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ISOPERIMETRIC SURFACES WITH BOUNDARY

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Abstract. We prove that many common combinations of soap films and soap bubbles that result from dipping polyhedral wire frames in a soap solution are minimizing with respect to their boundary and bubble volume. This can be thought of as a combination of the Plateau problem of least area for surfaces spanning a given boundary and the isoperimetric problem of least area for surfaces enclosing a given volume. Proof is given in arbitrary dimension using a combination of the mapping of Gromov, after Knothe, and the paired calibrations of Lawlor and Morgan.

In this paper, we prove that many common soap-film-like surfaces that enclose a volume and have a regular polytope "wireframe" as boundary are mass minimizing with respect to their boundary and enclosed volume. Examples of such surfaces are shown in Figure 1. We give a direct proof using the flux of specialized vector fields to model the mass of each competitor a la Gromov [5] and Lawlor and Morgan [8].

We propose the name of "equitent problem" for an area minimization question in which competitors must enclose a given volume ("equal content") as well as span a given boundary ("equal extent"). Equitent problems are thus a generalization of Plateau problems (fixed boundary) and isoperimetric problems (fixed enclosed volume). The equitent problem appears largely unexplored to date; we give what we believe to be the first major results.

1. Knothe-Gromov

In 1989 Mikhail Gromov [5], following work of Herbert Knothe [7], described a beautiful method for proving isoperimetric theorems. He used what is now called the Knothe-Rosenblatt rearrangement, developed independently by Knothe [7] and Rosenblatt [10], which we now describe. Construct an area-preserving map F from a competitor U to the round ball B of radius r (centered at the origin) of the same volume by sending 'slices' of U to 'slices' of B, and repeating the same process in each slice. Specifically, let this map be given by

$$F : U \to B,$$
$$(y_1, y_2, \dots, y_n) \mapsto (z_1, z_2, \dots, z_n),$$

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such that

$$\frac{\mathcal{H}^{n-j+1}U \cap \{x_1 = y_1, \ldots, x_{j-1} = y_{j-1}, x_j \leq y_j\}}{\mathcal{H}^{n-j+1}U \cap \{x_1 = y_1, \ldots, x_{j-1} = y_{j-1}\}}$$
$$= \frac{\mathcal{H}^{n-j+1}B \cap \{x_1 = z_1, \ldots, x_{j-1} = z_{j-1}, x_j \leq z_j\}}{\mathcal{H}^{n-j+1}B \cap \{x_1 = z_1, \ldots, x_{j-1} = z_{j-1}\}}$$

for all $1 \leq j \leq n$. Thus F_j is seen to depend only on the variables $\{x_1, x_2, \ldots, x_j\}$. Scaled volume is preserved set-wise under each 'slice,' with the last coordinate mapped (locally) linearly on the final 'slice' of a portion of a line in U mapped to a line segment in B. Volume is then preserved pointwise. In addition, if $U = B$, then F is simply the identity on B.

This scheme makes DF lower triangular with positive diagonal entries. Since $\prod_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \det DF = 1$, the arithmetic-geometric mean inequality implies that $\operatorname{tr} DF \geq n$. Thinking of F as a vector field on U, $\operatorname{tr} DF = \operatorname{div} F \geq n$. Now since the image of F is a ball of radius r, the length of the vector $\frac{1}{r}F(\mathbf{x})$ is always at most 1. The divergence theorem then gives us that

$$\mathcal{H}^{n-1}\partial U = \int_{\partial U} 1 \geq \int_{\partial U} \frac{1}{r} F \cdot \mathbf{n} = \int_U \frac{1}{r} \operatorname{div} F \geq \frac{n}{r} \cdot \mathcal{H}^n U,$$

with equality for the ball. The standard isoperimetric inequality in $\mathbb{R}^n$ results.

If we replace B with any convex set, this scheme still works to create a volume preserving map F with divergence at least n. The details of this construction were worked out by Brothers and Morgan in [2]. They also provide a version of the divergence theorem with weakened hypotheses and show that F satisfies these.

