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THE RADON-NIKODYM PROPERTY FOR SOME BANACH ALGEBRAS RELATED TO THE FOURIER ALGEBRA

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Abstract. The Radon-Nikodym property for the Banach algebras $A_p^r(G) = A_p \cap L^r(G)$, where $A_2(G)$ is the Fourier algebra, is investigated. A complete solution is given for amenable groups G if $1 < p < \infty$ and for arbitrary G if $p = 2$ and $A_2(G)$ has a multiplier bounded approximate identity. The results are new even for $G = \mathbb{R}^n$.

1. Introduction

The Fourier algebra of the torus $\mathbb{T}$, namely $A(\mathbb{T})$, is in fact $\ell^1(\mathbb{Z})$ and as such has the Radon-Nikodym property (RNP), a property possessed by any Banach space which is isomorphic to an ℓ^1 space (see the sequel and [DiU]). However, $L^1(\mathbb{R})$, hence $A(\mathbb{R})$, does not possess this property.

A Banach space X has the RNP iff its unit ball wants to be weakly compact, but just cannot make it, as beautifully put orally by Jerry Uhl.

In fact X has the RNP iff any norm bounded closed convex subset Y is the norm closed convex hull of its strongly exposed points.

For dual Banach spaces X the RNP is equivalent to any such Y being the norm closed convex hull of its extreme points, i.e., X having the Krein-Milman property (KMP) (see [DiU], p. 128, for at least 17 conditions which are equivalent to the RNP).

In what follows, G will always denote a locally compact group. The results are new even if $G = \mathbb{R}^n$.

Clearly $A(G)$ has the RNP if G is compact abelian, and yet $A(G)$ does not have the RNP if G is abelian but noncompact. However, if K is any compact subset of the abelian group G, then $A_K(G) = \{u \in A(G);\ \operatorname{spt} u \subset K\}$ does have the RNP (here $\operatorname{spt} u = \operatorname{cl}\{x \in G; u(x) \neq 0\}$ and cl denotes closure). The above results can be proved using the Fourier transform and other tools of abelian harmonic analysis. These are not available anymore if G is not abelian.

For any G, $\forall 1 < p < \infty$, let $A_p(G)$ denote the Figa-Talamanca-Herz Banach algebra as defined by Herz in [Hz1], thus generated by $L^{p'} * L^{p\vee}$; see the sequel. Hence $A_2(G)$ is the Fourier algebra of G (i.e., $A(G)$ as described by Eymard in [Ey1]).

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We have proved in [Gr2] that for any G and any compact $K \subset G$ and any $1 < p < \infty$, $A_K^p = \{u \in A_p(G);\ \operatorname{spt} u \subset K\}$ has the RNP. This seems to be the first paper in which the RNP has been studied for function subalgebras of $A_p(G)$.

It is the purpose of this paper to investigate the RNP for the Figa-Talamanca-Herz-Lebesgue Banach algebras $A_p^r(G) = A_p \cap L^r(G)$, $\forall 1 \le r \le \infty$, equipped with the norm

$$\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}, \quad \text{see [Gr1]}.$$

We give a complete solution to the problem in the case when G is unimodular and either is amenable and $1 < p < \infty$, or $p = 2$ and $A_2(G)$ has a multiplier bounded approximate identity.

For abelian G, the Banach algebras $A_2^r(G) = \left\{f \in L^1(\hat{G}); \hat{f} \in L^r(G)\right\}$ have been around for a long time. Their study started in a beautiful paper by Larsen, Liu and Wang [LLW]. The first paper in which $A_p^1(G)$, for *non-abelian* G was studied is by Lai and Chen [LCh]. It was this paper which gave us the impetus to study in [Gr1] functional analytic properties of the Banach algebras $A_p^r(G)$, for arbitrary G.

It has been proved by W. Braun in an unpublished manuscript [Br] that $A_p^1(G)$ has the RNP if G is amenable, employing the method used in our paper [Gr2].

The main result of this paper is the following:

Theorem 2.2. *Let G be unimodular and either*

(1) $1 < p < \infty$ *and G is amenable or*
(2) $p = 2$ *amd $A_2(G)$ has a multiplier bounded approximate identity.*

Then $\forall 1 \le r \le \max(p, p')$, $A_p^r(G)$ has the RNP.

If G is $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})$, then $A_2^r(G)$, $\forall 2 < r \le \infty$, does not have the RNP. Hence the above is the best one can do.

