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REAL QUADRATIC FUNCTION FIELDS OF RICHAUD-DEGERT TYPE WITH IDEAL CLASS NUMBER ONE

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Abstract. We determine all real quadratic function fields of Richaud-Degert type with ideal class number one.

1. Introduction

It is an interesting problem to determine the class numbers of real quadratic number fields. Mollin and Williams ([MW]) determined all the possible real quadratic fields of Richaud-Degert type with class number one under the assumption of generalized Riemann hypothesis. Sasaki ([Sa2]) introduced the generalized Ono invariant and gave a criterion for a real quadratic field to have class number one using these invariants, extending Rabinovitch's theorem for imaginary quadratic fields. This criterion is very useful to determine real quadratic fields of particular type with class number one, e.g., Richaud-Degert type. In Sasaki's theory, the continued fraction expansion of $\sqrt{d}$ or $\frac{1+\sqrt{d}}{2}$ plays a crucial role, and the continued fraction expansion of $\sqrt{d}$ or $\frac{1+\sqrt{d}}{2}$ of Richaud-Degert type is rather simple. The usefulness of Richaud-Degert type lies mostly on the fact that the period of the continued fraction expansion of the Richaud-Degert type quadratic irrational is short, and so the fundamental unit of the Richaud-Degert type real quadratic field is small.

In this article we study the analogous problems in function fields. Let $k = \mathbb{F}_q(T)$ and $D = A^2 + a$ with $A \in \mathbb{A} = \mathbb{F}_q[T]$ $a \in \mathbb{F}_q^*$ when q is odd, which is called *of Chowla type*. Feng and Hu ([FH]) gave a criterion for a real quadratic function field $k(\sqrt{D})$ to be of Chowla type with ideal class number one and determined all such D so that the ideal class number of $k(\sqrt{D})$ is 1. In this paper we extend this result to Richaud-Degert type, that is, D is of the form $A^2 + B$ with $B \mid A$, following Sasaki's idea. For this, we defined the analogue of generalized Ono invariants and obtained a function field analogue of Sasaki's criterion. Wang and Zhang ([WZ1], [WZ2], [WZ3]) also studied the ideal class groups of real quadratic function fields, and Richaud-Degert type real quadratic function fields provide nice examples of their study.

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In the final section, the characteristic 2 case is considered, and we determine all real quadratic function fields of Richaud-Degert type with ideal class number 1 in this case too.

2. Continued fractions

In this section we review some basic properties of continued fractions in function fields. For the details we refer to C. Friesen's thesis ([F1]) or A. Stein's technical report on continued fractions in real quadratic function fields ([Ste]).

Let q be a power of an odd prime. We fix the following notation:

$k = \mathbb{F}_q(X), \qquad k_\infty = \mathbb{F}_q((1/X)), \quad \mathbb{A} = \mathbb{F}_q[X].$

$x \in k_\infty$, $[x] :=$ the polynomial part of x.

$\|x\| =$ the normalized absolute value on k_∞.

Note that if $A \in \mathbb{A}$, then $\|A\| = q^{\deg A}$. Then:

$f_0(x) := x$ and, if $f_i(x) \neq [f_i(x)]$, let $f_{i+1}(x) := (f_i(x) - [f_i(x)])^{-1}$.

$f_n(x)$ is called the nth *iterate* of x, and $a_n = a_n(x) := [f_n(x)]$ is called the nth *partial denominator* of x. Write

$$[a_0; a_1, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_n}}}.$$

Let

$$Q_{-1}(x) := 0, \quad Q_0(x) := 1, \quad P_{-1}(x) := 1, \quad P_0(x) := a_0(x),$$
$$Q_n(x) = a_n(x)Q_{n-1}(x) + Q_{n-2}(x), \qquad P_n(x) = a_n(x)P_{n-1}(x) + P_{n-2}(x).$$

Then

$$\frac{P_n}{Q_n} = [a_0; a_1, \ldots, a_n].$$

Lemma 2.1 ([F1, Lemma 1.0]). *For $x \in k_\infty \setminus k$,*

$$x = \frac{P_{n-1}f_n + P_{n-2}}{Q_{n-1}f_n + Q_{n-2}},$$
$$\|Q_n\| = \|a_1 a_2 \cdots a_n\|,$$
$$Q_n P_{n-1} - P_n Q_{n-1} = (-1)^n,$$
$$\|x - P_n/Q_n\| = \frac{1}{\|Q_n Q_{n-1}\|},$$
$$\|P_{n+1} - Q_{n+1}x\| < \|P_n - Q_n x\|.$$

Lemma 2.2 ([F1, Lemma 1.6, 1.7]). *Let $x \in k_\infty \setminus k$ and $A, B \in \mathbb{A}$. Then:*

i) *If $\|x - A/B\| < \|x - P_n/Q_n\|$, then $\|B\| > \|Q_n\|$.*
ii) *If $\|xB - A\| < \|xQ_n - P_n\|$, then $\|B\| \geq \|Q_{n+1}\|$.*
iii) *If $\|x - A/B\| < 1/\|B\|^2$, then $A = cP_n$ and $B = cQ_n$ for some $n \geq 0$ and $c \in \mathbb{F}_q^*$.*

An extension K of k is called *real* if the infinite place ∞ of k splits completely in K. Note that a quadratic extension $K = k(\sqrt{D})$ with D monic square-free in $\mathbb{A}$ is real if and only if $\deg D$ is even. An element $x \in k_\infty$ is called *real quadratic irrational* if $x \notin k$ is quadratic over k.