2. Paired calibration

In 1993, Lawlor and Morgan [8] gave a simple proof that in every dimension, the cone over the $(n-2)$-skeleton of the regular simplex is area-minimizing. In $\mathbb{R}^2$ this is the length-minimizing Y-shaped figure of three edges meeting at 120-degree angles, and in $\mathbb{R}^3$ it is the union of isosceles triangles from each edge of a regular tetrahedron into its center of mass. This result had already been proved by Jean Taylor in 1975 [11] for $\mathbb{R}^3$. However, the result by Lawlor and Morgan was new in higher dimensions.

A simple description of their "paired calibration" technique is to line up (in $\mathbb{R}^3$, say) four heat lamps shining directly toward the faces of a regular tetrahedron. Consider that the heat rays from a lamp are parallel with constant intensity, and are completely absorbed when they hit a surface. Now place the proposed minimizing cone in this configuration for a set time, and measure the temperature at each point of the cone. Replace the cone with any competing surface for the same amount of time, and measure its temperature. Because each piece of the cone has the ideal surface angle at which to absorb heat from the two lamps that shine on that piece (analogous to facing a single lamp at 90 degrees), we know that the cone is at least as hot, pointwise, as the competing surface. On the other hand, because both competitors span the same boundary, the total heat absorbed is the same for both. Equal heat absorbed and hotter temperature pointwise means the cone must have less surface area over which to distribute the heat. This approach is made rigorous by representing heat absorption by the flux integrals of a set of constant vector fields.

We will combine the Knothe-Gromov vector field approach with paired calibration.

3. The surfaces

A uniform polytope in $\mathbb{R}^n$ (or uniform polyhedron in $\mathbb{R}^3$) is a polytope with symmetric vertices made of uniform polytope facets of one dimension down. The uniform polytopes in two dimensions are the regular polygons. In particular the edges of a uniform polytope are all of equal length, and the vertices of a uniform polytope all have equal distance from its center of mass. Examples of uniform polyhedra in $\mathbb{R}^3$ include the Platonic and Archimedean solids.

Definition 3.1 (A family of soap-film-like surfaces with enclosed volume)**.** Let Γ be a convex uniform polytope of dimension $m \leq n$ of unit edge length embedded in $\mathbb{R}^n$ and centered at the origin. Let $\mathbf{p}_1, \ldots, \mathbf{p}_k$ be the vertices of Γ. Then $\sum_i \mathbf{p}_i = \mathbf{0}$ and $\|\mathbf{p}_i - \mathbf{p}_j\| = 1$ for all vertices $\mathbf{p}_i$ and $\mathbf{p}_j$ that are adjacent in Γ.

For each $\mathbf{p}_i$ let C_i be the set of points of $\mathbb{R}^n$ lying strictly closer to $\mathbf{p}_i$ than to any other $\mathbf{p}_j$, and define K to be the complement of $\bigcup C_i$ (if $k = 1$, let $K = \emptyset$). We note then that C_i and C_j share boundary non-trivially (on a set of non-zero $\mathcal{H}^{n-1}$ measure) if and only if $\mathbf{p}_i$ and $\mathbf{p}_j$ are adjacent vertices in Γ.

Next, choose $r \geq 0$ and define B as the intersection of the open balls of radius r centered at $-r\,\mathbf{p}_i$. Let S be the boundary of B and let β be the volume of B.

Finally, define

$$M = M(\Gamma, r) = (K \setminus \overline{B}) \cup S.$$

Figure 1 contains examples of M for various polytopes Γ. We note that in our construction the faces of B correspond to the vertices of Γ and the vertices of B correspond to the faces of Γ. Thus when this construction is applied to a uniform polytope Γ, the bubble B will be homeomorphic to the dual polytope of Γ. This is why the bubble resulting from Γ = an icosahedron in Figure 1 below looks like a dodecagon.

