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ON COMPACTNESS OF THE $\overline{\partial}$-NEUMANN PROBLEM AND HANKEL OPERATORS

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ABSTRACT. Let $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, where Ω_1 and Ω_2 are two smooth bounded pseudoconvex domains in $\mathbb{C}^n, n \geq 3$, such that $\overline{\Omega}_2 \subset \Omega_1$. Assume that the $\overline{\partial}$-Neumann operator of Ω_1 is compact and the interior of the Levi-flat points in the boundary of Ω_2 is not empty (in the relative topology). Then we show that the Hankel operator on Ω with symbol ϕ, H^{Ω}_{ϕ}, is compact for every $\phi \in C(\overline{\Omega})$ but the $\overline{\partial}$-Neumann operator on Ω is not compact.

Let Ω be a domain in $\mathbb{C}^n$ and $A^2(\Omega)$ denote the Bergman space on Ω, the space of square integrable holomorphic functions on Ω. The Bergman projection, the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$, is denoted by P^{Ω} and the Hankel operator with symbol $\phi \in L^{\infty}(\Omega)$, denoted by H^{Ω}_{ϕ}, is defined as $H^{\Omega}_{\phi}(f) = \phi f - P^{\Omega}(\phi f)$ for $f \in A^2(\Omega)$.

The $\overline{\partial}$-Neumann problem is solving $\Box u = v$, where $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, on square integrable $(0,1)$-forms and $\overline{\partial}^*$ is the Hilbert space adjoint of $\overline{\partial}$. We will denote the solution operator to $\Box$ on a domain Ω, the $\overline{\partial}$-Neumann operator on Ω, by N^{Ω}. On bounded pseudoconvex domains, Hörmander [Hör65] showed that N is a bounded operator on $L^2_{(0,1)}(\Omega)$, and Kohn [Koh63] showed that $P^{\Omega} = I - \overline{\partial}^* N^{\Omega}\overline{\partial}$. Therefore, $H^{\Omega}_{\phi}(f) = \overline{\partial}^* N^{\Omega}(f\overline{\partial}\phi)$ for $f \in A^2(\Omega)$ and $\phi \in C^1(\overline{\Omega})$. We refer the reader to [CS01, Str10] for more information about the $\overline{\partial}$-Neumann problem and to [ČŞ09] (and the references therein) for more information on compactness of Hankel operators on Bergman spaces.

Given Kohn's formula, it is natural to expect strong connections between N^{Ω} and Hankel operators on $A^2(\Omega)$. For example, if Ω is bounded and pseudoconvex, and if N^{Ω} is compact on $L^2_{(0,1)}(\Omega)$, then H^{Ω}_{ϕ} is compact on $A^2(\Omega)$ for all $\phi \in C(\overline{\Omega})$ (see [Has08, Theorem 3] and [Str10, Proposition 4.1]). We are interested in the converse, which is a question of Fu and Straube [FS01, Remark 2]: does compactness of H^{Ω}_{ϕ} on $A^2(\Omega)$ for all $\phi \in C(\overline{\Omega})$ imply that N^{Ω} is compact on $L^2_{(0,1)}(\Omega)$? The answer to this question is still open in general. However, if Ω is allowed to be non-pseudoconvex we can show that the answer is no (see Theorem 1 below).

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We call Ω an annulus type domain if $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, where Ω_1 and Ω_2 are smooth, bounded, pseudoconvex, and $\overline{\Omega}_2 \subset \Omega_1$. The following theorem of Shaw, contained in [Sha10, Theorem 3.5], guarantees that the $\overline{\partial}$-Neumann operator exists on annulus type domains in $\mathbb{C}^n$ for $n \geq 3$ and it is connected to the Bergman projection the same way as it is on bounded pseudoconvex domains.

