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LARGE SUBGROUPS OF A FINITE GROUP OF EVEN ORDER

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ABSTRACT. It is shown that if G is a group of even order with trivial center such that $|G| > 2|C_G(t)|^3$ for some involution $t \in G$, then there exists a proper subgroup H of G such that $|G| < |H|^2$. If $|G| > |C_G(t)|^3$ and $k(G)$ is the class number of G, then $|G| \leq k(G)^3$.

1. INTRODUCTION

Let G be a group of even order and t an involution of G. R. Brauer and K.A. Fowler proved in [1] that if G is a finite simple group of even order, then $|G| < (|C_G(t)|^2)!$; i.e. there are only finitely many finite simple groups with a given centralizer of an involution. It is well-known that if G has at least two non-conjugate involutions, then the order of G is strictly less than $|C_G(t)|^3$ for some involutio'n t of G. Using the classification of the finite simple groups, H. Yamaki proved in [7] that this also holds in general, even if G has only one class of involutions. Moreover, also using the classification, A. Lev has proved in [5] that every finite group G has a subgroup H such that $|H| > |G|^{1/2}$.

Recently K. Harada and M. Miyamoto in [3] obtained the following bound, the proof of which does not depend on the classification.

Let G be a finite group with a single class of involutions. For $g \in G$ the *extended centralizer* $C^*_G(g)$ of G is defined by $C^*_G(g) = \{x \in G \mid g^x = g^{\pm 1}\}$. Let Ω_π be the set of all π-elements of G, where π is a set of odd primes satisfying a certain restriction connected with the structure of the Grünberg-Kegel graph of G, which we omit here (see [3]).

Let t be an involution of G and $H = C_G(t)$. If the Sylow 2-subgroups of G are non-cyclic and not (generalized) quaternion, then the following holds:

$$|G| < |H|^3 + m_\pi |H|^2,$$

where $m_\pi = \max\{|H|, |t^G \cap C^*_G(g)| \mid 1 \neq g \in \Omega_\pi\}$. In particular, if $|H| = m_\pi$, then $|G| < 2|C_G(t)|^3$.

In the following we will show that a similar result, not depending on the classification of the finite simple groups, can be obtained using the following result due to E.P. Wigner [6], Theorem 2 (see also [2]).

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Let G be a finite group, and for $g \in G$ let $\zeta(g)$ be the number of the square roots of g in G, i.e. the cardinality of the set of the elements $x \in G$ such that $x^2 = g$. Then Wigner's inequality is as follows:

$$\sum_{g\in G} |C_G(g)|^2 \geq \sum_{g\in G} \zeta(g)^3. \tag{1}$$

Using this we will prove the following.

Theorem 1.1. *Let G be a finite group of even order with $O_2(Z(G)) = 1$ and let t be an involution in G. Then either $|G| < 2|C_G(t)|^3$ or there exists a proper subgroup H of G such that $|G| \leq |H|^2$.*

Another result of this type is the following:

Theorem 1.2. *Let G be a finite group of even order and let t be an involution in G. If $|G| > 2|C_G(t)|^3$, then there exists an element $1 \neq x \in G$ such that $|G : C_G(x)| \leq k(G) - 1$, where $k(G)$ is the class number of G. Moreover, either $|G| < |C_G(t)|^3$ or $|G| \leq k(G)^3$.*

2. Proofs

Proof of Theorem 1.1. Clearly $\zeta(1)$ is the number of elements of order 2 or 1, and so $|G : C_G(t)| < \zeta(1)$. Hence, by Wigner's inequality, we have

$$\sum_{g\in G} |C_G(g)|^2 > \frac{|G|^3}{|C_G(t)|^3}. \tag{2}$$

The left side of (2) can be rewritten as $\sum_{i=1}^{k} |G||C_G(y_i)|$, where $k = k(G)$ and the y_i are the representatives of the conjugacy classes of G, $y_k = 1$.

