PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 1, January 2012, Pages 309–323
S 0002-9939(2011)10896-7
Article electronically published on May 16, 2011

ON THE COMPLEXITY OF THE RELATIONS OF ISOMORPHISM AND BI-EMBEDDABILITY

LUCA MOTTO ROS

(Communicated by Julia Knight)

Abstract. Given an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$, that is, the collection of the countable models of some $\mathcal{L}_{\omega_1\omega}$-sentence, denote by $\cong_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ the analytic equivalence relations of, respectively, isomorphisms and bi-embeddability on $\mathcal{C}$. Generalizing some questions of A. Louveau and C. Rosendal, in a paper by S. Friedman and L. Motto Ros they proposed the problem of determining which pairs of analytic equivalence relations (E, F) can be realized (up to Borel bireducibility) as pairs of the form $(\cong_{\mathcal{C}}, \equiv_{\mathcal{C}})$, $\mathcal{C}$ some $\mathcal{L}_{\omega_1\omega}$-elementary class (together with a partial answer for some specific cases). Here we will provide an almost complete solution to such a problem: under very mild conditions on E and F, it is always possible to find such an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$.

1. Introduction

An equivalence relation E defined on a Polish space (or, more generally, on a standard Borel space) X is said to be *analytic* if it is an analytic subset of $X \times X$. Analytic equivalence relations arise very often in various areas of mathematics and are usually connected with important classification problems; see e.g. the preface of [Hjo00] for a brief but informative introduction to this subject. The most popular way to measure the relative complexity of two analytic equivalence relations E and F is given by the notions of *Borel reducibility* and *Borel bireducibility* (in symbols $\leq_B$ and $\sim_B$, respectively): $E \leq_B F$ if there is a Borel function f between the corresponding domains which reduces E to F, that is, such that $x\,E\,y \iff f(x)\,F\,f(y)$ for every x, y in the domain of E, and $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$. (We will denote by $<_B$ the strict part of $\leq_B$.) Intuitively, $E \leq_B F$ means that E is not more complicated than F, so $E \sim_B F$ means that E and F have the same complexity.

Similar definitions and terminology will also be applied to *analytic quasi-orders* (i.e. reflexive and transitive relations R on a standard Borel space X which are analytic subsets of $X \times X$), and when dealing with an analytic quasi-order R we will also often consider the analytic equivalence relation $E_R = R \cap R^{-1}$ canonically induced by R.

Received by the editors April 15, 2010 and, in revised form, November 10, 2010.
2010 *Mathematics Subject Classification.* Primary 03E15.
Key words and phrases. Analytic equivalence relation, isomorphism, (bi-)embeddability, Borel reducibility.
The author would like to thank the FWF (Austrian Research Fund) for generously supporting this research through project number P 19898-N18.

A nice example of an analytic equivalence relation is the following: consider an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$, that is, the collection of all countable models of some sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$, with $\mathcal{L}$ some countable language. Assuming that all these models have domain ω (the set of natural numbers), we can canonically identify each of them with an element of the Polish space of $\mathcal{L}$-structures $\mathrm{Mod}(\mathcal{L})$ (which is homeomorphic to the Cantor space), and by a well-known theorem of Lopez-Escobar (see e.g. [Kec95, Theorem 16.8]), $\mathcal{C} \subseteq \mathrm{Mod}(\mathcal{L})$ is an $\mathcal{L}_{\omega_1\omega}$-elementary class if and only if $\mathcal{C}$ is Borel and invariant under isomorphism. This easily implies that the relation of isomorphism $\cong_{\mathcal{C}}$ between elements of $\mathcal{C}$ becomes an analytic equivalence relation (relations of this form will be simply called *isomorphism relations*).

If in the previous definition we replace isomorphisms with (logical) embeddings between elements of $\mathcal{C}$, we get the analytic quasi-order $\sqsubseteq_{\mathcal{C}}$ of embeddability on $\mathcal{C}$, which in turn canonically induces the analytic equivalence relation $\equiv_{\mathcal{C}}$ of bi-embeddability between elements of $\mathcal{C}$. The possible relationships between $\cong_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ were first investigated in [FMR09], where the authors constructed various $\mathcal{L}_{\omega_1\omega}$-elementary classes $\mathcal{C}$ satisfying certain conditions on $\cong_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ to answer some questions posed by Louveau and Rosendal in [LR05]. In particular, in [FMR09] it is shown that given an arbitrary analytic equivalence relation F there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\cong_{\mathcal{C}} \sim_B \mathrm{id}(\mathbb{R})$, where $\mathrm{id}(\mathbb{R})$ denotes the identity relation on $\mathbb{R}$, and $\equiv_{\mathcal{C}} \sim_B F$. After those examples, the following problem was formulated:

Problem. Consider an arbitrary pair of analytic equivalence relations (E, F). Is it possible to find an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $E \sim_B \cong_{\mathcal{C}}$ and $F \sim_B \equiv_{\mathcal{C}}$?

Similarly, one can consider the analogous question regarding a pair (E, R) consisting of an analytic equivalence relation and an analytic quasi-order. Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $E \sim_B \cong_{\mathcal{C}}$ and $R \sim_B \sqsubseteq_{\mathcal{C}}$?

For ease of exposition, if such a $\mathcal{C}$ exists we will say that $\mathcal{C}$ *represents* the pairs (E, F) or (E, R), respectively. The problem of giving a complete and general characterization of those (E, F) and (E, R) which can be represented by an $\mathcal{L}_{\omega_1\omega}$-elementary class was considered in [FMR09] to be a potentially difficult problem. First we must notice that there are some obvious limitations to the possibility of having such a representation. For example, since there are many analytic equivalence relations which are not even Borel reducible to an isomorphism relation, we should at least ask that E be a *quasi-isomorphism relation*, i.e. that E be Borel bireducible with some isomorphism relation on some $\mathcal{L}_{\omega_1\omega}$-elementary class (by [Hod93, Theorem 5.5.1], such a class can be assumed to always consist of connected graphs) or, equivalently, to an equivalence relation induced by the Borel action of a closed subgroup of the symmetric group S_∞; see [BK96, Theorems 2.3.5 and 2.7.3]. In contrast, no *a priori* condition must be put on F or R since in [FMR09] it is shown that any analytic equivalence relation (resp. any analytic quasi-order) is actually Borel bireducible with the bi-embeddability (resp. embeddability) relation on a corresponding $\mathcal{L}_{\omega_1\omega}$-elementary class; see Theorem 3.2.

A less trivial, but still easy, restriction that must be put on the pairs (E, F) and (E, R) is given by the following "cardinality" consideration. Denote by $\mathrm{id}(n)$, where $1 \leq n \leq \omega$, an arbitrary analytic equivalence relation with exactly n classes. Given an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$, since $\equiv_{\mathcal{C}}$ is by definition coarser than $\cong_{\mathcal{C}}$, the "cardinality" of $\mathcal{C}/_{\equiv_{\mathcal{C}}}$ cannot exceed the "cardinality" of $\mathcal{C}/_{\cong_{\mathcal{C}}}$, that is:

- if $\cong_{\mathcal{C}} \leq_B \mathrm{id}(\omega)$, then $\equiv_{\mathcal{C}} \leq_B \cong_{\mathcal{C}}$;

- if $F \leq_B \equiv_{\mathcal{C}}$, then $F \leq_B \cong_{\mathcal{C}}$, where F is one of $\mathrm{id}(1), \dots, \mathrm{id}(\omega), \mathrm{id}(\mathbb{R})$.

As we will see, if Vaught's Conjecture is true (equivalently, by Silver's dichotomy, if every E which is a quasi-isomorphism relation is $\leq_B$-comparable with $\mathrm{id}(\mathbb{R})$), then these are quite surprisingly the unique obstructions to obtain a representation of the pairs (E, F) and (E, R). In fact, Theorems 3.3 and 3.4 (which constitute the main results of this paper) show that given a quasi-isomorphism relation E and an analytic quasi-order R such that either $\mathrm{id}(\mathbb{R}) \leq_B E$ or $E_R \leq_B E, \mathrm{id}(\mathbb{R})$, there exists an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ with the property that $\cong_{\mathcal{C}} \sim_B E$ and $\sqsubseteq_{\mathcal{C}} \sim_B R$. In particular, if E is a quasi-isomorphism relation and either $E \leq_B \mathrm{id}(\omega)$ or $\mathrm{id}(\mathbb{R}) \leq_B E$, then there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ representing the pair (E, F) (resp. (E, R)) *if and only if* either $F \leq_B E$ (resp. $E_R \leq_B E$) or $\mathrm{id}(\mathbb{R}) \leq_B E$. The same kind of result can be obtained by considering homomorphisms, weak-homomorphisms or weak-epimorphisms instead of embeddings; see Sections 3 and 4.