Write, for a real quadratic irrational x with discriminant D,

$$f_n(x) = \frac{p_n(x) + \sqrt{D}}{q_n(x)}.$$

Lemma 2.3 ([F1, Lemma 1.20]). *We have*

$$p_{n+1} = a_n q_n - p_n,$$
$$q_{n+1} q_n = D - p_{n+1}^2,$$
$$(q_0 P_{n-1} - p_0 Q_{n-1})^2 - D Q_{n-1}^2 = (-1)^n q_0 q_n.$$

Lemma 2.4 ([F1, Lemma 1.24, 1.25]). *Let $D \in \mathbb{A}$ be a square-free monic polynomial of even degree.*

i) *Let $N \in \mathbb{A}$ with $\|N\| < \|D\|$. Then*

$$A^2 - DB^2 = N$$

has a solution in coprime $A, B \in \mathbb{A}$ if and only if $N = c^2(-1)^n q_n(\sqrt{D})$ for some $c \in \mathbb{F}_q^*$ and some $n \geq 0$.

ii) *For $x = \sqrt{D}$, we have*

$$\|a_n\| > 1, \quad \|p_n\| = \|\sqrt{D}\|, \quad \text{and} \quad sgn(p_n) = 1,$$
$$\|\sqrt{D} + p_n\| = \|\sqrt{D}\|, \qquad \|a_n q_n\| = \|\sqrt{D}\|.$$

For a quadratic irrational x there exist $c \in \mathbb{F}_q^*$, $n_0 \geq 0$ and $r \geq 1$ such that

$$a_{n+r} = c a_n$$

for all $n \geq n_0$. The smallest such r is called the *quasi-period* of x.

Lemma 2.5 ([Ste, Theorem 5.1]). *Let D be a square-free monic polynomial of even degree in $\mathbb{A}$ and r be the quasi-period of $\sqrt{D}$. Then $P_{r-1} + Q_{r-1}\sqrt{D}$ is a fundamental unit of $k(\sqrt{D})$.*

Proposition 2.6 ([F1, Corollary 4.1]). *Let I be an O_K-ideal with $N(I) < \|\sqrt{D}\|$, where $N(I)$ is the cardinality of O_K/I. Then I is principal if and only if $I = \langle p_n(\sqrt{D}) + \sqrt{D}, q_n(\sqrt{D})\rangle$ for some $n \geq 0$.*

3. Generalized Ono invariant

T. Ono defined a number p_d, which now is called the *Ono invariant*, for each square-free negative integer d and raised a question comparing p_d and the class number h_d of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ ([O]). The question was solved by R. Sasaki ([Sa1]). Later Sasaki ([Sa2]) generalized the Ono invariant to the real quadratic field and obtained a result similar to the imaginary case. In this section we obtain the function field analogue of Sasaki's result in real quadratic fields. The function field analogue of Sasaki's result in the imaginary case is also true, and the proof is almost the same.

Let K be a global function field over a finite field $\mathbb{F}_q$. Then K corresponds to an algebraic curve $\mathcal{C}$ over $\mathbb{F}_q$. Let S be a finite set of places of K and O_S be the ring of functions in K which are regular away from S. We call O_S the ring of S-integers of K, which is a Dedekind domain. Let $\mathfrak{a}$ be an ideal of O_S, which can be written as $\mathfrak{a} = \prod \mathfrak{p}_i^{n_i}$ with $\mathfrak{p}_i$ prime ideal of O_S. The prime $\mathfrak{p}_i$ corresponds to place (or point) P_i of K (or $\mathcal{C}$). We associate a divisor $A = \sum n_i P_i$ of $\mathcal{C}$ to the ideal $\mathfrak{a}$. For

a divisor $A = \sum n_P P$ of $\mathcal{C}$, the finite part $(A)_{fin}$ of A is defined to be $\sum_{P \notin S} n_P P$. A divisor $A = \sum n_P P$ is called *effective* if $n_P \geq 0$ for every $P \in \mathcal{C}$.

The following lemma follows easily from the Riemann-Roch Theorem.