Theorem (Shaw)**.** *Let $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, where Ω_1 and Ω_2 are two smooth bounded pseudoconvex domains in $\mathbb{C}^n, n \geq 3$, such that $\overline{\Omega}_2 \subset \Omega_1$. Then*

i) *N_q^Ω exists on $L^2_{(0,q)}(\Omega)$ for $1 \leq q \leq n-2$,*
ii) *$\overline{\partial}^* N^\Omega$ is the canonical solution operator for $\overline{\partial}$,*
iii) *$P^\Omega = I - \overline{\partial}^* N^\Omega \overline{\partial}$.*

In fact Shaw ([Sha10, Theorem 3.5]) showed that the $\overline{\partial}$-Neumann operator is bounded on $(0,1)$-forms for $n \geq 2$. However, the space of harmonic forms $\mathcal{H}^\Omega_{(0,1)}$, defined in the next section, is infinite dimensional when $n = 2$ and trivial when $n \geq 3$. Hence, when $n = 2$, items ii) and iii) in Shaw's theorem above are not valid.

The following theorem is our main result. We note that $B(p,r)$ denotes the open ball centered at p with radius r, and a point p, in the boundary of a smooth domain $\Omega \subset \mathbb{C}^n$, is called Levi-flat if the Levi form of Ω, the restriction of the complex Hessian of a defining function onto a complex tangent space, is constant zero at p. We denote the boundary of a domain Ω by $b\Omega$.

Theorem 1. *Let $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, where Ω_1 and Ω_2 are two smooth bounded pseudoconvex domains in $\mathbb{C}^n, n \geq 3$, such that $\overline{\Omega}_2 \subset \Omega_1$. Assume that the $\overline{\partial}$-Neumann operator N^{Ω_1} is compact on $L^2_{(0,1)}(\Omega_1)$ and that there exists a ball, $B(p,r)$ centered at $p \in b\Omega_2$ with radius $r > 0$ such that $B(p,r) \cap b\Omega_2$ is a Levi-flat surface. Then the Hankel operator H^Ω_ϕ is compact on $A^2(\Omega)$ for every $\phi \in C(\overline{\Omega})$ but the $\overline{\partial}$-Neumann operator N^Ω is not compact on $L^2_{(0,1)}(\Omega)$.*

See Remark 3 for an explanation of why we stated the above theorem for domains in $\mathbb{C}^n$ for $n \geq 3$.

Remark 1. Hankel operators are closely connected to a very important class of operators called Toeplitz operators. The Toeplitz operator on $A^2(\Omega)$ with symbol $\phi \in L^\infty(\Omega)$, denoted by T^Ω_ϕ, is defined as $T^\Omega_\phi f = P^\Omega(\phi f) = \phi f - H^\Omega_\phi f$ for $f \in A^2(\Omega)$. Let $\phi \in C(\overline{\Omega})$ such that $\phi(z) \neq 0$ for $z \in b\Omega_1$. Choose $\psi, \phi_1 \in C(\overline{\Omega})$ such that $\phi_1(z) = \phi(z)$ for $z \in b\Omega_1$ and $|\phi_1| > 0$ on $\overline{\Omega}$, and $\psi = 1/\phi_1$. Then

$$(T^\Omega_\phi - T^\Omega_{\phi_1} + T^\Omega_{\phi_1})T^\Omega_\psi = T^\Omega_{\phi-\phi_1}T^\Omega_\psi + T^\Omega_{\phi_1\psi} - P^\Omega M_{\phi_1} H^\Omega_\psi = I + K,$$

where $K = T^\Omega_{\phi-\phi_1}T^\Omega_\psi - P^\Omega M_{\phi_1}H^\Omega_\psi$. Now assume that Ω, Ω_1, and Ω_2 are as in Theorem 1. Then K is a compact operator ($T^\Omega_{\phi-\phi_1}$ is compact because $\phi_1 - \phi = 0$ on the outer boundary of Ω and H^Ω_ψ is compact by Theorem 1). Hence, T^Ω_ϕ is Fredholm for any $\phi \in C(\overline{\Omega})$ with the property that $\phi(z) \neq 0$ for $z \in b\Omega_1$. The Fredholm property of Toeplitz operators on some pseudoconvex domains in $\mathbb{C}^n$ has been studied by several authors (see, for example, [Ven72, HI97]).