The results above can also be naïvely interpreted as a proof that the complexities of the relations of isomorphism and bi-embeddability on some $\mathcal{L}_{\omega_1\omega}$-elementary class are (almost) independent from each other: in fact, given any isomorphism relation $\cong_{\mathcal{E}}$ and an arbitrary quasi-order R on $\mathcal{E}$ such that $\cong_{\mathcal{E}} \subseteq E_R$ (so that R can *potentially* be the embeddability relation on $\mathcal{E}$), then the above-mentioned results show that unless both $\cong_{\mathcal{E}}$ and E_R are $\leq_B$-incomparable with $\mathrm{id}(\mathbb{R})$ there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that[1] $\cong_{\mathcal{C}} \sim_B \cong_{\mathcal{E}}$ and $\sqsubseteq_{\mathcal{C}} \sim_B R$. This also means that almost all the possible mutual relationships between the isomorphism and the (bi-)embeddability relations can actually be realized with a suitable $\mathcal{L}_{\omega_1\omega}$-elementary class.

While proving our main result, we will deal with the notion of classwise Borel isomorphism, which plays a key role in the proofs of the results of this paper. This notion (which is strictly finer than Borel bireducibility) slightly strengthens some variants of Borel reducibility already introduced in [FS89], and we feel that the applications we are going to present can be viewed as evidence that such a notion is natural, interesting and useful in the study of analytic equivalence relations and quasi-orders.

The paper is organized as follows. In Section 2 we will prove some basic results about classwise Borel isomorphisms and classwise Borel embeddability which will be useful in subsequent sections (but which may also be of independent interest). In Section 3 we will prove the main results of this paper (Theorems 3.3 and 3.4), and finally in Section 4 we will show how to extend the results of the previous section to the case of weak-epimorphisms.

We assume that the reader is quite familiar with the standard terminology and basic results about analytic equivalence relations and Borel reducibility. References for these topics are, for example, [Kec95], [BK96], [Hjo00] and [Gao09]. Part of the main techniques that will be used in this paper were first introduced in [FMR09] and, partially, in [FS89]. For the reader's convenience, throughout this paper we will recall the main results and constructions coming from those papers, but we refer to the original works for proofs and detailed explanations.

[1] Our result is even stronger. In fact, we obtain that there is a pair of functions *simultaneously* witnessing both $\cong_{\mathcal{C}} \simeq_{cB} \cong_{\mathcal{E}}$ (see Section 2 for the definition of $\simeq_{cB}$) and $\sqsubseteq_{\mathcal{C}} \sim_B R$.

2. Classwise Borel isomorphism and classwise Borel embeddability

The present section contains a basic analysis of classwise Borel isomorphism and classwise Borel embeddability (see Definitions 2.1 and 2.2) and is mainly motivated by the fact that some of the properties presented here will be used in Theorem 3.3. Nevertheless, the results of this section are also interesting *per se*, as they constitute a study of some basic properties of these natural and useful notions, and a modest contribution to the study of orbit equivalence relations (i.e. of those analytic equivalence relations which are induced by a Borel action of a Polish group on some standard Borel space).

Definition 2.1 ([FMR09])**.** Let E, F be two analytic equivalence relations on standard Borel spaces X, Y, respectively. We say that E is *classwise Borel isomorphic*[2] to F ($E \simeq_{cB} F$ in symbols) if there are Borel reductions $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ of E into F and F into E, respectively, such that their factorings to the quotient spaces $\hat{\varphi}\colon X/_E \to Y/_F$ and $\hat{\psi}\colon Y/_F \to X/_E$ are bijections and satisfy $\hat{\varphi} = \hat{\psi}^{-1}$.

In other words, $E \simeq_{cB} F$ if and only if there is a bijection $f\colon X/_E \to Y/_F$ such that both f and f^{-1} admit Borel liftings.

Classwise Borel isomorphism *strictly refines* Borel bireducibility. An example of this phenomenon was first given in [FMR09] by considering the $\sim_B$-equivalence class of $\mathrm{id}(\mathbb{R})$, but this result will be extended in Theorem 2.10 to the $\sim_B$-equivalence class of *any* orbit equivalence relation which Borel reduces $\mathrm{id}(\mathbb{R})$. However, as we will see in Theorems 2.5 and 2.8, the two notions coincide if we restrict our attention to some special case, e.g. to the class of Borel orbit equivalence relations.

Definition 2.2. Given two analytic equivalence relations E, F on standard Borel spaces X, Y, respectively, we say that E *classwise Borel embeds* into F ($E \sqsubseteq_{cB} F$ in symbols) just in case there is a Borel F-saturated subset $Y_E \subseteq Y$ such that $E \simeq_{cB} F \restriction Y_E$.

The notion of classwise Borel embeddability is not far from what in [Gao09] is called *faithful Borel reducibility* (a notion first introduced in [FS89]). However, classwise Borel embeddability is a strictly stronger notion because we require the existence of a sort of "inverse" (modulo equivalence classes) of the reduction from E to F.

Another interesting property of classwise Borel embeddability is that many popular classes of analytic equivalence relations are closed under this notion of reducibility. If for example we consider the class of isomorphism relations, or even the broader class of orbit equivalence relations, we have that any analytic equivalence relation which classwise Borel embeds into an element of this class is actually classwise Borel isomorphic to an element of the same class. This should be contrasted with the fact that it is still an open (and seemingly hard) problem to determine if every analytic equivalence relation which is Borel reducible to an element of one of the above mentioned classes is Borel bireducible with a member of the same class. To the best of our knowledge, [Gao09, Theorem 11.3.9] and Remark 2.7 are the unique results in this direction.

[2]This notion was introduced in [FMR09] as "Borel isomorphism". However, in the present paper we decided to adopt a variation of that name to avoid confusion with a different notion of Borel isomorphism which is now quite standard in the literature; see e.g. [Gao09].

Finally, a classwise Borel embedding between two isomorphism relations $\cong_{\mathcal{C}}$ and $\cong_{\mathcal{E}}$ can be seen as a sort of $\mathcal{L}_{\omega_1\omega}$*-interpretation* between the two elementary classes in the sense explained in [FS89, p. 897], that is, in the sense that it provides a correspondence between $\mathcal{L}_{\omega_1\omega}$-sentences. More precisely, for every $\mathcal{L}_{\omega_1\omega}$-sentence Φ there is an $\mathcal{L}_{\omega_1\omega}$-sentence Ψ such that the set of models in $\mathcal{C}$ satisfying Φ is mapped by the witnesses of $\cong_{\mathcal{C}} \sqsubseteq_{cB} \cong_{\mathcal{E}}$ into the set of models in $\mathcal{E}$ which satisfy Ψ in a bijective (up to isomorphism) and Borel way.

The next proposition shows that classwise Borel embeddability is the counterpart in terms of reducibility of classwise Borel isomorphism.

Proposition 2.3. *Let E, F be analytic equivalence relations. If $E \sqsubseteq_{cB} F$ and $F \sqsubseteq_{cB} E$, then $E \simeq_{cB} F$. Therefore $E \simeq_{cB} F$ if and only if $E \sqsubseteq_{cB} F \sqsubseteq_{cB} E$.*

Proof. It is enough to notice that we can apply the usual Schröder-Bernstein argument because if $\varphi\colon \mathrm{dom}(E) \to \mathrm{dom}(F)$ and $\psi\colon \mathrm{dom}(F) \to \mathrm{dom}(E)$ witness $E \sqsubseteq_{cB} F$, then for every E-invariant Borel $A \subseteq \mathrm{dom}(E)$ the F-saturation of $\varphi(A)$ is $\psi^{-1}(A)$, hence a Borel set (and similarly exchanging E and F). $\square$

It is not hard to see that if E is a countable analytic equivalence relation on X and F is an arbitrary Borel equivalence relation on Y, then $E \leq_B F$ implies $E \sqsubseteq_{cB} F$ (the converse to this fact is obvious). In fact, if f is a Borel reduction of E into F, consider the Borel set $Z = \{(x, y) \in X \times Y \mid f(x)\, F\, y\}$ and the map $g\colon Z \to Y\colon (x, y) \mapsto y$. By countability of E, the Borel map g is countable-to-1, so $\mathrm{range}(g)$ (which is the F-saturation of $\mathrm{range}(f)$) is Borel as well, and there is a Borel right inverse h of g. Therefore, $\mathrm{range}(g)$, f and the composition of h with projection on the first coordinate witness $E \sqsubseteq_{cB} F$. This easy observation can be extended with a completely different and more difficult argument to the case of an arbitrary orbit equivalence relation E; see [Gao09, (Proof of) Corollary 5.2.4].