A crucial step in the proof is the identification of $A_p^r(G)$ $\forall 1 \le r \le \max(p, p')$ as a dual Banach space, namely:

Theorem 2.1. *Let G be unimodular and either* (1) $p = 2$ *or* (2) *G is amenable or discrete, and $1 < p < \infty$. Then $W_p \cap L^r(G) = A_p \cap L^r(G) = A_p^r(G)$ is a dual Banach space, $\forall 1 \le r \le \max(p, p')$.*

The above equality fails even if $p = 2$ and $G = \mathbb{T}$, and $r > 2$.

2. Main results

In what follows, $A_p(G)$ will denote the Figa-Talamanca-Herz Banach algebra as defined in [Hz1]. (This algebra is denoted by $A_{p'}(G)$ in [Ey2] and in [Gr1].)

Hence $u \in A_p(G)$ if and only if $u = \sum u_n * v_n^\vee$, where $u_n \in L^{p'}(G)$, $v_n \in L^p(G)$, $\sum \|u\|_{L^{p'}} \|v_n\|_{L^p} < \infty$, the infimum of these being the norm of $u \in A_p$, and $p^{-1} + (p')^{-1} = 1$, $1 < p < \infty$.

Let $A_p^r(G) = A_p \cap L^r(G)$, with norm given by $\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}$ (see [Gr1] for their properties). Let r' be given by $r^{-1} + (r')^{-1} = 1$ and $r' = \infty$, if $r = 1$.

Denote by $PM_p(G) = A_p(G)^*$, the Banach space dual of $A_p(G)$, and let $PF_p(G)$ denote the *norm closure* of $L^1(G)$ in $PM_p(G)$. Let $W_p(G) = PF_p(G)^*$. $W_p(G)$ is a subspace of $L^\infty(G)$, a Banach algebra under pointwise operations, and an $A_p(G)$ module; see Cowling [Co2], pp. 91 and 94.

Proposition 2.1. *Let G be a locally compact group. Denote $W_p^r(G) = W_p \cap L^r(G)$, with norm given by $\|u\| = \|u\|_{W_p} + \|u\|_{L^r}$. Then $W_p^r(G)$ is the dual of a Banach space.*

Proof. Denote $X = PF_p(G)$, $Y = L^{r'}(G)$, with their appropriate norms, and let $Z = X \times Y$, with norm given by $\|(x,y)\| = \max(\|x\|, \|y\|)$. Then $Z^* = (X \times Y)^* = (X^* \times Y^*) = (W_p \times L^r)$, with norm $\|(u,f)\| = \|u\|_{W_p} + \|f\|_{L^r}$. Now consider the diagonal $N = \{(w,w); w \in W_p^r\}$ as a subspace of Z^*. N is isometric to the Banach space W_p^r. The last will be the dual of a Banach space if N is a w^* closed subspace of Z^*. Then $N = (Z/N_0)^*$, where N_0 is the annihilator of N in Z; see [Ru2], 4.7-4.9.

Hence let $(w_\alpha, w_\alpha) \to (w_0, f_0)$, w^* in Z^*, where $w_\alpha \in W_p^r$, $w_0 \in W_p$, $f_0 \in L^r$. Thus for any $\varphi \in PF_p$, $f \in L^{r'}$, $(w_\alpha, \varphi) + (w_\alpha, f) \to (w_0, \varphi) + (f_0, f)$. Now choose $a \in L^1 \cap L^{r'}$. By first choosing $\varphi = a$, $f = 0$ we get that $(w_\alpha, a) \to (w_0, a) = \int w_0 a \, \mathrm{d}x$. But if we choose $\varphi = 0$, $f = a$, we get that $(w_\alpha, a) \to (f_0, a) = \int f_0 a \, \mathrm{d}x$. It follows that for all $a \in L^1 \cap L^{r'}$, $\int (w_0 - f_0) a \, \mathrm{d}x = 0$. Since f_0 has σ compact support, it follows that $w_0 = f_0$ a.e. Hence, $w_0 \in W_p^r$, and W_p^r is a dual Banach space. □

Let $B_p(G)$ denote the pointwise multipliers of $A_p(G)$, with the multiplier norm. It is proved by Cowling in [Co2], Theorem 5, that $W_p(G) \subset B_p(G)$ and that $W_p(G)$ is isomorphically isometric to $B_p(G)$ if and only if G is amenable.

