QUANTITATIVE UNIQUENESS ESTIMATE FOR THE MAXWELL SYSTEM WITH LIPSCHITZ ANISOTROPIC MEDIA

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Abstract. We study quantitative uniqueness estimates for the time harmonic Maxwell system with Lipschitz anisotropic media. Our main results are a three-balls inequality and a minimal vanishing rate at a point of any nontrivial solution. The proof relies on a Carleman estimate with a divergence term.

1. Introduction

In this paper we study local properties of solutions of the time-harmonic Maxwell system with anisotropic media

\begin{align*}
\text{curl} H &= i\omega \varepsilon E \quad \text{in } \Omega, \\
\text{curl} E &= -i\omega \mu H \quad \text{in } \Omega.
\end{align*}

Here \( \Omega \) is an open subset of \( \mathbb{R}^3 \), \( \omega \in \mathbb{C} \setminus \{0\} \) and \( \varepsilon(x), \mu(x) \) are two real symmetric matrix-valued functions in \( \Omega \) satisfying

\begin{align*}
\lambda |\xi|^2 \leq \varepsilon(x) \xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \lambda |\xi|^2 \leq \mu(x) \xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \forall \ x \in \Omega, \ \xi \in \mathbb{R}^3,
\end{align*}

and

\begin{align*}
\|\varepsilon\|_{W^{1,\infty} (\Omega)} + \|\mu\|_{W^{1,\infty} (\Omega)} \leq M,
\end{align*}

where \( 0 < \lambda \leq 1 \) and \( M \) are positive constants.

We prove a quantitative uniqueness estimate in the form of a three balls inequality for \( (E, H) \) satisfying (1.1). As a by-product, we derive a minimal vanishing rate at an arbitrary point of \( \Omega \) for any nontrivial \( (E, H) \), which implies a unique continuation property for (1.1). When \( \Omega = \mathbb{R}^3 \), we also obtain a minimal decay rate of any nontrivial bounded \( (E, H) \) at infinity. We now state our main results. Let us denote \( B_R(x_0) = \{|x-x_0| < R\} \).
Theorem 1.1. Assume that (1.2) and (1.3) hold. Then there exist \( \rho, s > 0 \), depending on \( \lambda, M \), such that for any \( (E, H) \in (L^2_{\text{loc}}(\Omega))^6 \) solving (1.1) and \( r_0 < r_1 < r_2/2 < \rho \) with \( B_{r_2}(x_0) \subset \Omega \), we have

\[
\int_{B_{r_2}(x_0)} (|E|^2 + |H|^2)dx \leq C \left( \int_{B_{r_0}(x_0)} (|E|^2 + |H|^2)dx \right)^{\tau} \left( \int_{B_{r_2}(x_0)} (|E|^2 + |H|^2)dx \right)^{1-\tau},
\]

where \( C \) depends on \( \lambda, M, r_1, r_2, s \) and

\[
\tau = \frac{(2r_1)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}}.
\]

In fact, the constant \( C \) is explicitly given by

\[
C = \max\{C_{\lambda, M}, e^{2s[(r_1)^{-s}-(r_2/2)^{-s}]}\}
\]

with \( C_{\lambda, M} \) depending on \( \lambda \) and \( M \).

Corollary 1.2. Assume that the hypotheses in Theorem 1.1 hold. Let \( r_0, r_1, r_2 \) be described as above. Then for any nontrivial \((E, H) \in (L^2_{\text{loc}}(\Omega))^6 \) satisfying (1.1),

\[
\int_{B_{r_2}(x_0)} (|E|^2 + |H|^2)dx \geq \exp(-C r_0^{-s}) \int_{B_{r_2}(x_0)} (|E|^2 + |H|^2)dx,
\]

where \( C \) depends on \( \lambda, M, r_1, r_2, s \) and

\[
\frac{\int_{B_{r_2}(x_0)} (|E|^2 + |H|^2)dx}{\int_{B_{r_1}(x_0)} (|E|^2 + |H|^2)dx}.
\]

From Corollary 1.2, it immediately follows that if \( \Omega \) is connected and for some \( x_0 \in \Omega \), \((E, H) \) satisfies

\[
\int_{B_{r}(x_0)} (|E|^2 + |H|^2)dx \leq C_N \exp(-N r^{-s}), \quad \forall r \ll 1, \quad \forall N \in \mathbb{N},
\]

then \( E = H \equiv 0 \) in \( \Omega \). This is the unique continuation property with exponential vanishing rate for (1.1). Note that we do not impose any structural assumptions on the matrices \( \varepsilon \) and \( \mu \) at \( x_0 \).