Proposition 2.4 (Gao)**.** *Let E be an orbit equivalence relation and F be an arbitrary Borel equivalence relation. Then $E \leq_B F \iff E \sqsubseteq_{cB} F$.*

As a corollary of Propositions 2.4 and 2.3, we get that for Borel orbit equivalence relations the notions of Borel bireducibility and classwise Borel isomorphism coincide.

Theorem 2.5. *If E, F are Borel orbit equivalence relations. Then $E \sim_B F$ if and only if $E \simeq_{cB} F$.*

Notice that the results above cannot be extended to arbitrary orbit equivalence relations. In fact in [Gao01, Theorem 4] it is proved that for example the relation of isomorphism on countable graphs does not classwise Borel embed into (in fact it does not even faithful Borel reduce to) the relation of isomorphism on countable linear orders (or on "simple" countable trees), whereas all these isomorphism relations are S_∞-complete (and hence pairwise Borel bireducible) by [Hod93, Theorem 5.5.1] and [FS89, Theorems 1 and 3], respectively.

One of the main limitations of Proposition 2.4 is obviously that the equivalence relation F must be Borel. Our next goal will be to show that in some specific situations (that is, for some specific orbit equivalence relations E) such restrictions can be removed, albeit in this case we have to compensate for this with the requirement that F is an orbit equivalence relation as well. In the terminology of [BK96], (Y, a) is an *effective Borel G-space* if Y is a Δ^1_1 subset of a recursively presented

Polish space, G is a recursively presented Polish group with recursive composition and inverse functions, and a is a Δ^1_1 action of G on Y. Moreover, if y is any element of Y we denote by ω_1^y the first (countable) ordinal not recursive-in-y, and put $\omega_1^{Gy} = \inf\{\omega_1^z \mid z\, F_a\, y\}$, with F_a being the orbit equivalence relation induced by a on Y.

Lemma 2.6. *Assume E is an arbitrary equivalence relation on the Polish space X and F is an orbit equivalence relation on a standard Borel space Y. If $E \leq_B F$, then there is an E-invariant Borel comeagre set $C \subseteq X$ and an F-invariant Borel set $A \subseteq Y$ such that $F \restriction A$ is a Borel equivalence relation and $E \restriction C \leq_B F \restriction A$.*

Proof. Assume that $F = F_a$ is induced by the Borel action a of the Polish group G on the standard Borel space Y, and let f be a Borel reduction of E into F. Assume that (Y, a) is an effective Borel G-space and X a recursively presented Polish space (otherwise we relativize), and let p be a parameter such that f is a $\Delta^1_1(p)$-function.

We claim that $\{x \in X \mid \omega_1^{(f(x),p)} \leq \omega_1^p\}$ is comeagre in X. Granting this, $C = \{x \in X \mid \omega_1^{Gf(x)} \leq \omega_1^p\}$ is an E-invariant comeagre subset of X. Put $A = \{y \in Y \mid \omega_1^{Gy} \leq \omega_1^p\}$. Then A is Borel and F-invariant (so that $C = f^{-1}(A)$ is Borel as well), and by the relativized version of [BK96, Proposition 7.2.2] $F \restriction A$ is Borel. But by definition of C, $f \restriction C$ is a Borel map witnessing $E \restriction C \leq_B F \restriction A$.

It remains to prove the claim. Assume toward a contradiction that $B = \{x \in X \mid \omega_1^{(f(x),p)} > \omega_1^p\}$ is nonmeagre in X. Since B is a $\Pi^1_1(p)$ subset of X, by the Sacks-Tanaki Basis Theorem for nonmeagre $\Pi^1_1(p)$ sets (see [Mos80, Exercise 4F.20]) there would be a $\Delta^1_1(p)$-point $x_0 \in B$. But then $f(x_0)$ would be a $\Delta^1_1(p)$-point as well, so that $\omega_1^{f(x_0),p} \leq \omega_1^p$, a contradiction! $\square$

Remark 2.7. In particular, if an arbitrary equivalence relation E on a Polish space X is Borel reducible to an orbit equivalence relation (resp. an isomorphism relation or a countable equivalence relation), then there is an E-invariant Borel comeagre set $C \subseteq X$ such that $E \restriction C$ is Borel and is Borel bireducible with (in fact, classwise Borel isomorphic to) an orbit equivalence relation (resp. to an isomorphism relation or to a countable equivalence relation).

Call an analytic equivalence relation E on the standard Borel space X *invariant by comeagre subsets* if for every comeagre $C \subseteq X$ one has $E \leq_B E \restriction C$ (note that it is enough to restrict the attention to Borel comeagre sets C). Examples of invariant by comeagre subsets analytic equivalence relation are the following:

- $E = \mathrm{id}(\mathbb{R})$: this is because it is a classical result that any comeagre subset of $\mathbb{R}$ must contain a perfect subset;
- $E = E_0$: by a classical fact (see e.g. [Hjo00, Theorem 3.2]), for every comeagre $C \subseteq \mathrm{dom}(E_0)$ we have $E_0 \restriction C \not\leq_B \mathrm{id}(\mathbb{R})$ (otherwise $E_0 \leq_B \mathrm{id}(\mathbb{R})$). But $E_0 \restriction C$ is obviously Borel, so $E_0 \leq_B E_0 \restriction C$ by the Glimm-Effros Dichotomy (see e.g. [BK96, Theorem 3.4.2]).

On the contrary, Lemma 2.6 implies, in particular, that if E is an orbit equivalence relation which is *not* Borel, then it cannot be invariant by comeagre subsets.

Theorem 2.8. *Let E, F be orbit equivalence relations and E be invariant by comeagre subsets. Then $E \leq_B F \iff E \sqsubseteq_{cB} F$. In particular, the result holds with $E = \mathrm{id}(\mathbb{R})$ and $E = E_0$.*

Proof. One direction is obvious. For the other direction, apply Lemma 2.6 to E and F, use the fact that E is invariant by comeagre subsets, and then apply Proposition 2.4 to E and $F \restriction A$ to get $E \sqsubseteq_{cB} F \restriction A$. Since A is F-invariant and Borel, this means $E \sqsubseteq_{cB} F$ as well. □

Theorem 2.8 shows, in particular, that if E is either $\mathrm{id}(\mathbb{R})$ or E_0 and F is an orbit equivalence relation, then $E \leq_B F \iff E \sqsubseteq_{cB} F$. We are now going to show that in this case there are other natural conditions which are equivalent to the previous ones. Such conditions arise from the natural idea of considering disjoint unions of analytic equivalence relations. Given E, F on standard Borel spaces X, Y, respectively, we denote by $E \sqcup F$ the analytic equivalence relation on $X \sqcup Y$ (where $\sqcup$ denotes disjoint union) defined by $x\,(E \sqcup F)\,y$ if and only if either $x, y \in X$ and $x\,E\,y$ or else $x, y \in Y$ and $x\,F\,y$. Disjoint union seems a natural operation to be considered because if E, F belong to some natural class of equivalence relations (such as isomorphism relations, orbit equivalence relations, and so on), then $E \sqcup F$ is an analytic equivalence relation in the same class in which both E and F classwise Borel embed.