Remark 3.2. As $\|\mathbf{p}_i\| = \|\mathbf{p}_j\|$ for all i, j, we see that $B = \emptyset$ if and only if $\|\mathbf{p}_i\| < 1$ or $r = 0$. If this is the case, $M = K$ and any volume constraint is vacuous. The proof below will apply to either case. Convex regular polytopes for which $\|\mathbf{p}_i\| < 1$ and for which this construction may yield a non-empty bubble B include: ($m = 2$) k-gons with $k < 6$, ($m = 3$) all platonic solids except the regular dodecahedron, and ($m \geq 4$) the m-simplex and the m-orthoplex. Minimizers over convex uniform polytopes such that $\|\mathbf{p}_i\| \geq 1$ also exist, but need not be convex or have spherical caps as faces.

4. The minimization theorem

Theorem 4.1. *Given Γ and $r \geq 0$, the surface $M = M(\Gamma, r)$ is area-minimizing in the following sense.*

Let L_0 be the closure of a bounded open set in $\mathbb{R}^n$ such that $M_0 = M \cap L_0$ contains all of S. Let T be any compact surface (rectifiable set) such that

$$(M \setminus M_0) \cup T$$

divides $\mathbb{R}^n$ into k unbounded components T_i such that $C_i \setminus L_0 \subset T_i$ and T_i and T_j share boundary non-trivially only if $\mathbf{p}_i$ is adjacent to $\mathbf{p}_j$ in Γ. Let U be the union

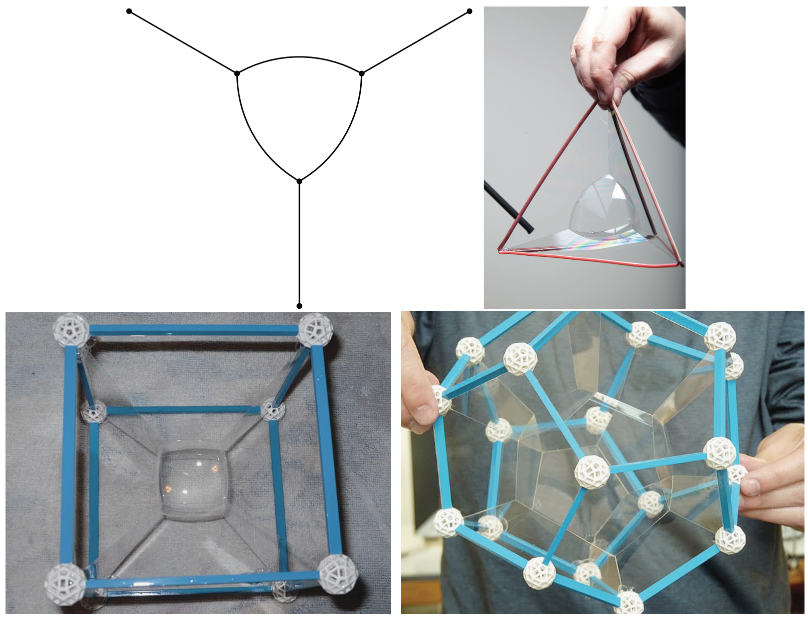

FIGURE 1. Examples of M for $\Gamma =$ an equilateral triangle, a regular tetrahedron, a regular octahedron, a regular icosahedron

of the compact components, and suppose that the volume of U is β, the same as the volume of B. Then

$$\mathcal{H}^{n-1}M_0 \leq \mathcal{H}^{n-1}T.$$

Proof. Take an arbitrary L_0 and define M_0 as in the statement of the theorem. Choose a competing surface T.

Label the unbounded components of $[(M \setminus M_0) \cup T]^C$ as $T_1, \ldots, T_k$ and label the union of the compact components as U, as in the statement of the theorem. For each i let

$$X_i = \partial U \cap \overline{T_i},$$

and for each $i \neq j$ let

$$W_{ij} = (\partial T_i) \cap (\partial T_j) \cap L_0.$$

Note then that $\mathcal{H}^{n-1}W_{ij} \neq 0$ only if $\mathbf{p}_i$ and $\mathbf{p}_j$ are adjacent in Γ. This labeling is illustrated in an example in Figure 2.