Lemma 3.1. *Let K be an arbitrary global function field of genus g and S be a nonempty finite set of places of K containing a place ∞ of degree 1. Let O_S be the ring of S-integers of K. Then every O_S-ideal class contains an O_S-ideal of degree at most g.*

Proof. Let $\mathfrak{a}$ be an O_S-ideal. Let A be the divisor of K associated to $\mathfrak{a}$ and $a = \deg \mathfrak{a} = \deg A$. Since the divisor $A + (g-a)\infty$ has degree g, there exists $f \in K$ with $B = A + (g-a)\infty + (f)$ effective and of degree g. Then $\deg(B)_{fin} \leq \deg B = g$. Thus the ideal $\mathfrak{b}$ associated to $(B)_{fin}$ is equivalent to $\mathfrak{a}$ and has degree $\leq g$. $\square$

Let D be a monic square-free polynomial of even degree $2d$. Let $q_i = q_i(\sqrt{D})$ and $q' = sgn(q_i)^{-1} q_i$. Let

$$a(D) = \{q'_0, \ldots, q'_m\},$$

where m is the period of $\sqrt{D}$. For a polynomial N, a sequence $(N_1, N_2, \ldots, N_\ell)$ of divisors of N is called an $a(D)$-*sequence* if

$$\deg N_1 > 0, \quad N_i \mid N_{i+1}, \quad N_i \neq N_j \text{ for } i \neq j, \tag{3.1}$$

$$N_j/N_i \notin a(D) \quad \text{if } \deg N/N_i < \deg(NN_i/N_j), \tag{3.2}$$

$$NN_i/N_j \notin a(D) \quad \text{if } \deg N/N_i > \deg(NN_i/N_j) \tag{3.3}$$

and

$$N_j/N_i \text{ or } NN_i/N_j \notin a(D) \quad \text{if } \deg N/N_i = \deg(NN_i/N_j). \tag{3.4}$$

Define the $a(D)$-degree $\deg_{a(D)} N$ of N with respect to the set $a(D)$ by

$$\deg_{a(D)} N = \max\{\ell : \text{ there exists an } a(D)\text{-sequence } (N_1, N_2, \ldots, N_\ell) \text{ of } N\}.$$

The *generalized Ono invariant* p_D of D is defined by

$$p_D = \max_{\deg A < d-1}\{\deg_{a(D)} P([\sqrt{D}] + A)\},$$

where $P(X) = D - X^2$. By Proposition 2.6 we get

Lemma 3.2. *Let A be a monic polynomial in $\mathbb{A}$ such that $A \mid B^2 - D$ and $\deg A < d$. If the ideal $[A, B + \sqrt{D}]$ is principal, then $A \in a(D)$.*

Now we have the following analogue of [Sa2], Main Theorem. The proof is almost the same as that in [Sa2], but we give it for the convenience of the reader.

Theorem 3.3. *Let D be a square-free monic polynomial of even degree. We have:*

i) $p_D \leq h_D$,
ii) $p_D = 1$ *if and only if* $h_D = 1$.

Proof. i) Let $p_D = \deg_{a(D)}(P(B))$ for some $B = [\sqrt{D}] + A$ with $\deg A < d-1$. Let $(N_1, \ldots, N_\ell)$, $\ell = p_D$ be a sequence of divisors of $N = P(B)$ as in the definition of the generalized Ono invariant. We will show that the ideal classes $\mathfrak{p}_i = [N_i, B + \sqrt{D}]$ are mutually distinct. Suppose that $\mathfrak{p}_i$ is equivalent to $\mathfrak{p}_j$, $i < j$. Then both $[N_j/N_i, B + \sqrt{D}]$ and $[NN_i/N_j, B + \sqrt{D}]$ are principal. Since $\deg N < 2d$, $min\{\deg N_j/N_i, \deg NN_i/N_j\} < d$. Then by Lemma 3.2, N_j/N_i or NN_i/N_j is in $a(D)$, which is a contradiction.

ii) We will show that if $p_D = 1$, then $h_D = 1$. Suppose that $h_D \geq 2$. Let $\mathfrak{p}$ be a nonprincipal prime ideal with smallest $\deg N(\mathfrak{p})$. Then $\deg N(\mathfrak{p}) \leq g = d-1 < d$ by Lemma 3.1, where g is the genus of $k(\sqrt{D})$. Then by [F1], Theorem 4.0, $\mathfrak{p} = [P, B+\sqrt{D}]$ with $\|B-\sqrt{D}\| < \|P\|$. Then we can write $B = [\sqrt{D}]+A$ with $\deg A < \deg P = \deg N(\mathfrak{p}) \leq d-1$. Let $\mathfrak{q} = [N/P, B+\sqrt{D}]$, where $N = D - B^2$. Note that P must be prime. Then $\mathfrak{p}\mathfrak{q} = [N, B+\sqrt{D}]$ is principal, and so $\mathfrak{q}$ is not principal. We will show that $P \notin a(D)$. Then $\{N/P, N\}$ is an a_D-sequence of length 2. Suppose $P = q_i \in a(D)$. Then $\mathfrak{a}_i = [q_i, p_i + \sqrt{D}]$ is principal by Proposition 2.6. Since P divides both $N(B+\sqrt{D})$ and $N(p_i+\sqrt{D})$, P divides $B^2 - p_i^2 = (B+p_i)(B-p_i)$. If $P \mid B-p_i$, i.e., $B-p_i = CP$ for some $C \in \mathbb{A}$, then $[P, B+\sqrt{D}] = [P, p_i+\sqrt{D}] = \mathfrak{a}_i$ is principal. If $P \mid B+p_i$, then $[P, B+\sqrt{D}] = [P, p_i-\sqrt{D}] = \mathfrak{a}_i'$, the conjugate of $\mathfrak{a}_i$, is principal. Thus we get a contradiction. $\square$

Example 1. Let $q = 3$ and $D = (T(T+1))^2 + T$.