Similar to the result in [9], using Corollary 1.2, one can also study the minimal decay rate of any nontrivial vanishing rate for (1.1). Note that we do not impose any structural assumptions on the matrices \( \varepsilon \) and \( \mu \) at \( x_0 \).

We have the following:

Corollary 1.3. Assume that the hypotheses of Theorem 1.1 hold. Let \((E, H) \in (L^2_{\text{loc}}(\mathbb{R}^3))^6 \) be a nontrivial solution to

\[
\begin{align*}
\text{curl} H &= i\omega \varepsilon E & \text{in } \mathbb{R}^3, \\
\text{curl} E &= -i\omega \mu H & \text{in } \mathbb{R}^3.
\end{align*}
\]

Suppose that there exists \( K > 0 \) such that

\[
\|E\|_{L^\infty(\mathbb{R}^3)} + \|H\|_{L^\infty(\mathbb{R}^3)} \leq K.
\]
Then there exist \( \rho_0 \) depending on \( \lambda, M, K \), and \( \epsilon > 1 \) depending on \( \lambda, M, \) and \( L \) depending on \( \int_{B_r(0)}(|E|^2 + |H|^2)dx, \lambda, M, r, \) such that for all \( 0 < r \leq \rho_0 \),

\[
M_r(t) \geq L^{e^{t/r}}.
\]

We now mention some related results. When \( \epsilon \) and \( \mu \) are \( C^2 \) smooth, the unique continuation property was proved by Leis [6]. If \( \epsilon \) and \( \mu \) are \( C^1 \) smooth, the uniqueness of the Cauchy problem for (1.1) was established by Eller and Yamamoto [4]. Their result implies that if \((E, H)\) vanishes in an open subset of \( \Omega \), then it vanishes identically in \( \Omega \). Our result is obviously an improvement of those in [6] and [4].

When both \( \epsilon \) and \( \mu \) are Lipschitz, the strong unique continuation property for (1.1) was proved by Okaji [10] and by Vogelsang [12]. A recent result by Colombini and Koch [3] also implies the strong unique continuation property for (1.1) when \( \epsilon \) and \( \mu \) are in the Gevrey class. However, it is important to point out that additional structural assumptions on \( \epsilon \) and \( \mu \) are required in [3], [10], and [12]. In view of the counterexamples by Alinhac [1], it seems that the strong unique continuation property for (1.1) may not hold without such assumptions.

As with previous unique continuation results for (1.1), our results are proved using a Carleman estimate, which we derive in Section 2. To handle the Lipschitz smoothness of coefficients, our Carleman estimate contains a divergence term. This type of Carleman estimate was first introduced for the Stokes equations in [5] and was also useful in treating the Lamé system with less regular coefficients [7], [8]. In Section 3, we first reduce the Maxwell system (1.1) to a weakly coupled second order elliptic system, following [6] and [2]. We then apply our Carleman estimate to this reduced system to obtain Theorem 1.1. The proof of Corollary 1.3 is given in Section 4.

2. Carleman estimate

Denote \( \varphi(x) = |x|^{-s} = r^{-s} \). The proof of our main result relies on the following Carleman estimate.

**Proposition 2.1.** Let \( P = \sum_{ij} a_{ij}(x) \partial_i \partial_j \) be a second order elliptic operator with Lipschitz coefficients \( \{a_{ij}\} \) satisfying

\[
\lambda |\xi|^2 \leq \sum_{ij} a_{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2 \quad \forall \ x \in \mathbb{R}^3, \ \xi \in \mathbb{R}^3.
\]

Assume that there exists \( M > 0 \) such that

\[
\|a\|_{L^\infty(\mathbb{R}^3)} + \|\nabla a\|_{L^\infty(\mathbb{R}^3)} \leq M.
\]

Then there exist constants \( s_0 = s_0(\lambda, M) \) such that if \( \beta \geq s \geq s_0, \epsilon > 0 \) we can find \( R = R(s, \lambda, M, \epsilon) \in (0, 1) \) so that for any \( u \in H^s_0(B_R \setminus \{0\}) \), \( f \in L^2_0(B_{2R} \setminus \{0\})^3 \) with \( Pu + \text{div} f \in L^2(B_{2R} \setminus \{0\}) \), the following inequality holds:

\[
\beta^3 s^4 \int r^{-3s+3} e^{2\beta \varphi} |u|^2 dx + \beta s^2 \int r^{-s+5} e^{2\beta \varphi} |\nabla u|^2 dx
\leq C_{\lambda, M} \int r^7 e^{2\beta \varphi} |Pu + \text{div} f|^2 dx + C_{\lambda, M} \beta^3 s^2 \int r^{-3s+5} e^{2\beta \varphi} |f|^2 dx.
\]