The next theorem has been proved in the case when G is abelian and $p = 2$ by Liu and van Rooij in [LiR], Corollary 2.6, using tools involving the Fourier transform. It has been proved in [BrFei] that $A_p^1(G)$ is a dual Banach space in the case when G is amenable, as pointed out by Braun in [Br]. A substantial improvement of these results is the next result, for which the abelian methods fail.

Theorem 2.1. *Let G be unimodular and either* (1) $p = 2$ *or* (2) *G is amenable or discrete, and $1 < p < \infty$. Then*

$$W_p \cap L^r(G) = A_p \cap L^r(G) = A_p^r(G), \quad \forall 1 \le r \le \max(p, p').$$

Hence $A_p^r(G)$ is a dual Banach space for all $1 \le r \le \max(p, p')$.

Remark 2.1. This theorem cannot be improved in terms of p and r, even for the torus $\mathbb{T}$ and $p = 2$.

In fact, as noted in [LiR], p. 40, it has been proved by Hewitt and Zuckerman in [HZ], Corollary 4.5, that if G is the torus or any 0-dimensional compact abelian group with dual group Γ, there exists a singular probability measure μ on G, such that $\mu * \mu$ is absolutely continuous and such that $\sum |\hat{\mu}(\gamma)|^{2r} < \infty$, $\forall r > 1$.

Hence, $W_2 \cap L^r(\Gamma) \neq A_2 \cap L^r(\Gamma)$, $\forall r > 2$.

Proof. Let U_α be a relatively compact base of neighborhoods of e and $V_\alpha = V_\alpha^{-1}$, and let there be open neighborhoods of e such that $V_\alpha^2 \subset U_\alpha$. Denote $s_\alpha = \lambda(V_\alpha)^{-1} 1_{V_\alpha}$. Let $t_\alpha = s_\alpha * s_\alpha^{\sim}$, where $f^{\sim}(g) = f(g^{-1})\Delta(g^{-1})$. Let $e_\alpha = t_\alpha * t_\alpha$; thus e_α is the square of a special operator, as in [Fe].

We now apply the lemma on p. 129 of [Fe] (which is an improvement of our lemma 1 on p. 461 of [GrL]) to $w \in W_p \cap L^p(G)$ and get that $\|e_\alpha * w - w\|_{W_p} \to 0$. However $e_\alpha \in L^{p'}$ and thus $e_\alpha * w \in L^{p'} * L^p(G)^\vee \subset A_p(G)$, since G is unimodular.

If $p = 2$, then the unit ball of $PF_2(G)$ is strongly dense in the unit ball of $PM_2(G) = A_2(G)^*$, even if G is not amenable, by Kaplanski's density theorem. If G is amenable and $1 < p < \infty$, the same holds true by Theorem 5 of Herz [Hz1]. It follows that the W_p norm restricted to $A_p(G)$ coincides with the A_p norm in both cases.

Now $e_\alpha * w \in A_p(G)$ is a W_p, hence an A_p norm Cauchy net. It follows that $w \in A_p(G)$. Hence $W_p \cap L^p(G) = A_p \cap L^p(G)$ in both cases. Now W_p contains only bounded functions; thus $W_p \cap L^r \subset W_p \cap L^p$, $\forall 1 \le r \le p$. Hence, for such r, $W_p \cap L^r \subset A_p \cap L^p \cap L^r = A_p \cap L^r = A_p^r$.

Now $W_p(G)^\vee = W_{p'}(G)$ and $A_p(G)^\vee = A_{p'}(G)$; see [Co2], p. 91. Since G is unimodular it is clear that

$$(W_p \cap L^r)^\vee = W_p^\vee \cap L^r = W_{p'} \cap L^r = A_{p'}^r, \quad \forall 1 \le r \le p'.$$

If $p \le 2$, then $W_p \cap L^r = (W_{p'} \cap L^r)^\vee = (A_{p'}^r)^\vee = A_p^r$, $\forall 1 \le r \le p'$.

If G is discrete, one can omit the amenability of G since then $L^{p'} \cup L^p = L^{p'} * \delta_e \cup \delta_e * L^p \subset A_p$. Let $s = \max(p, p')$. If $1 \le r \le s$, then $W_p \cap L^r \subset W_p \cap L^s \subset A_p$. Hence $W_p \cap L^r \subset A_p^r$. □

We will prove in what follows that $A_p^r(G)$, for $1 \le r \le \max(p, p')$, has, as a Banach space, the Radon-Nikodym property (RNP) for certain G. This fact has been proved for $A_p^1(G)$ if G is amenable by W. Braun in an unpublished preprint [Br] using our method in [Gr2], where we show that $A_K^p(G) = \{u \in A_p;\ \operatorname{spt} u \subset K\}$, for compact K, has the RNP.