It is easy to check that $a(D) = \{1, T\}$ and $[\sqrt{D}] = T(T+1)$. Let $B_a = T(T+1)+a$ with $0 \neq a \in \mathbb{F}_q$. Then $P(B_a) = 2a(T^2+T) + a^2 - T$ and

$$\begin{aligned} P(B_a)(0) &= a^2 \neq 0, \\ P(B_a)(1) &= a^2 + a - 1 = (a-1)^2 - 2 \neq 0, \\ P(B_a)(-1) &= a^2 + 1 \neq 0. \end{aligned}$$

Thus $P(B_a)$ is irreducible and $h_D = 1$ by Theorem 3.3 (ii).

Example 2. Let $q = 5$. Since $(T(T+1))^2 + T = T(T-1)(T^2+3T-1)$, $2 \mid h_D$ with $D = (T(T+1))^2 + T$.

Now consider $D = (T(T-1))^2 + T = T(T^3 - 2T^2 + T + 1)$. Note that $T^3 - 2T^2 + T + 1$ is irreducible over $\mathbb{F}_5$; thus h_D is odd by genus theory. We easily see that $a(D) = \{1, T\}$. Let $B = T(T-1)+1$. Then $P(B) = 2T(T-1)+1-T = (T-1)(2T-1)$. Thus $p_D > 1$ and $h_D > 1$ by Theorem 3.3 (ii).

Let $D = (T(T-2))^2+T$ and $B = T(T-2)+a$. Then $P(B) = a^2+2aT(T-2)-T$. Then $P(B)(1) = (a-1)^2 - 2 \neq 0$, $P(B)(-1) = a^2 - a + 1 = (a+2)^2 - 3 \neq 0$, $P(B)(2) = a^2 - 2 \neq 0$ and $P(B)(-2) = a^2 + a + 2 = (a-2)^2 - 2 \neq 0$. Hence $h_D = 1$.

4. Real quadratic function fields of Richaud-Degert type

A monic polynomial D of degree $2d$ can be written uniquely as $A^2 + B$ with $\deg A = d$ and $\deg B < d$. We say that the real quadratic function field $k(\sqrt{D})$ is of *Richaud-Degert type*, or R-D type for short, if D is of the form $A^2 + B$ with $\deg B < \deg A$ and $B \mid A$. In this case we write $D = (MN)^2 + aM$ with M, N monic, $a \in \mathbb{F}_q^*$ and $\deg N > 0$. If $D = (MN)^2 + aM$ is of R-D type, then the continued fraction expansion of $\sqrt{D}$ is easily seen to be

$$[MN; \overline{2a^{-1}N, 2MN}].$$

Note that $a(D) = \{1, M\}$ if $\deg M > 0$ and $a(D) = \{1\}$ if M is constant.

The following theorem is a generalization of Theorem 3.5 of [FH] to R-D type.

Proposition 4.1. *Let $K = k(\sqrt{D})$ be a real quadratic function field with monic square-free polynomial D of degree $2d$. Suppose that h_D is odd. Then the following are equivalent:*

i) *For any monic irreducible polynomial* P *in* $\mathbb{A}$ *with* $\deg P < d$*, we have* $\left(\frac{D}{P}\right) \neq 1$*. Here* $\left(\frac{A}{P}\right)$ *is the Legendre symbol and we define* $\left(\frac{A}{P}\right) = 0$ *iff* $P \mid A$*.*

ii) *For any* $C \in \mathbb{A}$ *and an irreducible polynomial* $P \nmid D$ *with* $\deg C < \deg P < d$*, we have* $C^2 - D \not\equiv 0 \mod P$*.*

iii) $h_D = 1$ *and* D *is of R-D type, say* $D = (MN)^2 + aM$*.*

Proof. (i) $\iff$ (ii) is easy, and we omit the proof of it.

(i) $\Longrightarrow$ (iii): Let P be a prime dividing D with $\deg P < d$. Since h_D is odd, the prime of K lying over P is principal. By Lemma 2.1 of [FH], $h_D = 1$. We write $D = A^2 + B$ with $\deg B < d$. If $\deg B = 0$, we are done. Assume that $\deg B > 0$. Let P be an irreducible factor of B. Then clearly $D \equiv A^2 \pmod P$. Since $(\frac{D}{P}) \neq 1$, P should divide D, and so $P \mid A$. Since $D = A^2 + B$ is square-free, B must be square-free and $B \mid A$.