Proposition 2.9. *Let E be an orbit equivalence relation and F be either* $\mathrm{id}(\mathbb{R})$ *or* E_0*. Then the following are equivalent:*[3]

i) $F \leq_B E$*;*
ii) $F \sqsubseteq_{cB} E$*;*
iii) $E \sim_B E' \sqcup F$ *for some analytic equivalence relation* E'*;*
iv) $E \sim_B E \sqcup F$*.*

Proof. i) ⇒ ii) by Theorem 2.8. ii) ⇒ iii) because if $Y_F \subseteq \mathrm{dom}(E) = Y$ is Borel, E-invariant, and such that $F \simeq_{cB} E \restriction Y_F$, then iii) is obviously satisfied with $E' = E \restriction (Y \setminus Y_F)$. Now let E' witness iii): since clearly $F \sqcup F \leq_B F$, then $E \sqcup F \sim_B E' \sqcup F \sqcup F \sim_B E' \sqcup F \sim_B E$, so iv) holds. Finally, iv) ⇒ i) because $F \leq_B E \sqcup F \sim_B E$. □

Proposition 2.9 allows us to extend the example given in [FMR09] of a pair of analytic equivalence relations which are Borel bireducible but not classwise Borel isomorphic to the context of arbitrary orbit equivalence relations (this result should also be contrasted with Theorem 2.5 above).

Theorem 2.10. *Let E be an orbit equivalence relation such that* $\mathrm{id}(\mathbb{R}) \leq_B E$*. Then there is an analytic equivalence relation $F \sim_B E$ such that $F \not\simeq_{cB} E$.*

Proof. Let X be the domain of E, and let $B \subseteq \mathbb{R} \times \mathbb{R}$ be a Borel set with nonempty vertical sections and with no Borel uniformization. Consider the Borel equivalence relation E_B on B given by the vertical sections of B; we claim that $F = E \sqcup E_B$ works. Clearly $E \leq_B E \sqcup E_B$. Moreover, since the projection on the first coordinate witnesses $E_B \leq_B \mathrm{id}(\mathbb{R})$, we have $E \sqcup E_B \leq_B E \sqcup \mathrm{id}(\mathbb{R})$. However, by Proposition 2.9 $E \sqcup \mathrm{id}(\mathbb{R}) \leq_B E$, whence $E \sqcup E_B \leq_B E$.

Finally, assume towards a contradiction that E and $E \sqcup E_B$ are classwise Borel isomorphic, and let $\varphi\colon X \to X \sqcup B$ and $\psi\colon X \sqcup B \to X$ be witnesses of this fact. The set $X' = \varphi^{-1}(B)$ is Borel and E-saturated, so that $E' = E \restriction X'$ is a Borel orbit equivalence relation. Moreover, the composition of φ with the projection

[3]Since in both cases $F \sqcup F \simeq_{cB} F$ by Proposition 2.4 and the fact that $F \sqcup F \sqsubseteq_{cB} F$, in this proposition we could replace all occurrences of $\leq_B$ and $\sim_B$ with, respectively, $\sqsubseteq_{cB}$ and $\simeq_{cB}$.

on the first coordinate shows that E' is smooth, so by a theorem of Burgess (see e.g. [Gao09, Corollary 5.4.12]) there is a Borel selector $s\colon X' \to X'$ for E'. This implies that $f = \varphi \circ s \circ (\psi \restriction B)\colon B \to B$ is a well-defined Borel function and that $\mathrm{range}(f) = \{b \in B \mid b = f(b)\}$ is a Borel uniformization of B, a contradiction. $\square$

Given two analytic equivalence relations E, F, say that E *essentially refines*[4] F if and only if there is an analytic equivalence relation $E' \supseteq E$ such that $E' \simeq_{cB} F$. The following technical result will be used in the next section.

Proposition 2.11. *Let E be an orbit equivalence relations and F be either* $\mathrm{id}(\mathbb{R})$ *or E_0. If $F \leq_B E$, then E essentially refines F. Moreover, the converse holds if $F = \mathrm{id}(\mathbb{R})$.*

Proof. Under our assumption, $F \sqsubseteq_{cB} E$ by Proposition 2.9. Let X be the domain of E and $X_F \subseteq X$ be Borel E-invariant and such that $F \simeq_{cB} E \restriction X_F$. Define $E' \supseteq E$ on X by letting $x\,E'\,y$ if and only if either $x, y \notin X_F$ or $x\,E\,y$. Then $F \sqsubseteq_{cB} E'$ and $E' \sqsubseteq_{cB} F \sqcup \mathrm{id}(1) \sqsubseteq_{cB} F$ so that $E' \simeq_{cB} F$ by Proposition 2.3. The extra fact about $\mathrm{id}(\mathbb{R})$ follows from the fact that any witness of $\mathrm{id}(\mathbb{R}) \leq_B E'$ witnesses $\mathrm{id}(\mathbb{R}) \leq_B E$ as well. $\square$

3. The main result

In this section we will show our main result: if E is a quasi-isomorphism relation and R is an analytic quasi-order such that at least one of E, E_R has either countably or perfectly many equivalence classes, then there is an $\mathcal{L}_{\omega_1\omega}$-elementary class representing the pair (E, R) if and only if either $E_R \leq_B \mathrm{id}(\mathbb{R}), E$ or $\mathrm{id}(\mathbb{R}) \leq_B E$. Notice that this implies the corresponding result for pairs (E, F) consisting of two analytic equivalence relations. Therefore, from this point onward we will just consider the case of pairs of the form (E, R).

We first consider the basic case, namely when $E_R \sim_B \mathrm{id}(1)$. From this point onward, $\hat{\mathcal{L}}$ will denote the language of graphs consisting of just one binary relation symbol, while $\mathcal{L}$ will denote the language of *ordered graphs*, that is, a language consisting of two binary relation symbols. In particular, the interpretation of the second symbol (in a certain structure) will always be called the *order (relation)* of the structure.[5] Finally, an *ordered set-theoretical tree* is a set-theoretical tree with an extra transitive (binary) relation on its nodes.

Theorem 3.1. *Let E be a quasi-isomorphism relation. Then there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ consisting of ordered set-theoretical trees whose order is an equivalence relation (so that, in particular, it is reflexive) such that $E \sim_B \cong_{\mathcal{C}}$ and $\sqsubseteq_{\mathcal{C}} = \equiv_{\mathcal{C}} \sim_B \mathrm{id}(1)$ (in fact, if E itself is an isomorphism relation, then $E \simeq_{cB} \cong_{\mathcal{C}}$).*

Proof. Let $\hat{\mathcal{C}}$ be an arbitrary $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class such that $\cong_{\hat{\mathcal{C}}} \sim_B E$. For every $x \in \hat{\mathcal{C}}$, construct the set theoretical tree $\hat{T}_x$ on ${}^{<\omega}\omega \sqcup \omega$ in the following way. Consider the tree ${}^{<\omega}\omega$ with the inclusion relation. For any $s \in {}^{<\omega}\omega$, let $s^{\mathbb{E}}$ be the sequence $\langle s(2i) \mid 2i < |s| \rangle$. If $s \neq \emptyset$, denote by $\mathrm{rp}(s)$ the pair $(s(n), s(m))$ (also called the *relevant pair* of s), where n, m are such that $|s| = \langle n, m \rangle + 1$ and $\langle \cdot, \cdot \rangle$

[4]This condition is potentially stronger than just requiring the existence of an analytic equivalence relation $E' \supseteq E$ such that $E' \sim_B F$. However, if F is a countable analytic equivalence relation, then the two notions actually coincide.