Remark 2. Hankel operators can also be expressed as commutators of the Bergman projection with multiplication operators. These commutators proved to be useful in the proof of the complex version of Hilbert's seventeenth problem (see [CD97]). For more information about relations between the commutators and the $\overline{\partial}$-Neumann problem we refer the reader to [Str10, Chapter 4.1]. The computation is as follows:

$$\langle P^{\Omega}(\phi g), h\rangle_{L^2(\Omega)} = \langle \phi g, h\rangle_{L^2(\Omega)} = \langle g, H^{\Omega}_{\overline{\phi}} h\rangle_{L^2(\Omega)} = \langle (H^{\Omega}_{\overline{\phi}})^* g, h\rangle_{L^2(\Omega)}$$

for $\phi \in L^\infty(\Omega), h \in A^2(\Omega)$, and $g \perp A^2(\Omega)$. Hence for any $f \in L^2(\Omega)$ we have

$$\begin{aligned}[][M_\phi, P^{\Omega}](f) &= [M_\phi, P^{\Omega}](P^{\Omega} f) + [M_\phi, P^{\Omega}]((I - P^{\Omega})f) \\ &= H^{\Omega}_{\phi} P^{\Omega} f - P^{\Omega} M_\phi (I - P^{\Omega}) f \\ &= H^{\Omega}_{\phi} P^{\Omega} f - (H^{\Omega}_{\overline{\phi}})^* (I - P^{\Omega}) f.\end{aligned}$$

When $f \in A^2(\Omega), P^{\Omega} f = f$, and $(I - P^{\Omega})f = 0$, whence $H^{\Omega}_{\phi} = [M_\phi, P^{\Omega}]$ on $A^2(\Omega)$. Note that $(H^{\Omega}_{\overline{\phi}})^* : L^2(\Omega) \to A^2(\Omega)$ and it is compact if and only if $H^{\Omega}_{\overline{\phi}}$ is compact. Therefore, H^{Ω}_{ϕ} is compact on $A^2(\Omega)$ if and only if $[M_\phi, P^{\Omega}]$ is compact on $L^2(\Omega)$. We note that similar calculations as well as related issues appeared in [Has08] on pseudoconvex domains (see also [CD97, FS01]).

Corollary 1. *Let Ω, Ω_1, and Ω_2 be as in Theorem 1. Then the commutator $[M_\phi, P^{\Omega}]$ is compact on $L^2(\Omega)$ for every $\phi \in C(\overline{\Omega})$ but the $\overline{\partial}$-Neumann operator N^{Ω} is not compact on $L^2_{(0,1)}(\Omega)$.*

Example 1. Here, we give an explicit example. Let $\lambda_1(t) = 0$ for $t \leq 0$ and $\lambda_1(t) = e^{-1/t}$ for $t > 0$ and

$$\lambda(z_1, z_2, z_3) = \lambda_1\left(|z_1|^2 - \frac{1}{4}\right) + \lambda_1\left(|z_2|^2 - \frac{1}{4}\right) + \lambda_1\left(|z_3|^2 - \frac{1}{4}\right) - e^{-3}.$$

One can check that λ_1 is a convex function on $(-\infty, 1/2)$. Let us define

$$\Omega = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 < 9 \text{ and } \lambda(z_1, z_2, z_3) > 0\right\}.$$

So $\Omega_1 = B(0,3), \Omega_2 = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \lambda(z_1, z_2, z_3) < 0\right\}$, and $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ is a smooth bounded annulus type domain in $\mathbb{C}^3$. By construction $b\Omega \cap B((0, 0, \sqrt{7}/\sqrt{12}), 1/3)$ is a Levi-flat surface. Then Theorem 1 and Corollary 1 imply that $[M_\phi, P^{\Omega}]$ is compact on $L^2(\Omega)$ for every $\phi \in C(\overline{\Omega})$ (hence H^{Ω}_{ϕ} is compact on $A^2(\Omega)$ for every $\phi \in C(\overline{\Omega})$) but N^{Ω} is not compact on $L^2_{(0,1)}(\Omega)$.

Proof of Theorem 1

Let

$$\mathcal{H}^{\Omega}_{(0,1)} = \ker(\Box^{\Omega}) = \{u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) : \overline{\partial} u = 0,\ \overline{\partial}^* u = 0\} \subset L^2_{(0,1)}(\Omega).$$

We call $\mathcal{H}^{\Omega}_{(0,1)}$ the space of harmonic $(0,1)$-forms and denote by H^{Ω} the orthogonal projection from $L^2_{(0,1)}(\Omega)$ onto $\mathcal{H}^{\Omega}_{(0,1)}$. The following lemmas will be useful in the proof of Theorem 1.