Theorem 2.2. *Let G be unimodular and either*

(1) $1 < p < \infty$ *and G is amenable or*
(2) $p = 2$ *and $A_2(G)$ have a multiplier bounded approximate identity.*

Then $\forall 1 \le r \le \max(p, p')$, $A_p^r(G)$ has the RNP.

Remark 2.2. If G is $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})$, then $A_2^r(G)$, $\forall 2 < r \le \infty$, does not have the RNP. Yet $A_2(G)$ has a multiplier bounded approximate identity (see [CaHa], Remark 3.8) and G is unimodular (see [HRI] (15.30)). Hence the above is the best one can do.

Proof. Assume at first that G is separable metric. Let $K \subset G$ be compact and let $A_p^r[K] = \{u \in A_p^r(G);\ \operatorname{spt} u \subset K\}$, $A_p[K] = \{u \in A_p(G);\ \operatorname{spt} u \subset K\}$. Then the identity $I : A_p^r[K] \to A_p[K]$ is 1-1, onto and continuous; hence it is bicontinuous. $A_p(G)$ is norm separable; hence so are $A_p^r[K]$, $A_p[K]$. Now let K_n be compact and satisfy $K_n \subset \operatorname{int} K_{n+1}$, $\bigcup K_n = G$. Let $C_m = \{v_n^m;\ n \ge 1\}$ be norm dense in $A_p^r[K_m]$; then $\bigcup C_m$ is norm dense in $A_p^r(G)$. Since if G is amenable, then $A_p(G)$ has a bounded approximate identity, in both cases (1), (2), $A_p(G)$ has a multiplier bounded approximate identity. By [Gr1], p. 408, $A_p^r \cap C_c(G)$ is norm dense in $A_p^r(G)$. Hence let $v \in A_p^r(G)$ and $\varepsilon > 0$. Let $u \in A_p^r \cap C_c(G)$ be such that $\|v - u\|_{A_p^r} < \varepsilon$. Then $\operatorname{spt} u \subset K_i$ for some $i > 0$. Let v_j^i satisfy $\|u - v_j^i\|_{A_p^r} < \varepsilon$. Then $\|v - v_j^i\|_{A_p^r} < 2\varepsilon$.

Now, separable dual Banach spaces have the RNP (see [DiU], p. 218); hence $A_p^r(G)$ has the RNP for $1 \le r \le \max(p, p')$.

Assume now that G is arbitrary. By [DiU], p. 217, we need only prove that every separable closed subspace X of $A_p^r(G)$ has the RNP. Hence, let $\{u_k\}$ be norm dense in X. Each u_k has σ-compact support; hence there exists a σ-compact subgroup H such that $u_k = 0$ off H for all k. By [Hz1], p. 106, $A_p(H)$ can be identified isometrically with the subalgebra $\{u \in A_p(G);\, u = 0 \text{ off } H\}$. It is readily seen now that the same is the case for the algebras $A_p^r(H)$, $A_p^r(G)$.

Denote $1_H(x) = 1$, (0) if $x \in H$, $(x \notin H)$. Then by [Hz1], p. 106, the map $u \to 1_H u$, from $A_p(G) \to A_p(H)$, is a retract; thus $\|1_H u\|_{A_p(H)} \le \|u\|_{A_p(G)}$. It readily follows that if $\{u_\alpha\}$ is a multiplier bounded approximate identity for $A_p(G)$, then $\{1_H u_\alpha\}$ is such for $A_p(H)$. Since G is unimodular, so is H.

Thus we can and shall hence assume that G is σ-compact. We will prove that the above separable subspace X has the RNP.

Let $V_n = \big\{x; \|l_x u_k - u_k\|_{A_p^r} + \|r_x u_k - u_k\|_{A_p^r} < 1/n, \forall k \le n\big\}$, where $l_x u(y) = u(xy)$, $r_x u(y) = u(yx)$. These are open neighborhoods of the identity e of G. Hence there exists a compact normal subgroup N such that $N \subset \bigcap\{V_n;\, 1 \le n < \infty\}$ and such that G/N is separable metric; see Lemma 2.1. Thus, by [Hz1], p. 106, Prop. 6, $A_p(G/N)$ can be identified isometrically with $\{u \in A_p(G); l_x u = r_x u = u, \forall x \in N\}$. It follows readily that this is also the case for the algebra $A_p^r(G/N)$ by noting Weil's formula in [RS], Theorem 3.4.6. Hence the Banach space X is a subspace of the Banach algebra $A_p^r(G/N)$. Since subspaces of RNP spaces have the RNP (see [DiU], p. 217), it is enough to prove that $A_p^r(G/N)$ has the RNP.