[5]This is a little abuse of terminology. In fact, as we will see, such a relation will be required to be just a transitive relation and not an order in the usual sense.

is any bijection between $\omega \times \omega$ and ω such that $n, m \leq \langle n, m \rangle$. Now adjoin a new unique terminal successor taken from ω to s just in case either $|s|$ is even or else $|s|$ is odd and $\mathrm{rp}(s^{\mathbb{E}})$ is an edge of x, ensuring that at the end of this process every $n \in \omega$ is linked to some $s \in {}^{<\omega}\omega$. Finally, define T_x by adjoining the following equivalence relation E_x (which actually is independent from x) on the nodes of $\hat{T}_x$:

$$s \, E_x \, t \iff (s, t \notin {}^{<\omega}\omega) \vee (s = t = \emptyset) \vee (s, t \in {}^{<\omega}\omega \setminus \{\emptyset\} \wedge \mathrm{rp}(s^{\mathbb{E}}) = \mathrm{rp}(t^{\mathbb{E}})).$$

Following [FS89, proof of Theorem 1.1.1], one can easily check that $x \cong y \iff T_x \cong T_y$. (Any isomorphism between $x, y \in \hat{\mathcal{C}}$ can be lifted to an isomorphism of ${}^{<\omega}\omega$ into itself respecting the equivalence relations E_x and E_y and can then be naturally extended to an isomorphism between T_x and T_y. Conversely, from any isomorphism between T_x and T_y one can recover by a back and forth argument an isomorphism between x and y.) Let $j_{\mathcal{L}} \colon S_\infty \times \mathrm{Mod}(\mathcal{L}) \to \mathrm{Mod}(\mathcal{L})$ be the standard logic action of S_∞ on $\mathrm{Mod}(\mathcal{L})$. Arguing as in the proof of [FMR09, Theorem 4.1], we can then find a Borel $B \subseteq S_\infty$ such that the map $h \colon (x, b) \mapsto j_{\mathcal{L}}(b, T_x)$ defined on $\hat{\mathcal{C}} \times B$ is injective and that for every $x \in \hat{\mathcal{C}}$ and $q \in S_\infty$ there are $x' \in \hat{\mathcal{C}}$ and $b \in B$ such that $j_{\mathcal{L}}(q, T_x) = j_{\mathcal{L}}(b, T_{x'})$, Therefore, the range of this map is Borel and coincides with the saturation under isomorphism of $\{T_x \mid x \in \hat{\mathcal{C}}\}$, i.e. it is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$. Moreover $\cong_{\mathcal{C}} \sim_B \cong_{\hat{\mathcal{C}}}$, the equivalence being witnessed by the Borel map $x \mapsto T_x$ and, for the other direction, by the composition of h^{-1} with the projection on the first coordinate.

It remains to prove that $T_x \sqsubseteq T_y$ for every $x, y \in \hat{\mathcal{C}}$. This can be easily done by first constructing an embedding of $T_x \cap {}^{<\omega}\omega$ into $T_x \cap \{s \in {}^{<\omega}\omega \mid |s| \text{ is even}\}$ (use the fact that for every $s, t \in {}^{<\omega}\omega$ there is $t \subseteq v \in {}^{<\omega}\omega$ such that $|v|$ is even and $\mathrm{rp}(s^{\mathbb{E}}) = \mathrm{rp}(v^{\mathbb{E}})$) and then extending it to T_x using the fact that each s of even length always has a successor not in ${}^{<\omega}\omega$. $\square$

We will now discuss the case in which $\mathrm{id}(\mathbb{R}) \leq_B E$. Recall from [FMR09] that a *combinatorial tree* is a connected acyclic graph, while an *ordered* combinatorial tree is a combinatorial tree with an extra transitive (binary) relation defined on its nodes. We need the following result from [FMR09, Theorems 3.3 and 3.5].

Theorem 3.2 ([FMR09])**.** *For every analytic quasi-order R, there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ consisting of ordered combinatorial trees whose order relation is a strict well-founded order (so that, in particular, it is irreflexive) such that $\sqsubseteq_{\mathcal{C}} \sim_B R$ and $\cong_{\mathcal{C}} \simeq_{cB} \mathrm{id}(\mathbb{R})$ (and moreover $E_R \simeq_{cB} \equiv_{\mathcal{C}}$).*

Theorem 3.3. *Let E be a quasi-isomorphism relation on the standard Borel space X such that $\mathrm{id}(\mathbb{R}) \leq_B E$ and R be an arbitrary analytic quasi-order on the standard Borel space Y. Then there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $E \sim_B \cong_{\mathcal{C}}$ and $R \sim_B \sqsubseteq_{\mathcal{C}}$ (in fact, if E itself is an isomorphism relation, then $E \simeq_{cB} \cong_{\mathcal{C}}$ and $E_R \simeq_{cB} \equiv_{\mathcal{C}}$).*

Proof. We can assume that $X = Y = \mathbb{R}$. Let $\mathcal{C}'$ be given by applying Theorem 3.2 to R so that $\sqsubseteq_{\mathcal{C}'} \sim_B R$ and $\cong_{\mathcal{C}'} \simeq_{cB} \mathrm{id}(\mathbb{R})$, and let $\varphi_0 \colon \mathcal{C}' \to \mathbb{R}$ and $\psi_0 \colon \mathbb{R} \to \mathcal{C}'$ witness the classwise Borel isomorphism. Then apply Theorem 3.1 to E to get an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}''$ such that $\cong_{\mathcal{C}''} \sim_B E$ and $\equiv_{\mathcal{C}''} \sim_B \mathrm{id}(1)$. Since $\mathrm{id}(\mathbb{R}) \leq_B E$, $\cong_{\mathcal{C}''}$ essentially refines $\mathrm{id}(\mathbb{R})$ by Proposition 2.11, so let $E' \supseteq \cong_{\mathcal{C}''}$ be a Borel equivalence relation on $\mathcal{C}''$ which is classwise Borel isomorphic to $\mathrm{id}(\mathbb{R})$ and let $\varphi_1 \colon \mathcal{C}'' \to \mathbb{R}$ and $\psi_1 \colon \mathbb{R} \to \mathcal{C}''$ be witnesses to this fact. Notice that $\psi_1 \circ \varphi_0 \colon \mathcal{C}' \to \mathcal{C}''$ and

$\psi_0 \circ \varphi_1 \colon \mathcal{C}'' \to \mathcal{C}'$ witness $\cong_{\mathcal{C}'} \simeq_{cB} E'$. Now consider the set

$$W = \{(x, z, g) \in \mathcal{C}' \times \mathcal{C}'' \times G \mid \psi_0(\varphi_1(z)) \cong_{\mathcal{C}'} x\},$$

where G is the closed subset of S_∞ consisting of those g such that for all $n, m \in \omega$, if n, m have the same parity, then $n \leq m \iff g(n) \leq g(m)$. Notice that W is Borel because $\cong_{\mathcal{C}'}$ is a Borel equivalence relation, and define S and F on W by

$$(x_1, z_1, g_1)\, S\, (x_2, z_2, g_2) \iff x_1 \sqsubseteq x_2 \iff x_1 \sqsubseteq_{\mathcal{C}'} x_2,$$
$$(x_1, z_1, g_1)\, F\, (x_2, z_2, g_2) \iff z_1 \cong z_2 \iff z_1 \cong_{\mathcal{C}''} z_2.$$

Obviously, the projections on the first and on the second coordinate witness, respectively, $S \leq_B \sqsubseteq_{\mathcal{C}'}$ and $F \leq_B \cong_{\mathcal{C}''}$. Moreover, the Borel map sending $z \in \mathcal{C}''$ to $(\psi_0(\varphi_1(z)), z, \mathrm{id})$ (which is an element of W by definition) witnesses $\cong_{\mathcal{C}''} \leq_B F$. Now consider the Borel map h sending $x \in \mathcal{C}'$ to $(x, \psi_1(\varphi_0(x)), \mathrm{id})$. Since $\psi_0(\varphi_1(\psi_1(\varphi_0(x)))) = \psi_0(\varphi_0(x)) \cong_{\mathcal{C}'} x$, we have that $h(x) \in W$, and obviously h reduces $\sqsubseteq_{\mathcal{C}'}$ to S. Therefore we get $S \sim_B \sqsubseteq_{\mathcal{C}'}$ ($\sim_B R$) and $F \sim_B \cong_{\mathcal{C}''}$ ($\sim_B E$), and hence it will be enough to find an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\sqsubseteq_{\mathcal{C}} \sim_B S$ and $\cong_{\mathcal{C}} \sim_B F$. Define the Borel map f from W into the space of $\mathcal{L}$-structures on ω by sending $w = (x, z, g)$ into $j_{\mathcal{L}}(g, x \oplus z)$, where $x \oplus z$ is the structure on ω obtained by "copying" in the obvious way x on the even numbers and z on the odd numbers. Let $w_1 = (x_1, z_1, g_1)$ and $w_2 = (x_2, z_2, g_2)$ denote arbitrary elements of W.

Claim 3.3.1. f reduces S to $\sqsubseteq$ and F to $\cong$.

