text{End} H^0(D, E|_D)$, or by the projective bundle with fibers $\mathbb{P} H^0(D, E|_D)$.

We first compute the exponent of $\beta$, i.e., the order of $\beta$ as an element in the torsion group $\text{Br}(\mathcal{M}(r,\xi))$. This will in particular re-prove results of [BH] Section 5).

**Proposition 3.2.** Let $\xi$ be a real point of the Picard variety $\text{Pic}^d(X)$.

i) If $\xi$ is a real line bundle, then $\beta \in \text{Br}(\mathcal{M}(r,\xi))$ has exponent $\gcd(r, \chi)$.

ii) If $\xi$ is quaternionic, then $\beta \in \text{Br}(\mathcal{M}(r,\xi))$ has exponent $\gcd(2r, \chi)$. 


Proof: An integer \( n \in \mathbb{Z} \) annihilates the class \( \beta \in H^2_{\text{et}}(M(r, \xi), G_m) \) of the gerbe \( \mathcal{M}(r, \xi)^1 \) if and only if there is a line bundle \( \mathcal{L} \) over \( \mathcal{M}(r, \xi)^1 \) which has weight \( n \); see for example [Ho, Lemma 4.9]. Hence the claim follows from Proposition 2.2.  \( \square \)

We denote by \( \mathbb{Z} \cdot \beta \subseteq \text{Br}(M(r, \xi)) \) the subgroup generated by the class \( \beta \) in (3.1). Let

\[
\begin{align*}
(3.2) \\
f : M(r, \xi) \longrightarrow \text{Spec}(\mathbb{R})
\end{align*}
\]

be the structure morphism. Recall that \( \text{Br}(\mathbb{R}) \cong \mathbb{Z}/2 \), the nontrivial element being the class \([H] \in \text{Br}(\mathbb{R})\) of the quaternion algebra \( H = \mathbb{R} \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k \).

Theorem 3.3. Let \( \xi \) be a real point of \( \text{Pic}^d(X) \), with \( d \) odd if \( g_X = r = 2 \). We have

\[
\text{Br}(M(r, \xi)) = \begin{cases} \\
\mathbb{Z} \cdot \beta \oplus f^*(\text{Br}(\mathbb{R})) \cong \mathbb{Z}/\gcd(r, \chi) \oplus \mathbb{Z}/2 & \text{if } \xi \text{ is a real line bundle,} \\
\mathbb{Z} \cdot \beta & \text{if } \xi \text{ is quaternionic.}
\end{cases}
\]

Proof. The structure morphism \( f \) in (3.2) yields a Leray spectral sequence

\[
(3.3) \\
E_2^{p,q} = H^p_{\text{et}}(\mathbb{R}, R^q f_*G_m) \Rightarrow H^{p+q}_{\text{et}}(M(r, \xi), G_m).
\]

We have \( R^1 f_*G_m = \text{Pic}(M(r, \xi)_C) \cong \mathbb{Z} \). The action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2 \) on it is trivial, for example because it preserves ampleness. From this we deduce

\[
E_2^{1,1} = H^1_{\text{et}}(\mathbb{R}, \mathbb{Z}) = \text{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0.
\]

Hence the spectral sequence (3.3) provides in particular an exact sequence

\[
H^1_{\text{et}}(M(r, \xi), G_m) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2_{\text{et}}(M(r, \xi), G_m) \longrightarrow E_2^{0,2}.
\]

Using \( f_*G_m = G_m \) and \( R^2 f_*G_m = \text{Br}(M(r, \xi)_C) \), we thus obtain an exact sequence

\[
\text{Pic}(M(r, \xi)) \xrightarrow{g^1} \text{Pic}(M(r, \xi)_C) \longrightarrow \text{Br}(\mathbb{R}) \xrightarrow{f^*} \text{Br}(M(r, \xi)_C),
\]

where \( g^1 \) and \( g^2 \) are pullback maps along the projection \( g : M(r, \xi)_C \longrightarrow M(r, \xi) \). Note that \( g^2 \) is surjective, since \( g^2(\beta) = \beta_C \) generates \( \text{Br}(M(r, \xi)_C) \) by [BBGN].

Suppose that \( \xi \) is a real line bundle. Then \( g^1 \) is surjective due to Proposition 2.3, so \( f^* \) is injective. Since \( \beta \) has the same exponent as its image \( \beta_C \) by Proposition 3.2, it follows that \( \text{Br}(M(r, \xi)) \) is the direct sum of its subgroups \( \mathbb{Z} \cdot \beta \) and \( f^*(\text{Br}(\mathbb{R})) \), as required.

Now suppose that \( \xi \) is quaternionic and that \( \chi' = \chi/\gcd(r, \chi) \) is even. Then \( f^* \) is injective as before, but the exponent \( \gcd(2r, \chi) \) of \( \beta \) is twice the exponent \( \gcd(r, \chi) \) of its image \( \beta_C \). Hence \( \gcd(r, \chi) \cdot \beta = f^*([H]) \), and the class \( \beta \) generates \( \text{Br}(M(r, \xi)) \).

Finally, suppose that \( \xi \) is quaternionic and that \( \chi' \) is odd. Then the cokernel of \( g^1 \) has two elements according to Proposition 2.3, so \( f^* \) is the zero map, and \( g^2 \) is an isomorphism. In particular, the class \( \beta \) again generates \( \text{Br}(M(r, \xi)) \).  \( \square \)
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