(iii) $\Longrightarrow$ (i): Let P be a monic irreducible polynomial with $\deg P < d$ and $(\frac{D}{P}) = 1$. Then P splits in K; that is $(P) = \mathfrak{p}\mathfrak{p}'$. Since $h_D = 1$, we have $\mathfrak{p} = (U + V\sqrt{D})$, $\mathfrak{p}' = (U - V\sqrt{D})$ and

$$U^2 - V^2 D = cP \quad (c \in \mathbb{F}_q^*).$$

Then using Lemma 2.4 and the fact that $a(D) = \{1, M\}$, we see that $P = M$, which is impossible, since we have assumed that $\left(\frac{D}{P}\right) = 1$. □

The case where M is constant is contained in [FH]. We assume that $\deg M > 0$. Then the fundamental unit is $(2MN^2 + a) + 2N\sqrt{D}$. Therefore the regulator R_D of O_{K_D} is $2d - \deg M$.

Remark 4.2. If h_D is odd, then from genus theory either D is irreducible or D is a product of two irreducibles of odd degree. Thus, if $D = (MN)^2 + aM$ and h_D is odd, then M and $MN^2 + a$ must both be irreducible of odd degree.

Proposition 4.3. *Let* $D = (MN)^2 + aM$ *with* $m = \deg M > 0$ *and* $h_D = 1$*. Then*

- $q = 3$ *and* $d \leq 5$*.*
- $q = 5$ *and* $d \leq 3$*.*
- $q = 7$ *and* $d \leq 2$*.*

Proof. In the proof of [FH], Theorem 4.1, we have the inequality

$$h(O_K)R_K \geq \frac{(q-1)(q^{2d-3} + 1 - 2(d-1)q^{\frac{2d-3}{2}})}{(2d-3)(q^{d-1} - 1)},$$

where R_K is the regulator. Then a simple computation gives the result by taking account of the regulator $R_K = 2d - m$. □

If $D = (MN)^2 + aM$ is of R-D type, then $a(D) = \{1, M\}$. To classify all real quadratic function fields of R-D type with $h_D = 1$, we need to verify $p_D = 1$ for all possible cases in Proposition 4.3. That is, we have to check that $P_D(B)$ is irreducible or a product of M and an irreducible polynomial for any polynomial B of the form $MN + C$ with $\deg C < d - 1$. Thus combining Theorem 3.3 and Proposition 4.3 we can determine all real quadratic function fields of R-D type with class number 1.

With the aid of a computer program (Maple) we determined all Richaud-Degert type real quadratic function fields with class number 1. Note that if $\deg D = 2$, then the genus of K_D is 0 and the class number is 1.

Theorem 4.4. *Let $K_D = k(\sqrt{D})$ with $D = (MN)^2 + aM$ square-free monic, $a \in \mathbb{F}_q^*$, $\deg M > 0$ and $\deg MN > 1$. Then the ideal class number h_D of K_D is 1 if and only if (M, N, a) is one of the following:*

i) $q = 3$, $(T+\alpha, T+\alpha+\beta, \beta)$, $(T+\alpha, (T+\alpha+\beta)^2, \beta)$, *where* $\alpha \in \mathbb{F}_q$, $\beta \in \mathbb{F}_q^*$.
ii) $q = 5$, $(T+\alpha, T+\alpha+\beta, \gamma)$, *where* $\alpha \in \mathbb{F}_q$, $\beta \in \mathbb{F}_q^*$ *and* $\gamma = \beta+2$ *if* $\beta = 1, 2$, $\gamma = \beta - 2$ *if* $\beta = -1, -2$.
iii) $q = 7$, $(T + \alpha, T + \alpha, 2)$, $(T + \alpha, T + \alpha, -2)$, *where* $\alpha \in \mathbb{F}_q$.

5. Characteristic 2 case

In this section we assume that the characteristic of k is 2. Most properties of continued fractions in odd characteristic also hold for this case. We refer to [Z] for details.

Let y satisfy the equation

$$P_{(A,B)}(X) := X^2 + AX + B = 0,$$

with $A, B \in \mathbb{A} = \mathbb{F}_q[T]$ and A monic. We require that

$$X^2 + AX + B \equiv 0 \mod C^2 \tag{5.0}$$

has no solution in $\mathbb{A}$ for each nonconstant divisor C of A, so that the ring of integers of $k(y)$ is $\mathbb{A}[y]$. See Lemma 5.2 below. We write $K_{(A,B)}$ for $k(y)$.

Lemma 5.1. *Every quadratic extension K of k is of the form $K_{(A,B)}$ with A, B as above.*

Proof. It is known that any quadratic function field K in characteristic 2 is of the form $K = k(z)$, where z satisfies the equation

$$z^2 + z + \frac{B'}{A'},$$

where $\gcd(A', B') = 1$, and $A' = \prod P_i^{e_i}$ with P_i prime and e_i odd ([HL], §2). Let $A = \prod P_i^{\frac{e_i+1}{2}}$ and $B = B' \prod P_i$. Then $y = Az$ satisfies the equation

$$y^2 + Ay + B = 0,$$

and $K = k(y)$. It suffices to show that the equation

$$X^2 + AX + B \equiv 0 \mod P_i^2 \tag{1}$$

has no solution in $\mathbb{A}$. If $e_i > 1$, then (1) becomes

$$X^2 + B_i P_i \equiv 0 \mod P_i^2, \tag{2}$$

with $P_i \nmid B_i$. It is clear that (2) has no integral solutions. Now suppose that $e_i = 1$. Then (1) becomes

$$X^2 + A_i P_i X + B_i P_i \equiv 0 \mod P_i^2, \tag{3}$$

with $P_i \nmid A_i$, $P_i \nmid B_i$. If (3) has an integral solution $C \in \mathbb{A}$, then $C \equiv 0 \mod P_i$, and then $B_i \equiv 0 \mod P_i$, which is a contradiction. Thus every quadratic function field in characteristic 2 is of the form $K_{(A,B)}$ with A, B as above. □

We also assume that the field $K_{(A,B)} = k(y)$ is real.