That G/N is unimodular can be directly proved; see [Ga], Theorem V.3.15 and the remarks on p. 267. We will prove that $A_p(G/N)$ has a multiplier bounded approximate identity and then use the first part above to finish the proof.

By [Hz1], Prop. 6 on p. 106, $A_p(G/N)$ can and shall be identified with the subalgebra of $A_p(G)$, given by $B = \{u \in A_p(G);\ \ell_x u = r_x u,\ \forall x \in N\}$, where $\ell_x u(y) = u(xy)$, $r_x u = u(yx)$. The onto map $M_N u(x) = \int u(xy)\,\mathrm{d}\lambda_N(y)$, where λ_N is the Haar measure of N, from $A_p(G) \to A_p(G/N)$ is a Banach space retract; i.e., $\|M_N u\|_{A_p(G/N)} \le \|u\|_{A_p(G)}$, $\forall u \in A_p(G)$. Moreover M_N is the identity on $A_p(G/N)$. It can be directly shown that $M_N(uv) = uM_N(v)$, $\forall u \in A_p(G/N)$, $v \in A_p(G)$.

Now let $\{u_\alpha\}$ be a multiplier bounded approximate identity in $A_p(G)$. We claim that $\{M_N u_\alpha\}$ is such in $A_p(G/N)$, since, if $v \in A_p(G/N)$, then

$$\|(M_N u_\alpha)v - v\|_{A_p(G/N)} = \|M_N(u_\alpha v) - M_N v\|_{A_p(G/N)} \le \|u_\alpha v - v\|_{A_p(G)} \to 0.$$

Also,

$$\|(M_N u_\alpha)v\|_{A_p(G/N)} = \|M_N(u_\alpha v)\|_{A_p(G/N)} \le \|u_\alpha v\|_{A_p(G)} \le K\|v\|_{A_p(G)},$$

for some $K > 0$, since $\{u_\alpha\}$ is a multiplier bounded approximate identity in $A_p(G)$. □

To complete the above proof we still need to prove the following lemma, which is an improvement of Theorem 8.7 in [HRI], where it is assumed that G is compactly generated. The proof is inspired by Prop. 6 in [Hz1]. Lemma 2.1 gives, in addition, a new proof for [HRI] (8.7).

Lemma 2.1. *Let G be a σ-compact locally compact group and let U_n, $n = 1, 2, \ldots$, be neighborhoods of the identity. Then there exists a compact normal subgroup $N \subset \bigcap\{U_n; 1 \le n < \infty\}$, such that G/N is separable metric.*

Proof. Let $V_1 = V_1^{-1}$ be a relatively compact neighborhood of e such that $V_1^4 \subset U_1$. If $V_1, \dots, V_{n-1}$ were chosen, choose the neighborhood $V_n = V_n^{-1}$ of e such that $V_n^4 \subset \{\bigcap U_k,\ 1 \le k \le n\} \cap V_{n-1}$. Let $f_n \in C_c(G)$ be such that $0 \le f_n \le 1$, $f_n = 1$, on V_n and $f_n = 0$ off V_n^2. Denote $v_n(x) = \|\lambda(x)f_n - f_n\|_\infty$, where $\lambda(x)f(y) = f(x^{-1}y)$. Then, in the language of [Hz1], p. 107, v_n is a continuous invariant pseudometric on G; i.e., it satisfies $0 \le v_n < \infty$, $v_n(e) = 0$, $v_n(x) = v_n(x^{-1})$ and $v_n(xy) \le v_n(x) + v_n(y)$, for all x, y.

We note that $\lambda(x)f_n$ and f_n have disjoint supports if $x \notin V_n^4$; thus $v_n(x) = \|f_n\|$ for such x. Hence the set $\{x; v_n(x) = 0\}$ is a compact, not necessarily normal, subgroup M_n of G. Now let $\omega(x) = \sum_n 2^{-n}[v_n(x)\big(1 + v_n(x)\big)^{-1}]$. Then $\omega(x)$ is again a continuous invariant pseudometric on G (since $g(t) = t/(1+t)$ is increasing on $[0, \infty)$), and $\omega(x) = 0$, iff $v_n(x) = 0$, $\forall n$, iff $x \in M_n \subset V_n^4 \subset U_n$, $\forall n$. Hence $\omega^{-1}(0) = M \subset \bigcap U_n$, where $M = \bigcap M_n$, is a compact, not necessarily normal, subgroup of G.