Proof of the claim. Assume first that $w_1\, S\, w_2$, that is, $x_1 \sqsubseteq x_2$. Since $z_1 \sqsubseteq z_2$ by the choice of $\mathcal{C}''$, we can glue these two embeddings into an embedding of $x_1 \oplus z_1$ into $x_2 \oplus z_2$, whence $f(w_1) \sqsubseteq f(w_2)$. Conversely, if $f(w_1) \sqsubseteq f(w_2)$, then $x_1 \oplus z_1 \sqsubseteq x_2 \oplus z_2$ as well. But any such embedding must send elements coming from x_1 into elements coming from x_2, as by Theorems 3.2 and 3.1 these are the unique vertices of, respectively, $x_1 \oplus z_1$ and $x_2 \oplus z_2$ which *are not* in order relation with themselves. This implies $x_1 \sqsubseteq x_2$, whence $w_1\, S\, w_2$.

Now assume $w_1\, F\, w_2$, so that $z_1 \cong z_2$. Since $E' \supseteq \cong_{\mathcal{C}''}$ and φ_1 reduces E' to $\mathrm{id}(\mathbb{R})$, we have that $\varphi_1(z_1) = \varphi_1(z_2)$, so that $x_1 \cong \psi_0(\varphi_1(z_1)) = \psi_0(\varphi_1(z_2)) \cong x_2$. Therefore one can glue these isomorphisms to witness $x_1 \oplus z_1 \cong x_2 \oplus z_2$, whence $f(w_1) \cong f(w_2)$. Conversely, assume that $f(w_1) \cong f(w_2)$, so that in particular $x_1 \oplus z_1 \cong x_2 \oplus z_2$. Since any isomorphism witnessing this fact must again map elements coming from z_1 into elements coming from z_2 (as these are the unique elements of the corresponding structure which *are* in order relation with themselves), from such an isomorphism one can recover an isomorphism between z_1 and z_2, whence $w_1\, F\, w_2$. □

Claim 3.3.2. f is injective and range(f) is saturated.

Proof of the claim. Assume first that $f(w_1) = f(w_2)$, and observe that for every $h_1, h_2 \in G$, if $\{h_1(2n+1) \mid n \in \omega\} = \{h_2(2n+1) \mid n \in \omega\}$, then $h_1 = h_2$. Since $k = g_i(2n+1) \iff k$ is in order relation with itself ($i = 1, 2$, $n, k \in \omega$) by Theorems 3.2 and 3.1 and the definition of f, from $f(w_1) = f(w_2)$ we get $\{g_1(2n+1) \mid n \in \omega\} = \{g_2(2n+1) \mid n \in \omega\}$, and hence we can conclude $g_1 = g_2$. But this implies $x_1 \oplus z_1 = x_2 \oplus z_2$, whence $x_1 = x_2$ and $z_1 = z_2$.

For the second part, it is enough to show that range(f) is the saturation of $\{\psi_0(\varphi_1(z)) \oplus z \mid z \in \mathcal{C}''\}$. One direction is obvious. For the other direction, note that for each $h \in S_\infty$ there are $g \in G$ and $p, q \in S_\infty$ such that $h(2n) = g(2p(n))$ and

$h(2n+1) = g(2q(n)+1)$ for every $n \in \omega$, so that $j_{\mathcal{L}}(h, \psi_0(\varphi_1(z)) \oplus z) = j_{\mathcal{L}}(g, x' \oplus z')$, where $x' = j_{\mathcal{L}}(p, \psi_0(\varphi_1(z))) \in \mathcal{C}'$ and $z' = j_{\mathcal{L}}(q, z) \in \mathcal{C}''$. But since $z' \cong z$ and $E' \supseteq \cong_{\mathcal{C}''}$, we have $\varphi_1(z) = \varphi_1(z')$, so that $x' \cong_{\mathcal{C}'} \psi_0(\varphi_1(z)) = \psi_0(\varphi_1(z'))$. Therefore $(x', z', g) \in W$ and $j_{\mathcal{L}}(h, \psi_0(\varphi_1(z)) \oplus z) = f(w)$; hence we are done. □

Since $W = \mathrm{dom}(f)$ is Borel and f is Borel and injective, then $\mathrm{range}(f)$ is Borel and f^{-1} is a Borel function reducing $\sqsubseteq$ to S and $\cong$ to F. Since $\mathrm{range}(f)$ is also saturated, $\mathrm{range}(f) = \mathcal{C}$ for some $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$. Therefore we get the desired result. □

Using the technique developed in the previous proof, we can now deal with the case $E_R \leq_B \mathrm{id}(\omega), E$.

Theorem 3.4. *For every $1 \leq n \leq \omega$, every analytic quasi-order R such that $E_R \sim_B \mathrm{id}(n)$, and every quasi-isomorphism relation E such that $\mathrm{id}(n) \leq_B E$, there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $E \sim_B \cong_{\mathcal{C}}$ and $R \sim_B \sqsubseteq_{\mathcal{C}}$ (in fact, if E itself is an isomorphism relation, then $E \simeq_{cB} \cong_{\mathcal{C}}$ and $E_R \simeq_{cB} \equiv_{\mathcal{C}}$).*

Proof. The case $n = 1$ is Theorem 3.1; thus we will consider just the case $1 < n \leq \omega$. Apply Theorem 3.2 and let $\mathcal{C}'$ be an $\mathcal{L}_{\omega_1\omega}$-elementary class such that $R \sim_B \sqsubseteq_{\mathcal{C}'}$ (and $\cong_{\mathcal{C}'} \simeq_{cB} \mathrm{id}(\mathbb{R})$, so that $\cong_{\mathcal{C}'}$ is Borel). Moreover, let $\mathcal{C}''$ be an $\mathcal{L}_{\omega_1\omega}$-elementary class such that $E \sim_B \cong_{\mathcal{C}''}$ and $\equiv_{\mathcal{C}''} \sim_B \mathrm{id}(1)$ (such a $\mathcal{C}''$ exists by Theorem 3.1). Choose pairwise nonisomorphic $z_1, \ldots, z_n \in \mathcal{C}''$ (this is possible because $\mathrm{id}(n) \leq_B E$). Then $\hat{\mathcal{C}}'' = \mathcal{C}'' \setminus \bigcup\{[z_i]_{\cong} \mid 1 \leq i < n\}$ is a nonempty Borel invariant set, so it is an $\mathcal{L}_{\omega_1\omega}$-elementary class. Now choose $x_1, \ldots, x_n \in \mathcal{C}'$ to be pairwise $\equiv_{\mathcal{C}'}$-inequivalent and such that every other $x \in \mathcal{C}'$ is $\equiv_{\mathcal{C}'}$-equivalent to some of these x_i's (this is possible since $\equiv_{\mathcal{C}'} \sim_B E_R \sim_B \mathrm{id}(n)$). Now put

$$W = \{(x, z, g) \in \mathcal{C}' \times \mathcal{C}'' \times G \mid (z \in \hat{\mathcal{C}}'' \wedge x \cong_{\mathcal{C}'} x_n) \vee \bigvee_{1 \leq i < n} (z \cong z_i \wedge x \cong_{\mathcal{C}'} x_i)\},$$

where G is defined as in the proof of Theorem 3.3. Now letting S, F and f be defined as in that proof, by (almost) the same argument one gets that $S \sim_B \sqsubseteq_{\mathcal{C}'} \sim_B R$, $F \sim_B \cong_{\mathcal{C}''} \sim_B E$, f reduces S to $\sqsubseteq$ and E to $\cong$ (this is essentially because if $(x, z, g), (x', z', g') \in W$ and $z \cong z'$, then $x \cong x'$), f is injective and $\mathrm{range}(f)$ is saturated, so that taking $\mathcal{C} = \mathrm{range}(f)$ we obtain the result. □

We now want to make some comments on possible variations of our main result. First of all, notice that in both Theorem 3.3 and Theorem 3.4 we can also have that the resulting $\mathcal{C}$ consists of ordered combinatorial trees. In fact, given $x \in C'$ (where $\mathcal{C}'$ is as in the proof of Theorem 3.3), call *root* of x the least vertex with respect to the (strict well-founded) order relation on x. By inspecting the proof of Theorem 3.2, it is easy to check that any embedding (and consequently any isomorphism) between $x_1, x_2 \in \mathcal{C}'$ must send the root of x_1 to the root of x_2. Moreover, by applying [FMR09, Theorem 4.1] to the $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}''$ defined in the proof of Theorem 3.3, we get that such $\mathcal{C}''$ can be assumed to consist of ordered combinatorial trees with the further property that for each $z \in \mathcal{C}''$ there is a unique element which is in order relation just with itself (such an element is called the *root* of z) and that for every $z_1, z_2 \in \mathcal{C}''$ there is an embedding between them which sends the root of z_1 to the root of z_2 (this easily follows from the construction given in [FMR09, Theorem 4.1]). Define the Borel set W as above and redefine f to be the Borel function sending $(x, z, g) \in W$ to $j_{\mathcal{L}}(g, x \hat{\oplus} z)$, where $x \hat{\oplus} z$ is the *ordered combinatorial tree* obtained by first considering $x \oplus z$ and then

linking the root of x to the root of z. It is now straightforward to check that the proofs of Theorems 3.3 and 3.4 can be carried out with this new definition of f (by the properties of $\mathcal{C}'$ and $\mathcal{C}''$ described above).