Lemma 5.2. *Let A, B, y be as above and $K = K_{(A,B)}$. Then:*

i) *The ring $\mathbb{B}$ of integers of K is $\mathbb{A}[y]$.*
ii) *A prime P of $\mathbb{A}$ is ramified in K if and only if P divides A.*

iii) *The genus of* K *is* $\deg A - 1$.
iv) *One can take* B *so that* $\deg B < \deg A$.

Proof. i) Let $u + vy \in \mathbb{B}$ with $u, v \in k$. We need to show that $u, v \in \mathbb{A}$. Since $u + vy \in \mathbb{B}$, $Tr(u + vy) = vA \in \mathbb{A}$ and $Nm(u + vy) = u^2 + uvA + v^2B \in \mathbb{A}$. Write $u = \frac{U_1}{U_2}$ and $v = \frac{V_1}{V_2}$ with $(U_1, U_2) = (V_1, V_2) = 1$. Then $V_2 \mid A$ and

$$\frac{U_1^2}{U_2^2} + \frac{U_1V_1}{U_2V_2}A + \frac{V_1^2}{V_2^2}B \in \mathbb{A}.$$

We must have $v_P(U_2) \leq v_P(V_2)$ for any prime P, which implies that $U_2 \mid V_2$. If V_2 is not a constant, then $\frac{U_1V_2}{U_2V_1} \mod V_2^2$ is a solution to

$$X^2 + AX + B \equiv 0 \mod V_2^2,$$

which contradicts condition (5.0). Thus U_2, V_2 are constants, and we are done.

ii) It is easy to see that $\frac{y}{A}$ satisfies the equation

$$X^2 + X + \frac{B}{A^2}.$$

If a prime P divides A, then by condition (5.0) $P^2 \nmid B$. Thus, for $P \mid A$, $v_P(\frac{B}{A^2}) < 0$. The result follows from Proposition III.7.8, (c) of [Sti].

iii) follows from the proof of ii) and [Sti], Proposition III.7.8, (d).

iv) If we normalize the quadratic equation for K as in [HL] by

$$X^2 + X + D = 0,$$

where $D = \frac{C}{\prod_{i=1}^r P_i^{e_i}}$ with e_i odd, $(C, P_i) = 1$ and $\deg C < \deg \prod_{i=1}^r P_i^{e_i}$, since K is real. Then K is also obtained by

$$X^2 + A_1X + B_1 = 0,$$

where $A_1 = \prod_i P_i^{\frac{e_i+1}{2}}$ and $B_1 = C \prod_i P_i$. It is not hard to see that

$$X^2 + A_1X + B_1 \equiv 0 \mod P_i^2$$

is not solvable in $\mathbb{A}$. Thus we can take B so that $\deg B < 2 \deg A$. Then we can find $C \in \mathbb{A}$ with $\deg C < \deg A$ so that $B = AC + C^2 + D$ with $\deg D < \deg A$. Now replace y by $y + C$. □

From now on we always assume that $\deg B < \deg A$. For any $x \in K$ one can consider the continued fraction expansion of x. Thus a_i, P_i, Q_i are defined in the same way, and they satisfy the same properties as in the odd characteristic case. In particular, all the properties of continued fractions in odd characteristic mentioned in §2 hold true in this case too. We refer to [Z] for details. We state the following analogue of Lemma 2.3. We use the same notation as in the previous sections.

Lemma 5.3 (cf. [Z], §2)**.** *Let* $f_n(x) = \frac{p_n(x)+y}{q_n(x)}$. *Then*

$$p_{n+1} = a_nq_n + p_n + A,$$
$$q_{n+1}q_n = p_{n+1}^2 + p_{n+1}A + B,$$
$$P_{(A,B)}(\frac{p_0Q_{n-1} + q_0P_{n-1}}{Q_{n-1}}) = \frac{q_0q_n}{Q_{n-1}^2}.$$

Note that our notation is different from that of [Z]. We follow the notation of [F1]. P_i, Q_i (resp. p_i, q_i) in [Z] are p_i, q_i (resp. P_i, Q_i) in our notation. We get the following lemma by the same method as in [F1], Lemma 1.9, using the third equation in Lemma 5.3 and [Z], Theorem 9.