Lemma 1. *Let Ω be an annulus type domain in $\mathbb{C}^n$ for $n \geq 2$. Then N^{Ω} is compact on $L^2_{(0,1)}(\Omega)$ if and only if for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|u\|^2 \leq \varepsilon\left(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2\right) + \|H^{\Omega} u\|^2 + C_\varepsilon \|u - H^{\Omega} u\|^2_{-1} \tag{1}$$

for $u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*) \subset L^2_{(0,1)}(\Omega)$.

Proof. We note that $\overline{\partial}$ has closed range in $L^2_{(0,1)}(\Omega)$ (see [Sha10, Theorem 3.3]). Let us define

$$\Gamma = Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*) \cap \left(\mathcal{H}^{\Omega}_{(0,1)}\right)^{\perp}$$

($X^{\perp}$ denotes the orthogonal complement of X) and equip the space Γ with the graph norm. That is, $\|u\|^2_{\Gamma} = \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2$. Then the embedding $j : \Gamma \hookrightarrow L^2_{(0,1)}(\Omega)$ is continuous [Sha10]. Furthermore, $N = j \circ j^*$. (For a proof of this, see [Str10, Theorem 2.9]. Although pseudoconvexity is assumed in [Str10, Theorem 2.9], its proof applies in our situation as well because j is a bounded operator.) Hence N is compact if and only if j is compact and compactness of j is equivalent to the following estimate ([Str10, Proposition 4.2]): for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|u\|^2 \leq \varepsilon \left(\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2\right) + C_\varepsilon \|u\|^2_{-1} \text{ for } u \in \Gamma.$$

One can substitute $u - H^{\Omega}u$ instead of u above to show that the inequality above is equivalent to (1). □

Lemma 2. *Let Ω be an annulus type domain in $\mathbb{C}^n$ for $n \geq 3$ such that N^{Ω} exists and is compact on $L^2_{(0,1)}(\Omega)$. Let p be a boundary point of Ω and $r > 0$ such that $U = \Omega \cap B(p,r)$ is a pseudoconvex domain. Then N^U is compact on $L^2_{(0,1)}(U)$.*

Proof. We note that since $n \geq 3$ the space $\mathcal{H}^{\Omega}_{(0,1)}$ is trivial and the proof is essentially contained in [Str10, Proposition 4.4] once we know that $\mathcal{H}^{\Omega}_{(0,1)}$ is trivial. However, we will give the proof here for the convenience of the reader.

Lemma 1 implies that compactness of N^{Ω} is equivalent to the following estimate: for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|u\|^2_{\Omega} \leq \varepsilon(\|\overline{\partial}u\|^2_{\Omega} + \|\overline{\partial}^* u\|^2_{\Omega}) + C_\varepsilon \|u\|^2_{-1,\Omega} \quad \text{for} \quad u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*).$$

Let $\lambda_\varepsilon(z) = \dfrac{\|z-p\|^2 - r^2}{\varepsilon}$. One can check that $-2r + \varepsilon \leq \lambda_\varepsilon(z) \leq 2r + \varepsilon$ and

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda_\varepsilon(z)}{\partial z_j \partial \overline{z}_k} \xi_j \overline{\xi}_k \geq \frac{1}{\varepsilon}\|\xi\|^2$$

for $z \in V_\varepsilon = \{z \in \mathbb{C}^n \mid \text{dist}(z, bB(p,r)) < \varepsilon\}$ for $\xi \in \mathbb{C}^n$. Now we choose ϕ_ε as a smooth cutoff function ($0 \leq \phi_\varepsilon \leq 1$), $\phi_\varepsilon \equiv 1$ near $bB(p,r)$ and supported in V_ε. The triangle inequality implies that

$$\|u\|^2_U \leq 2\|\phi_\varepsilon u\|^2_U + 2\|(1-\phi_\varepsilon)u\|^2_U \text{ for } u \in L^2_{(0,1)}(U). \tag{2}$$