For each point $\mathbf{p}_i$ define a constant vector field $\mathbf{v}_i(\mathbf{x}) = \mathbf{p}_i$.

Use the Knothe-Gromov approach to define a volume-preserving map F from U to the convex set B, with $\operatorname{div} F \geq n$ wherever defined. Think of the map $\frac{1}{r}F$ as a vector field on U, so that

$$\operatorname{div} \frac{1}{r}F \geq \frac{n}{r}.$$

Notice now that every point $\mathbf{y} \in B$ is within distance r of $-r\mathbf{p}_i$ for all $1 \leq i \leq k$, which implies that $\left\| \frac{1}{r}\mathbf{y} + \mathbf{p}_i \right\| \leq 1$. So we see that the sum of vectors $\frac{1}{r}F + \mathbf{v}_i$ has length at most 1 anywhere. Recall that the difference vectors $\mathbf{v}_i - \mathbf{v}_j = \mathbf{p}_i - \mathbf{p}_j$ are also of length 1 on W_{ij} if W_{ij} has non-zero measure. From this point on, we ignore all W_{ij} of measure zero. Let us establish the convention that the (unit) normal

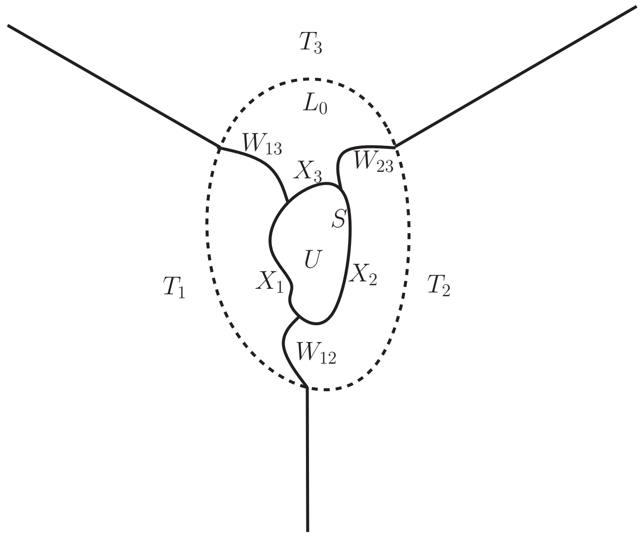

FIGURE 2. Illustration of labeling ($n = 2$, $k = 3$)

$\mathbf{n}_0$ to ∂U is outward pointing, and the normal $\mathbf{n}_{ij}$ to W_{ij} points toward the region with lower index. We may now compute:

$$\mathcal{H}^{n-1}T = \int_{\partial U} 1 + \sum_{i<j} \int_{W_{ij}} 1 \tag{1}$$

$$\geq \sum_i \int_{X_i} \left(\frac{1}{r}F + \mathbf{v}_i\right) \cdot \mathbf{n}_0 + \sum_{i<j} \int_{W_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{n}_{ij}. \tag{2}$$

We wish to rewrite (2) by considering the flux into the region T_i due to $\mathbf{v}_i$. This is the flux through $\partial T_i \cap T = (\bigcup_i W_{ij}) \cup X_i$, with normal $\mathbf{n}_i$ pointing into T_i. From the first sum we have the contribution of $\mathbf{v}_i$ through X_i. For each j we also have flux through W_{ij} from the second sum. Notice that if $i < j$, then $\mathbf{n}_{ij} = \mathbf{n}_i$, while if $i > j$, then $\mathbf{n}_{ij} = -\mathbf{n}_i$. Thus we may rewrite (2) as

$$\int_{\partial U} \frac{1}{r}F \cdot \mathbf{n}_0 + \sum_i \int_{\partial T_i \cap T} \mathbf{v}_i \cdot \mathbf{n}_i \tag{3}$$

$$= \int_U \operatorname{div} \frac{1}{r}F + \sum_i \int_{\partial T_i \cap T} \mathbf{v}_i \cdot \mathbf{n}_i \tag{4}$$

$$\geq \frac{n}{r}\text{Volume}(U) + \sum_i \int_{\partial T_i \cap T} \mathbf{v}_i \cdot \mathbf{n}_i. \tag{5}$$

Notice that any term in the sum in (5) is the flux of a constant vector field through a surface with fixed boundary. We see then that the quantity (5) is independent of T (with L_0 held constant).