Lemma 5.4. *Let A, B, y be as above. Let $N \in \mathbb{A}$ with $\deg N < \deg A$. Then*

$$U^2 + AUV + BV^2 = N$$

has a solution in coprime $U, V \in \mathbb{A}$ if and only if $N = a^2 q_n(y)$ for some $a \in \mathbb{F}_q^*$ and some $n \geq 0$.

Let y be as before and $q_i = q_i(y)$. Let

$$a(A, B) := \{q_0', \ldots, q_k'\},$$

where $q_i' = sgn(q_i)^{-1} q_i$. For a polynomial $N \in \mathbb{A}$, we define $\deg_{a(A,B)} N$ as before replacing $a(D)$ by $a(A, B)$. The *generalized Ono invariant* $p_{(A,B)}$ of $X^2 + AX + B$ or y is defined by

$$p_{(A,B)} = \max_{\deg C < d-1} \{\deg_{a(A,B)} P_{(A,B)}([y] + C)\},$$

where $d = \deg A$. Note that $[y] = A$.

Note that $[V, U + y]$ is an ideal of O_K if and only if $V \mid U^2 + AU + B$ and that the ideal norm $N([V, U + y]) = (V)$. As in the classical case, we see that

$$[U_1, V + y][U_2, V + y] = [\frac{U_1 U_2}{G}, G(V + y)],$$

where $G = \gcd(U_1, U_2, V)$.

As with the odd characteristic case we obtain the following lemma using Lemma 5.4.

Lemma 5.5. *Let $U, V \in \mathbb{A}$ be such that $V \mid U^2 + AU + B$ and $\deg V < \deg A$. If the ideal $[V, U + y]$ is principal and V is monic, then $V \in a(A, B)$.*

An ideal $\mathfrak{a} = [V, U+y]$ is called *reduced* if $\|U+y\| < \|V\| = \|N(\mathfrak{a})\| < \|U+A+y\|$ or $\|U + A + y\| < \|V\| = \|N(\mathfrak{a})\| < \|U + y\|]$.

Lemma 5.6. *$[V, U + y]$ is reduced if and only if $\|V\| < \|A\|$.*

Proof. Theorem 12 of [Z]. □

Theorem 5.7. *Let A, B, y be as above and write $h_{(A,B)}$ to denote the ideal class number of $K_{(A,B)} = k(y)$. Then we have:*

i) $p_{(A,B)} \leq h_{(A,B)}$.
ii) $p_{(A,B)} = 1$ *if and only if* $h_{(A,B)} = 1$.

Proof. i) Let $D = [y] + C$ with $\deg C < d - 1$. Then $P_{(A,B)}(D) = (y + [y])^2 + A(y + [y]) + AC + C^2$. Noting that $\deg(y + [y]) < 0$, we have $\deg P_{(A,B)}(D) < 2d$. Then the proof is exactly the same as the proof of Theorem 3.3 (i) using Lemma 5.5 instead of Lemma 3.2.

ii) We will show that if $p_{(A,B)} = 1$, then $h_{(A,B)} = 1$. Suppose that $h_{(A,B)} \geq 2$. Let $\mathfrak{p}$ be a nonprincipal prime ideal with smallest $\deg N(\mathfrak{p})$. Then $\deg N(\mathfrak{p}) \leq g = d - 1 < d$ by Lemma 3.1, where g is the genus of k_D. Then by Lemma 5.6, $\mathfrak{p} = [P, U+y]$ with $U = [y] + C$, $\deg C < \deg N(\mathfrak{p}) = \deg P \leq d - 1$. Let $\mathfrak{q} = [N/P, U + y]$, where $N = N(U + y) = U^2 + AU + B$. Note that P must be prime. Then $\mathfrak{p}\mathfrak{q} = [N, U + y]$ is principal, and so $\mathfrak{q}$ is not principal. We will show that $P \notin a(A, B)$. Then

$\{N/P, N\}$ is an $a_{(A,B)}$-sequence of length 2. Suppose $P = q_i \in a(A,B)$. Then $\mathfrak{a}_i = [q_i, p_i + y]$ is principal by Theorem 13 of [Z]. Since V divides both $N(U+y)$ and $N(p_i+y)$, P divides $U^2 + p_i^2 + A(U+p_i) = (U+p_i)(U+p_i+A)$. Then we can see that $[P, U+y] = \mathfrak{a}_i$ or $\mathfrak{a}_i'$, which is principal, and we get a contradiction. □

Let P be an irreducible polynomial. The Hasse symbol $\{\frac{A,B}{P}\}$ is defined by

$$y^{q^{\deg P}} \equiv y + \{\frac{A,B}{P}\}A \mod \mathfrak{p}$$

if $P \nmid A$ and $\{\frac{A,B}{P}\} = -1$ if $P \mid A$, where $\mathfrak{p}$ is a prime ideal of K lying over P. See [Ha], §3. Thus $\{\frac{A,B}{P}\} = 1, 0, -1$ according to whether P is inert, splits, or is ramified in $K_{(A,B)}$.

We say that (A,B) is of *Richaud-Degert type* if $B \mid A$. The following lemma is straightforward.