Let $u \in \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*) \subset L^2_{(0,1)}(U)$. Then $(1-\phi_\varepsilon)u \in \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*) \subset L^2_{(0,1)}(\Omega)$. Since the domain U is not C^2-smooth, a direct application of the Morrey-Kohn-Hörmander formula is not possible. However, one can use the Morrey-Kohn-Hörmander formula (with weight $\lambda_\varepsilon + \psi$) with the exhaustion procedure developed in [Str97] (see also [Str10, Corollary 2.15]) together with the fact that $\phi_\varepsilon u$ belongs to the domain of $\overline{\partial}^*$ on U to show that

$$\|\phi_\varepsilon u\|^2_U \lesssim \varepsilon(\|\overline{\partial}\,(\phi_\varepsilon u)\,\|^2_U + \|\overline{\partial}^*\,(\phi_\varepsilon u)\,\|^2_U). \tag{3}$$

In the inequality above we used generalized constants. That is, $A \lesssim B$ denotes that $A \leq cB$ where $c > 0$ is independent of quantities of interest. Thus, from (2) and (3) we get

$$\begin{aligned}\|u\|_U^2 &\lesssim \varepsilon \left(\|\overline{\partial}\,(\phi_\varepsilon u)\,\|_U^2 + \|\overline{\partial}^*\,(\phi_\varepsilon u)\,\|_U^2 \right) + \|(1-\phi_\varepsilon)u\|_U^2 \\ &\lesssim \varepsilon \left(\|\overline{\partial} u\|_U^2 + \|\overline{\partial}^* u\|_U^2 + \|(\nabla\phi_\varepsilon)u\|_U^2 \right) + \|(1-\phi_\varepsilon)u\|_U^2.\end{aligned}$$

$(1-\phi_\varepsilon)u$ and $\nabla\phi_\varepsilon u$ can be viewed as forms on Ω in $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$. Let us choose $\chi_\varepsilon \in C_0^\infty(B(p,r))$ such that $\chi_\varepsilon \equiv 1$ on the union of the support of $\nabla\phi_\varepsilon$ and the support of $1-\phi_\varepsilon$. Then

$$\|u\|_U^2 \lesssim \varepsilon \left(\|\overline{\partial} u\|_U^2 + \|\overline{\partial}^* u\|_U^2 \right) + C_{\phi_\varepsilon} \|\chi_\varepsilon u\|_U^2 \tag{4}$$

for $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \subset L^2_{(0,1)}(U)$. Now, we will try to estimate the last term in (4). We note that $\chi_\varepsilon u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \subset L^2_{(0,1)}(\Omega)$ for any $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \subset L^2_{(0,1)}(U)$. Compactness of N^Ω on $L^2_{(0,1)}(\Omega)$ implies that for all $\varepsilon' > 0$ there exists $C_{\varepsilon'} > 0$ such that

$$\begin{aligned}\|\chi_\varepsilon u\|_\Omega^2 &\leq \varepsilon' \left(\|\overline{\partial}(\chi_\varepsilon u)\|_\Omega^2 + \|\overline{\partial}^*(\chi_\varepsilon u)\|_\Omega^2 \right) + C_{\varepsilon'} \|\chi_\varepsilon u\|_{-1,\Omega}^2 \\ &\leq \varepsilon' (\|\chi_\varepsilon \overline{\partial} u\|_\Omega^2 + \|\chi_\varepsilon \overline{\partial}^* u\|_\Omega^2 + \|\nabla\chi_\varepsilon u\|_\Omega^2) + C_{\varepsilon'} \|u\|_{-1,U}^2.\end{aligned} \tag{5}$$

Now, let us estimate $\|\nabla\chi_\varepsilon u\|_\Omega^2$ in (5):

$$\|\nabla\chi_\varepsilon u\|_\Omega^2 \lesssim \|\nabla\chi_\varepsilon u\|_U^2 \lesssim C_\varepsilon \|u\|_U^2 \lesssim C_\varepsilon \left(\|\overline{\partial} u\|_U^2 + \|\overline{\partial}^* u\|_U^2 \right).$$

In the last step we used the basic estimate on U, $\|u\|_U^2 \leq C(\|\overline{\partial} u\|_U^2 + \|\overline{\partial}^* u\|_U)$.