Now let $G = \bigcup K_n$, where K_n are compact sets such that $e \in K_n \subset K_{n+1}$. For each n let $\omega_n(x) = \sup\big\{\omega(yxy^{-1}); y \in K_n\big\}$, again a continuous invariant pseudometric on G. Let $\tau = \sum 2^{-n}\omega_n(1+\omega_n)^{-n}$. Then τ is a continuous invariant pseudometric such that $\tau^{-1}(0) = N$ is a compact normal subgroup such that $N \subset M \subset \bigcap U_n$ and $d(x, y) = \tau(x^{-1}y)$ is a continuous (even left invariant) pseudometric (as in [HRI]) on G which induces a metric on G/N and which renders G/N with the quotient topology into a separable metric space.

Clearly $\tau(x) \ge \omega_1(x)/4$, $\forall x \in G$, since $0 \le \omega(x) \le 1$. If $\tau(x) = 0$, then $\omega(x) = 0$, since $e \in K_1$; hence $N \subset M$. In addition, if $a \in G$, then $a \in K_k$ for some k and $0 = \omega_k(x) \ge \omega(axa^{-1})$. Thus $\omega(yxy^{-1}) = 0$, $\forall y \in G$. But then $\omega_n(axa^{-1}) = \sup\big\{\omega(yaxa^{-1}y^{-1}); y \in K_n\big\} = 0$, for all n. Hence $\tau(axa^{-1}) = 0$ and $N = \tau^{-1}(0)$ is a compact normal subgroup.

Now let $d(x, y) = \tau(x^{-1}y)$ for x, y in G. Then d is a continuous left invariant pseudo-metric, as in [HRI], (8.1); i.e., $d(x, x) = 0$, $d(x, y) = d(y, x)$, and

$$0 \le d(x, y) \le d(x, z) + d(z, y), d(x, y) = d(ax, ay) \quad \forall x, y, z.$$

We note that $\tau(xux^{-1}) = 0$, $\forall u \in N$, $x \in G$; thus $d(x, ux) = 0$, for such u, x. Hence $d(ux, vy) \le d(x, y) + d(y, u^{-1}vy) = d(x, y)$, $\forall u, v \in N, x, y \in G$. But then $d(x, y) = d\big(u^{-1}ux, v^{-1}vy\big) \le d(ux, vy)$, for such x, y, u, v. Thus $D(xN, yN) = d(x, y)$ is a left invariant continuous metric on G/N, which clearly induces the quotient topology on G/N, since it does so on each compact subset. It is readily seen that G/N is second countable; thus it is separable metric. □

Proof of Remark 2.2. If $G = \mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})$, then $A_2^r(G) = A_2(G)$, $\forall r > 2$, by Kunze and Stein [KuS] and by Lipsman [Li], respectively; see also [Co1].

We will prove hence that $A_2(G)$ does not have the RNP if G is any of the above groups. By Keith Taylor's beautiful result in [Tay], Theorem 4.1, we only need to prove that the regular representation is not a direct sum of irreducible representations for the above groups G.

Assume the contrary. Then the Plancherel measure μ, which is unique once the Haar measure has been fixed by [Dix], (18.8.2), is necessarily an atomic measure; see [Dix] (18.8.5). However, this is not the case for the above G, as can be seen from Knapp [Kn], II.7, formulas (2.24) and (2.25) on p. 42. In fact one can get from there, by taking $h = f * f^*$, where f is a compactly supported smooth function on

G, that $\|f\|_{L^2}^2 = \int\limits_{G^\wedge} Tr\pi(f)\pi(f) * d\mu(\pi)$ where the measure μ in [Kn], II.7, (2.24) and (2.25) is not atomic. There are some atoms in (2.25), i.e., for $\mathrm{SL}(2,\mathbb{R})$, which are the "discrete series", but the atoms do not make up everything in the support of μ. □

Remark 2.3. The group $\mathrm{SL}(2,\mathbb{R}) \propto \mathbb{R}^n$, $\forall\ n \geq 2$, does not satisfy condition (2) of Theorem 2.2; see [Do]. Yet there are a multitude of non-amenable groups which do satisfy this condition; see [CaHa].

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