We claim that equality holds throughout when we apply this estimate to the conjectured minimal surface. When $T = M_0$, we have $F(\mathbf{x}) = \mathbf{x}$. The inward pointing unit normal to X_i at $\mathbf{x}$ is given by $\frac{1}{r}\mathbf{x} + \mathbf{p}_i = \frac{1}{r}F(\mathbf{x}) + \mathbf{v}_i$. Also, notice that W_{ij} consists of points equidistant from $\mathbf{p}_i$ and $\mathbf{p}_j$, so if $i < j$, the normal to W_{ij} is given by $\mathbf{p}_i - \mathbf{p}_j$. This shows that we have equality from (1) to (2). Also notice that if F is the identity function, then $\operatorname{div} F = n$, giving us equality in the last step (5).

Letting $G(T) = \sum_i \int_{X_i} \left(\frac{1}{r}F + \mathbf{v}_i\right) \cdot \mathbf{n}_0 + \sum_{i<j} \int_{W_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{n}_{ij}$ from (2) above, we see that this argument reduces to a proof that

$$\mathcal{H}^{n-1}T \geq G(T) \geq G(M_0) = \mathcal{H}^{n-1}M_0. \qquad \square$$

We have shown that M_0 is mass minimizing among all figures that contain the same volume, have the same boundary, and have the same connectivity (regions T_i and T_j for some competitor T share boundary non-trivially only if they do in M_0). If Γ is a regular simplex then, as Γ is also a complete graph, T_i shares non-trivial boundary in M_0 with all other T_j. Thus we have the following corollary:

Corollary 4.2. *If Γ is a regular simplex, then M_0 is minimizing as above but without the connectivity condition.*

It is an open problem however as to which figures $M(\Gamma, r)$ other than those generated from a regular simplex are minimizing in this more general setting.

Corollary 4.3. *If Γ' is the dual polytope to some convex uniform polytope Γ, then the cone over Γ' is minimizing among all other surfaces that share the same boundary and have the same connectivity.*

Proof. Apply Theorem 4.1 to $M(\Gamma, 0)$. $\square$

The above corollary is a stronger version of the main result of [3], which showed that stationary cones are minimizing over diffeomorphisms. Choe's results however apply to the more general class of polyhedral sets, and not just cones.

5. Metacalibration

The proof of Theorem 4.1 is an example of a method which we call *metacalibration*. Calibration arguments [6] typically have the form

$$\mathcal{H}^n M = \int_M \varphi = \int_{M'} \varphi \leq \mathcal{H}^n M'$$

for some particular differential form φ, showing that M is mass minimizing among all competitors M'. This method has been useful in solving fixed boundary problems, such as Plateau's problem (see [9], cf. Chapter 6). Metacalibration generalizes the traditional calibration argument by allowing φ to vary for each competitor. This allows metacalibration to tackle problems with fixed volume constraints. As demonstrated in this paper, metacalibration can also handle combinations of fixed boundary and fixed volume constraints.

Using a variation of the Knothe-Rosenblatt rearrangement in a metacalibration argument, the authors were able to prove the double bubble conjecture in the plane [4]. Optimal transport promises to be another tool for constructing metacalibrations. For instance, the Brenier map [1] has many of the same properties as the Knothe-Rosenblatt rearrangement, including the divergence criterion used in this paper. We hope that a suitable generalization of the Brenier map can be used to solve other isoperimetric problems such as the triple bubble conjecture.

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