Thus, combining (5) and the discussion after it with (4), we get a compactness estimate on U. That is, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|u\|_U^2 \lesssim \varepsilon \left(\|\overline{\partial} u\|_U^2 + \|\overline{\partial}^* u\|_U^2 \right) + C_\varepsilon \|u\|_{-1,U}^2 \tag{6}$$

for all $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \subset L^2_{(0,1)}(U)$. □

Remark 3. We chose the domain Ω in $\mathbb{C}^n$ for $n \geq 3$ because we do not know if the localization in the proof of Lemma 2 is possible when $n = 2$. If $\Omega \subset \mathbb{C}^2$ is an annulus type domain, then $\mathcal{H}^\Omega_{(0,1)}$ is an infinite dimensional space [Sha10, Theorem 3.5] whereas $\mathcal{H}^\Omega_{(0,1)}$ is trivial when Ω is pseudoconvex.

In the following lemma, R_V denotes the restriction operator onto V.

Lemma 3. *Let $\Omega = \Omega_1 \backslash \overline{\Omega}_2$, where Ω_1 and Ω_2 are two smooth bounded pseudoconvex domains in $\mathbb{C}^n, n \geq 3$, such that $\overline{\Omega}_2 \subset \Omega_1$ and $\phi \in C^1(\overline{\Omega}_1)$. Then $H^{\Omega_1}_\phi$ is compact on $A^2(\Omega_1)$ if and only if $H^\Omega_{R_\Omega(\phi)}$ is compact on $A^2(\Omega)$.*

Proof. Let us prove the necessity first. By Hartogs' extension theorem there exists a unique bounded extension operator $E^{\Omega_1}_\Omega : A^2(\Omega) \to A^2(\Omega_1)$. One can check that $R_\Omega H^{\Omega_1}_\phi E^{\Omega_1}_\Omega f$ solves $\overline{\partial} u = f\overline{\partial}\phi$ on Ω. Furthermore, since $H^\Omega_{R_\Omega(\phi)} f$ is the canonical solution (the solution with minimal L^2 norm) for $\overline{\partial} u = f\overline{\partial}\phi$ we have

$$(I - P^\Omega) R_\Omega H^{\Omega_1}_\phi E^{\Omega_1}_\Omega = H^\Omega_{R_\Omega(\phi)}.$$

Therefore, compactness of $H^{\Omega_1}_\phi$ on $A^2(\Omega_1)$ implies that $H^\Omega_{R_\Omega(\phi)}$ is compact on $A^2(\Omega)$.

To prove the converse assume that $H^{\Omega}_{R_\Omega(\phi)}$ is compact on $A^2(\Omega)$ and let U be a neighborhood of $p \in b\Omega_1$ such that $U \cap \Omega_1 = U \cap \Omega$ is a domain. Then (i) in [ČŞ09, Proposition 1(i)] implies that $H^{\Omega\cap U}_{R_{\Omega\cap U}(\phi)} R_{\Omega\cap U}$ is compact on $A^2(\Omega)$. We note that even though [ČŞ09, Proposition 1] is stated for pseudoconvex domains, (i) is still true for general domains. However, one can check that

$$(I - P^{\Omega_1\cap U}) H^{\Omega\cap U}_{R_{\Omega\cap U}(\phi)} R_{\Omega\cap U} = H^{\Omega_1\cap U}_{R_{\Omega_1\cap U}(\phi)} R_{\Omega_1\cap U}$$

on $A^2(\Omega_1)$ and hence $H^{\Omega_1\cap U}_{R_{\Omega_1\cap U}(\phi)} R_{\Omega_1\cap U}$ is compact on $A^2(\Omega_1)$. Now (ii) [ČŞ09, Proposition 1(ii)] implies that $H^{\Omega_1}_{\phi}$ is compact on $A^2(\Omega_1)$. □

Remark 4. We note that compactness of N^{Ω_1} on $A^2_{(0,1)}(\Omega_1)$ is equivalent to compactness of N^{Ω} on $A^2_{(0,1)}(\Omega)$. This can be seen as follows:

Range's formula, $N^{\Omega_1} = \left(\overline{\partial}^* N_2^{\Omega_1}\right)\left(\overline{\partial}^* N_2^{\Omega_1}\right)^* + \left(\overline{\partial}^* N^{\Omega_1}\right)^*\left(\overline{\partial}^* N^{\Omega_1}\right)$, together with the fact that $\left(\overline{\partial}^* N_2^{\Omega_1}\right)^* u = 0$ for $u \in A^2_{(0,1)}(\Omega_1)$, implies that $N^{\Omega_1} u = \left(\overline{\partial}^* N^{\Omega_1}\right)^*\left(\overline{\partial}^* N^{\Omega_1}\right) u$ for $u \in A^2_{(0,1)}(\Omega_1)$. Hence, $N^{\Omega_1}|_{A^2_{(0,1)}(\Omega_1)}$ is compact if and only if $\overline{\partial}^* N^{\Omega_1}|_{A^2_{(0,1)}(\Omega_1)}$ is compact. (Here $f|_X$ denotes the restriction of the operator f onto the space X.) Similarly, one can show that $N^{\Omega}|_{A^2_{(0,1)}(\Omega)}$ is compact if and only if $\overline{\partial}^* N^{\Omega}|_{A^2_{(0,1)}(\Omega)}$ is compact. On the other hand, Lemma 3 implies that compactness of $\overline{\partial}^* N^{\Omega_1}|_{A^2_{(0,1)}(\Omega_1)}$ is equivalent to compactness of $\overline{\partial}^* N^{\Omega}|_{A^2_{(0,1)}(\Omega)}$.

We will need the following theorem of Catlin. For a proof we refer the reader to the proof of Proposition 9 in [FS01] (see also [ŞS06]). We note that even though Catlin's Theorem in [FS01] is stated in $\mathbb{C}^2$, the same proof works for the following version in $\mathbb{C}^n$.

Theorem (Catlin)**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$, with Lipschitz boundary. Assume that $b\Omega$ contains an $(n-1)$-dimensional complex manifold. Then N^{Ω} is not compact on $L^2_{(0,1)}(\Omega)$.*

Proof of Theorem 1. The assumption that N^{Ω_1} is compact implies that $H^{\Omega_1}_{\phi}$ is compact for all $\phi \in C^1(\overline{\Omega}_1)$ (see [FS01, Proposition 4], [Str10, Proposition 4.1], and [Has08, Theorem 3]). Since any $\phi \in C^1(\overline{\Omega})$ can be extended as a C^1 function on $\mathbb{C}^n$, Lemma 3 implies that H^{Ω}_{ϕ} is compact for all $\phi \in C^1(\overline{\Omega})$. However, one can approximate any $\phi \in C(\overline{\Omega})$ uniformly on $\overline{\Omega}$ by C^1 functions. Therefore, we conclude that H^{Ω}_{ϕ} is compact on $A^2(\Omega)$ for all $\phi \in C(\overline{\Omega})$.

Now we will show that N^{Ω} is not compact. Shaw's Theorem, stated in the introduction, implies that N^{Ω} is a bounded operator on $L^2_{(0,1)}(\Omega)$. Assume that N^{Ω} is compact on $L^2_{(0,1)}(\Omega)$. Let us choose $p \in b\Omega_2$ and $r > 0$ so that $U = \Omega \cap B(p,r)$ is a domain that does not intersect $b\Omega_1$ and the (inner) boundary of Ω in $B(p,r)$ is Levi-flat. Hence U is a non-smooth bounded pseudoconvex domain. Lemma 2 implies that if N^{Ω} is compact on $L^2_{(0,1)}(\Omega)$, then N^U is compact on $L^2_{(0,1)}(U)$. Compactness of N^U implies that $\overline{\partial}$ has a compact solution operator on

$L^2_{(0,1)}(U)$. However, this contradicts Catlin's Theorem stated above. Hence, N^{Ω} is not compact on $L^2_{(0,1)}(\Omega)$. This contradiction with the assumption that N^{Ω} is compact completes the proof. □

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