Lemma 5.8. *Suppose that (A,B) is of R-D type. Then*

i) *If B is not a constant, then the continued fraction expansion of y is $[A; \overline{\frac{A}{B}, A}]$ and $\frac{A}{B}y + 1$ is the fundamental unit. $a(A,B) = \{1, B\}$ in this case.*
ii) *If B is a constant $c \in \mathbb{F}_q^*$, then the continued fraction expansion of y is $[A; \overline{\frac{A}{c}, A}]$ and y is the fundamental unit. $a(A,B) = \{1\}$ in this case.*

Now we have the following analogue of Proposition 4.1.

Proposition 5.9. *Let A, B, y be as before. Suppose that $h_{(A,B)}$ is odd. Then the following are equivalent:*

i) *For any monic irreducible polynomial P in $\mathbb{A}$ with $\deg P < \deg A = d$, we have $\{\frac{A,B}{P}\} \neq 0$.*
ii) *For any $U \in \mathbb{A}$ and an irreducible polynomial $P \nmid A$ with $\deg U < \deg P < d$, $U^2 + AU + B \not\equiv 0 \mod P$.*
iii) *$h_{(A,B)} = 1$ and (A,B) is of R-D type.*

Proof. i) $\Longrightarrow$ iii): As in Proposition 4.1, it is easy to see that $h_{(A,B)} = 1$. Let P be a prime dividing B. Then $P_{(A,B)}(X)$ is reducible modulo P. Since $\deg P \leq \deg B < \deg A = d$, $P \mid A$ by i). If $P^2 \mid B$, then $X^2 + AX + B \equiv 0 \mod P^2$ has a solution, which is a contradiction. Therefore, $B \mid A$.

iii) $\Longrightarrow$ i): Use Lemma 5.8 and Lemma 5.4. □

Remark 5.10. It is known from the genus theory of Artin-Schreier extensions, [HL], Theorem 3.3, $h_{(A,B)}$ is odd only if A is a power of an irreducible P. From the condition (5.0) on A, B the following two cases can happen for $h_{(A,B)}$ to be 1:

- $(A,B) = (P, a)$,
- $(A,B) = (P^i, aP)$,

where P is irreducible, $a \in \mathbb{F}_q^*$ and $i > 1$ is an integer.

Proposition 5.11. *Let (A,B) be of R-D type with $d = \deg A$ and $h_{(A,B)} = 1$.*

i) *If $(A,B) = (P, a)$, then:*
   - *$q = 2$ and $d \leq 9$,*
   - *$q = 4$ and $d \leq 3$,*
   - *$q = 8$ and $d = 1$.*

ii) *If* $(A, B) = (P^i, aP)$ *with* $i > 1$, *then:*
- $q = 2$ *and* $(\deg P, i) = (5, 2), (4, 2), (3, 2)(3, 3), (2, 2), (2, 3), (2, 4), (2, 5), (1, k)$ *with* $k = 2, \ldots, 10$;
- $q = 4$ *and* $(\deg P, i) = (1, 2), (1, 3)$.

Proof. The proof is exactly the same as that of Proposition 4.3. □

Now we can use Theorem 5.7 and Proposition 5.11 to determine all characteristic 2 R-D type quadratic function fields with class number 1. Note also that if $\deg A = 1$, then the class number is 1.

Theorem 5.12. *Suppose that the characteristic of* k *is* 2. *The only real quadratic function fields* $K_{(A,B)}$ *with* $B \mid A$ *and* $\deg A > 1$ *of ideal class number* 1 *are:*

$q = 2$, (A, B) *is one of the following:*
$(T^2 + T + 1, 1), (T^3 + T + 1, 1), (T^3 + T^2 + 1, 1)$,
$(T^4 + T^3 + 1, 1), (T^4 + T^3 + T^2 + T + 1, 1)$,
$((T + \alpha)^k, (T + \alpha))$ *with* $\alpha \in \mathbb{F}_2$, $k = 2, 3, 4$,
$((T^2 + T + 1)^2, T^2 + T + 1)$.

$q = 4$, (A, B) *is one of the following:*
$(T^2 + T + \alpha, 1), (T^2 + T + \alpha + 1, 1), (T^2 + \alpha T + 1, \alpha), (T^2 + \alpha T + \alpha, \alpha)$,
$(T^2 + (\alpha + 1)T + 1, \alpha + 1), (T^2 + (\alpha + 1)T + \alpha + 1, \alpha + 1)$,
$((T + \beta)^2, \gamma(T + \beta))$,

where α *a generator of* $\mathbb{F}_4^*$, $\beta \in \mathbb{F}_4$ *and* $\gamma \neq 0, 1$.

Remark 5.13. In number field case, Mollin ([M]) determined all real quadratic fields of R-D type with class number two assuming GRH. It is also an interesting problem to determine real quadratic function fields of R-D type with class number two. The method in Proposition 4.3 and Remark 5.10 can give bounds for q and d, which will be much larger than those in the class number 1 case. For complete determination one needs some criteria such as those in Theorem 3.3 and Lemma 5.6. See [M] in the number field case.

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