The second possible variation is given by the fact that we can replace embeddings with homomorphisms and weak-homomorphisms.[6] This is because in [FMR09, Theorem 3.5] it is proved that any weak-homomorphism between two elements of the $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ given by Theorem 3.2 is automatically an embedding. This fact, together with the trivial observation that each embedding is, in particular, a (weak-)homomorphism, shows that the proofs of Theorems 3.1, 3.3 and 3.4 are also proofs of the analogous results obtained by replacing $\sqsubseteq$ with the analytic quasi-order naturally induced by (weak-)homomorphism.

Finally, in the case of embeddings and homomorphisms we can further replace the language of ordered graphs $\mathcal{L}$ (which consists of two binary relation symbols) with the language of graphs $\hat{\mathcal{L}}$ and obtain that the $\mathcal{C}$ resulting from any of the theorems of this section consists of graphs. This is because, as already noted in the introduction, in [Hod93, Theorem 5.5.1] it is shown that the $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ can be bi-interpreted in an $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class $\hat{\mathcal{C}}$ consisting of connected graphs, and a careful inspection of the proof shows that both interpretations preserve the embeddability and the homomorphism relation (in fact, one can show that for graphs in $\hat{\mathcal{C}}$ each homomorphism is automatically an embedding). The case of weak-homomorphisms seems more difficult, as we do not even know if a statement analogous to Theorem 3.2 holds when replacing the language $\mathcal{L}$ with $\hat{\mathcal{L}}$.

We end this section with a question about the unique possibility left open by the limitations discussed in the introduction and in Theorems 3.3 and 3.4.

Question. What if Vaught's Conjecture is false, $E \sim_B \cong_{\mathcal{C}}$ for some $\mathcal{C}$ witnessing this failure, and F is such that $\mathrm{id}(\omega) <_B F$ but $\mathrm{id}(\mathbb{R}) \not\leq_B F$ (or, similarly, if R is an analytic quasi-order such that $\mathrm{id}(\omega) <_B E_R$ but $\mathrm{id}(\mathbb{R}) \not\leq_B E_R$)?

Notice that an answer to this question must necessarily employ different techniques, because a careful inspection of the proofs of Theorems 3.3 and 3.4 shows that such arguments can in general be applied to those pairs (E, R) for which there is an analytic equivalence relation $F \supseteq \cong_{\mathcal{E}}$ (where $\mathcal{E}$ is some $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class such that $E \sim_B \cong_{\mathcal{E}}$) which is Borel isomorphic to a refinement of E_R and has a Borel selector. However, the Borel transversal given by such a selector must either be countable or contain a perfect subset, and hence (E, R) must satisfy the hypotheses of Theorem 3.4 in the former case and the hypotheses of Theorem 3.3 in the other case.

4. Replacing embeddings with epimorphisms

Given two countable structures x, y, call a function between their domains a *(weak-)epimorphism* if it is a surjective (weak-)homomorphism, and put $x \preceq^{(w)epi} y$ if and only if there is a (weak-)epimorphism of y onto x (as usual, we will denote by $\preceq_{\mathcal{C}}^{(w)epi}$ the restriction of $\preceq^{(w)epi}$ to the $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$). It is easy to check that any quasi-order of the form $\preceq_{\mathcal{C}}^{(w)epi}$ is an analytic quasi-order. We

[6] A function f between (the domains of) two structures x, y with the same language is said to be a *homomorphism* if it preserves relations and functions in both directions and a *weak-homomorphism* if it preserves relations and functions just from the domain structure to the range structure. In particular, embeddings coincide with *injective* homomorphisms.

will now prove that given a pair (E, R) as in the previous section (that is, such that either $\mathrm{id}(\mathbb{R}) \leq_B E$ or $E_R \leq_B \mathrm{id}(\omega), E$), it is always possible to produce an $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class $\hat{\mathcal{C}}$ such that $E \sim_B \cong_{\hat{\mathcal{C}}}$ and $R \sim_B \preceq^{wepi}_{\hat{\mathcal{C}}}$ (this implies the analogous statement for $\mathcal{L}_{\omega_1\omega}$-elementary classes, since it is enough to adjoin the empty order relation to the elements of $\hat{\mathcal{C}}$). We do not know if a similar result holds for the quasi-order $\preceq^{epi}$; in fact, it is still an open problem whether this relation is complete for analytic quasi-orders.

First we have to consider the basic case (which is analogous to Theorem 3.1).

Theorem 4.1. *Let E be a quasi-isomorphism relation. Then there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $E \sim_B \cong_{\mathcal{C}}$ and $z_1 \preceq^{wepi} z_2$ for every $z_1, z_2 \in \mathcal{C}$ (in fact, if E itself is an isomorphism relation, then $E \simeq_{cB} \cong_{\mathcal{C}}$).*

Proof. Let $\hat{\mathcal{C}}$ be an $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class consisting of graphs such that $\cong_{\hat{\mathcal{C}}} \sim_B E$ and $x_1 \sqsubseteq x_2$ for every $x_1, x_2 \in \mathcal{C}'$ (such a class exists by Theorem 3.1 and the observations following Theorem 3.3). We partially modify the construction given in the proof of Theorem 3.1, and for $x \in \hat{\mathcal{C}}$ define a set-theoretical tree $\hat{R}_x$ on ${}^{<\omega}\omega \sqcup \omega$ as follows: consider the tree ${}^{<\omega}\omega$ with the inclusion relation. If $s \in {}^{<\omega}\omega \setminus \{\emptyset\}$ is such that $s(i) = 0$ for some $i < |s|$, then adjoin to s two distinct terminal successors taken from ω; while if $\emptyset \neq s \in {}^{<\omega}(\omega \setminus \{0\})$, then adjoin to s a unique terminal successor taken from ω just in case $n-1$ and $m-1$, where $\mathrm{rp}(s) = (n, m)$, are linked in the graph x (as in the original argument, at the end of the above construction each element of ω must be the immediate successor of exactly one element of ${}^{<\omega}\omega$). Now construct an ordered set-theoretical tree R_x by adjoining to $\hat{R}_x$ the equivalence relation E_x defined in the proof of Theorem 3.1.

First check that $x \cong y \iff R_x \cong R_y$. For one direction, if f is an isomorphism between x and y, first define $f'(0) = 0$ and $f'(n+1) = f(n) + 1$, lift f' to an isomorphism of ${}^{<\omega}\omega$ into itself (which necessarily respect the equivalence relations E_x and E_y), and then extend such an isomorphism in the obvious way to an isomorphism of R_x and R_y. For the other direction, given an isomorphism g between R_x and R_y, recover by a back and forth argument an isomorphism between x and y — it is enough to use the fact that a sequence $s \in {}^{<\omega}(\omega \setminus \{0\})$ must be sent into an element of ${}^{<\omega}(\omega \setminus \{0\})$, because such an s can have at most one terminal successor, while every sequence which contains a 0 has two distinct terminal successors.