Suppose first that the center of G is trivial. Then we have

$$(k-1)|G||\ C_G(x_0)| + |G|^2 > \frac{|G|^3}{|C_G(t)|^3}, \tag{3}$$

where $|C_G(x_0)| = \max_{i=1}^{k-1} |C_G(y_i)|$. Dividing both sides by $|G|$, we obtain

$$(k-1)|C_G(x_0)| > |\ G|(\frac{|G|}{|C_G(t)|^3} - 1). \tag{4}$$

If $|G| > 2|C_G(t)|^3$, then $(k-1)|C_G(x_0)| > |G|$. Hence

$$|G : C_G(x_0)| < (k-1). \tag{5}$$

Using the class equation, we have $(y_k = 1)$

$$\sum_{i=1}^{k} 1/|C_G(y_i)| = 1 = \sum_{i=1}^{k-1} 1/|C_G(y_i)| + 1/|G|. \tag{6}$$

This implies that $(k-1)/|C_G(x_0)| + 1/|G| \leq 1$ and thus $|C_G(x_0)| > k-1$. By (5) this gives the following bound:

$$|G| < (k-1)|C_G(x_0)| < |C_G(x_0)|^2. \tag{7}$$

Thus in the case $Z(G) = 1$ Theorem 1.1 is proved.

Let K be the hypercenter of G. Since $O_2(Z(G)) = 1$, then K is of odd order. Denote by $\overline{G}$ the group G/K and use the standard bar convention. If $|\bar{G}| \leq 2|C_{\bar{G}}(\bar{t})|^3$, then $|C_G(t)| = |K||C_{\bar{G}}(\bar{t})|$ and $|G| = |K||\ \bar{G}| < 2|K|^3|C_{\bar{G}}(\bar{t})|^3$, and there is nothing to prove.

Hence we may assume that $|\bar{G}| > 2|\ C_G(t)|^3$. By the above consideration, there exists a proper subgroup $\bar{H}$ in $\bar{G}$ such that $|\bar{H}|^2 > |\bar{G}|$. Hence the full preimage H of $\bar{H}$ in G has the property $|H|^2 > |G|$. The theorem is proved.

Proof of Theorem 1.2. Recall that the conjugacy class with representative a is called *real* if a is conjugate with its inverse. If G is a finite group, we denote by $k_R(G)$ the number of all real classes of G.

We need the following equality, also due to E.P. Wigner (Theorem 1 in [6]):

$$\sum_{g\in G} \zeta(g)^2 = k_R(/G)|G|. \tag{8}$$

In particular, for an involution t in G the following holds:

$$\frac{|G|}{|C_G(t)|^2} \le k_R(G) \le k(G).$$

Using the class equality, we see that there exists an element $x \in G$ such that $|C_G(x)| > k(G)$. Therefore

$$\frac{|G|}{|C_G(t)|^2} \le k(G) < |C_G(x)|. \tag{9}$$

If $|C_G(t)| > k(G)$, we have that $|G| < |C_G(t)|^3$. If $|G| > |C_G(t)|^3$, then $|C_G(t)| \le k(G)$, and this implies $|G| \le k(G)^3$.

Assume that $|G| > 2|C_G(t)|^3$ for some involution t in G. Then, as in the proof of the first part of Theorem 1, there exists a non-trivial element $x \in G$ such that $|G : C_G(x)| \le k(G) - 1$.

Remark 1. If G is a non-abelian finite simple group, then it follows from (9) that $|G| \le (|C_G(t)|^2)!$.

Remark 2. Using the classification of finite simple groups it was proved in [4] that there is no non-abelian finite simple group G such that $|G| < k(G)^3$, with the exception of $G = L_2(q)$ with q even. However, in this case the centralizer of an involution is of order q. Therefore for every non-abelian finite simple group G and every involution $t \in G$ we have $|G| \le |C_G(t)|^3$.

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