Then argue as in the proof of Theorem 3.1 to show that the saturation under isomorphism of $\{R_x \mid x \in \hat{\mathcal{C}}\}$ forms an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\cong_{\mathcal{C}} \sim_B \cong_{\hat{\mathcal{C}}}$, so that it remains only to prove that for $z_1, z_2 \in \mathcal{C}$ one has $z_1 \preceq^{wepi} z_2$. Clearly, it is enough to show that for $x, y \in \hat{\mathcal{C}}$ there is a weak-epimorphism from R_y onto R_x. Let f be an embedding from x into y, and define g by letting $g(0) = 0$, $g(n+1) = m+1$ if $f(m) = n$, and $g(n+1) = 0$ otherwise. Now lift coordinatewise the function g to a surjection $\hat{g}$ from ${}^{<\omega}\omega$ onto itself. Note that if $s = \langle s_0, \ldots, s_n \rangle \in {}^{<\omega}(\omega \setminus \{0\})$ has a terminal successor in R_x, then the sequence $t = \langle f(s_0 - 1) + 1, \ldots, f(s_n - 1) + 1 \rangle$ has a terminal successor in R_y and is such that $\hat{g}(t) = s$, while if $s = \langle s_0, \ldots, s_n \rangle$ is such that $s_i = 0$ for some $i \leq n$, then the sequence $t = \langle t_0, \ldots, t_n \rangle$ defined by $t_i = 0$ if $s_i = 0$ and $t_i = f(s_i - 1) + 1$ otherwise is such that $\hat{g}(t) = s$ and both s and t have exactly two distinct terminal successors in R_x and R_y, respectively. Therefore one can extend $\hat{g}$ in the obvious way to a weak-epimorphism h from R_y onto R_x. $\square$

Corollary 4.2. *For every quasi-isomorphism relation E there is an $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class $\hat{\mathcal{C}}$ consisting of connected graphs such that $E \sim_B \cong_{\hat{\mathcal{C}}}$ and $z_1 \preceq^{wepi} z_2$ for every $z_1, z_2 \in \hat{\mathcal{C}}$ (in fact, if E itself is an isomorphism relation, then $E \simeq_{cB} \cong_{\hat{\mathcal{C}}}$).*

Proof. Assume $\mathcal{L} = \{P_0, P_1\}$, with P_0, P_1 binary relation symbols. Then it is enough to (bi-)interpret the class $\mathcal{C}$ given by the proof of Theorem 3.1 in a class $\hat{\mathcal{C}}$ of connected graphs as explained in [Hod93, Theorem 5.5.1] and check that even if in general such interpretation does not preserve $\preceq^{wepi}$, it is still true that if there is a weak-epimorphism h from $y \in \mathcal{C}$ onto $x \in \mathcal{C}$ *such that for every n, m in P_i-relation in x ($i = 0, 1$) there are $l \in h^{-1}(n), k \in h^{-1}(m)$ which are in P_i-relation in y* (which is the case for the elements of the class $\mathcal{C}$ constructed above), then there is a weak-epimorphism from the interpretation of y onto the interpretation of x. □

To prove the statement analogous to Theorems 3.3 and 3.4, we need to replace Theorem 3.2 with the following result obtained in [CMMR10, Section 5.1].

Theorem 4.3 ([CMMR10])**.** *For every analytic quasi-order R there is an $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class $\hat{\mathcal{C}}$ consisting of connected graphs such that $\preceq^{wepi}_{\hat{\mathcal{C}}} \sim_B R$ and $\cong_{\hat{\mathcal{C}}} \simeq_{cB} \mathrm{id}(\mathbb{R})$ (and moreover $\approx^{wepi}_{\hat{\mathcal{C}}} \simeq_{cB} E_R$, where $\approx^{wepi}_{\hat{\mathcal{C}}}$ is the equivalence relation associated to $\preceq^{wepi}_{\hat{\mathcal{C}}}$).*

We can now repeat the proofs of Theorems 3.3 and 3.4 and show the following.

Theorem 4.4. *Let E be a quasi-isomorphism relation and R be an arbitrary analytic quasi-order such that either $\mathrm{id}(\mathbb{R}) \leq_B E$ or $E_R \leq_B \mathrm{id}(\omega), E$. Then there is an $\hat{\mathcal{L}}_{\omega_1\omega}$-elementary class $\hat{\mathcal{C}}$ such that $E \sim_B \cong_{\hat{\mathcal{C}}}$ and $R \sim_B \preceq^{wepi}_{\hat{\mathcal{C}}}$ (in fact, if E itself is an isomorphism relation, then $E \simeq_{cB} \cong_{\hat{\mathcal{C}}}$ and $E_R \simeq_{cB} \approx^{wepi}_{\hat{\mathcal{C}}}$).*

Proof. Carry out the proofs of Theorems 3.3 and 3.4 by replacing the original $\mathcal{C}'$ and $\mathcal{C}''$ with the classes $\hat{\mathcal{C}}'$ and $\hat{\mathcal{C}}''$ obtained by applying, respectively, Theorem 4.3 to R and Corollary 4.2 to E and by redefining the quasi-order S on W using $\preceq^{wepi}$ instead of $\sqsubseteq$. By inspecting those proofs, it is enough to show that

(a) given $w_1 = (x_1, z_1, g_1), w_2 = (x_2, z_2, g_2) \in W$, any weak-epimorphism h of $f(w_2) = j_{\hat{\mathcal{L}}}(g_2, x_2 \oplus z_2)$ onto $f(w_1) = j_{\hat{\mathcal{L}}}(g_1, x_1 \oplus z_1)$ (hence, in particular, any isomorphism between $f(w_2)$ and $f(w_1)$) must send elements coming from x_2 (resp. z_2) into elements coming from x_1 (resp. z_1), and
(b) the vertices of $f(w_i)$ coming from x_i (and hence also the vertices of $f(w_i)$ coming from z_i, where $i = 1, 2$) can be recognized in an intrinsic way, i.e. using a property of the structure $f(w_i)$ which does not use any knowledge about w_i.

By the proofs of Theorem 4.3 and [Hod93, Theorem 5.5.1], the vertices of $f(w_i)$ coming from x_i ($i = 1, 2$) are uniquely determined by the property of belonging to a nontrivial (i.e. of size ≥ 3) clique (this gives part (b) above). However, it is easy to check that h must send each nontrivial clique into a clique of greater or equal size (so that vertices of $f(w_2)$ coming from x_2 are sent by h into vertices of $f(w_1)$ coming from x_1) and that, consequently, vertices of $f(w_2)$ coming from z_2 are sent by h into vertices of $f(w_1)$ coming from z_1 because h is surjective and must send connected subgraphs of $f(w_2)$ to connected subgraphs of $f(w_1)$. This gives part (a) and concludes our proof. □

References

[BK96] Howard Becker and Alexander S. Kechris. *The descriptive set theory of Polish group actions*, volume 232 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996. MR1425877 (98d:54068)

[CMMR10] Riccardo Camerlo, Alberto Marcone and Luca Motto Ros. Invariantly universal analytic quasi-orders. Submitted, 2010.

[FMR09] Sy-David Friedman and Luca Motto Ros. Analytic equivalence relations and bi-embeddability. J. Symbolic Logic, 76(1):243–266, 2011.

[FS89] Harvey Friedman and Lee Stanley. A Borel reducibility theory for classes of countable structures. *J. Symbolic Logic*, 54(3):894–914, 1989. MR1011177 (91f:03062)

[Gao09] Su Gao. *Invariant Descriptive Set Theory*, volume 293 of Monographs and Textbooks in Pure and Applied Mathematics. CRC Press, Taylor & Francis Group, 2009. MR2455198

[Gao01] Su Gao. Some dichotomy theorems for isomorphism relations of countable models. *J. Symbolic Logic*, 66(2):902–922, 2001. MR1833486 (2002g:03074)

[Hjo00] Greg Hjorth. *Classification and Orbit Equivalence Relations*, volume 75 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000. MR1725642 (2000k:03097)

[Hod93] W. A. Hodges. *Model Theory*, volume 42 of Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 1993. MR1221741 (94e:03002)

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*, volume 156 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995. MR1321597 (96e:03057)

[LR05] Alain Louveau and Christian Rosendal. Complete analytic equivalence relations. *Trans. Amer. Math. Soc.*, 357(12):4839–4866 (electronic), 2005. MR2165390 (2006d:03081)

[Mos80] Yiannis N. Moschovakis, *Descriptive Set Theory*. North-Holland, Amsterdam-New York-Oxford, 1980. MR561709 (82e:03002)

Mathematisches Institut–Abteilung für Mathematische Logik, Albert-Ludwigs-Universität Freiburg, Eckerstrasse, 1, D-79104 Freiburg im Breisgau, Germany

E-mail address: `luca.motto.ros@math.uni-freiburg.de`