**Remark.** It is clear that this estimate also holds when \( P \) is of divergence form, i.e., \( P = \sum_{ij} \partial_i (a_{ij} \partial_j) \).
Proof. Using standard approximation arguments, it is easy to see that we can
assume that \( u \in C_0^\infty(B_R \setminus \{0\}) \) and \( \text{div} f \in L_0^2(B_R \setminus \{0\}) \). Let \( m = s - 5 \) and
\( v = r^{-m/2} e^{\beta \varphi} u, \ g = r^{-m/2} e^{\beta \varphi} f \).

Then we have
\[
\begin{align*}
& r^\frac{m}{2} e^{\beta \varphi} (Pu + \text{div} f) \\
& = A + B + D + r^\frac{m}{2} \text{div} g + r^\frac{m}{2} \left( \frac{m}{2} r^{-2} + \beta sr^{-s-2} \right) x \cdot g,
\end{align*}
\]
where
\[
A = r^\frac{m}{2} a_{ij} \partial_{ij} v, \\
B = 2r^\frac{m}{2} \left( \frac{m}{2} r^{-2} + \beta sr^{-s-2} \right) a_{ij} \partial_i x_j v, \\
D = r^\frac{m}{2} v a_{ij} q_{ij}, \\
q_{ij} = x_i x_j \left( \beta^2 s^2 r^{-2s-4} - 7 \beta sr^{-s-4} + \frac{m(m-4)}{4} r^{-s} \right) \\
+ \delta_{ij} \left( \frac{m}{2} r^{-2} + \beta sr^{-s-2} \right).
\]

Let \( w \in H_0^1(B_R \setminus \{0\}) \) be a solution of
\[
\partial_i (a_{ij} \partial_j w) = \text{div} g
\]
and let
\[
A' = r^\frac{m}{2} a_{ij} \partial_{ij} (v + w), \\
B' = 2r^\frac{m}{2} \left( \frac{m}{2} r^{-2} + \beta sr^{-s-2} \right) a_{ij} x_j \partial_i (v + w).
\]

Then we have
\[
\begin{align*}
& r^\frac{m}{2} e^{\beta \varphi} (Pu + \text{div} f) - (A' + B' + D) \\
& = r^\frac{m}{2} \partial_i a_{ij} \partial_j w - 2r^\frac{m}{2} \left( \frac{m}{2} r^{-2} + \beta sr^{-s-2} \right) a_{ij} x_j \partial_i w \\
& + r^\frac{m}{2} e^{\beta \varphi} \left( \frac{m}{2} r^{-2} + \beta sr^{-s-2} \right) x \cdot f.
\end{align*}
\]

Using Lemma \ref{lemma2.2} below, it follows that
\[
(2.2) \quad \left( \int |r^\frac{m}{2} e^{\beta \varphi} (Pu + \text{div} f) - (A' + B' + D)|^2 \, dx \right)^{\frac{1}{2}} \\
\leq \beta^2 s^2 \int r^{-2s+5} e^{2\beta \varphi} |f|^2 \, dx + C_M \beta^2 s^2 \int r^{-s} |\nabla w|^2 \, dx \\
\leq \beta^2 s^2 \int r^{-2s+5} e^{2\beta \varphi} |f|^2 \, dx + C_M \beta^2 \int r^{-2s-\epsilon} |g|^2 \, dx \\
\leq C_M \beta^2 \int r^{-3s+5-\epsilon} e^{2\beta \varphi} |f|^2 \, dx.
\]
We have \( \int |A' + B' + D|^2 \, dx \geq 2 \text{Re} \int (A'B' + B'D) \, dx \). First consider
\[
I := 2 \text{Re} \int A'B' \, dx
\]
\[= 4 \text{Re} \int a_{ij} a_{kl} x_j \left( \frac{m}{2} r^s + \beta s \right) \partial_i (v + w) \partial_k (v + w) \, dx \]
\[= -4 \text{Re} \int \partial_i \left[ \left( \frac{m}{2} r^s + \beta s \right) a_{kl} a_{ij} x_j \right] \partial_k (v + w) \partial_i (v + w) \, dx \]
\[+ 2 \text{Re} \int \partial_i \left[ \left( \frac{m}{2} r^s + \beta s \right) a_{kl} a_{ij} x_j \right] \partial_k (v + w) \partial_i (v + w) \, dx \]

Here in the last identity we have used integration by parts three times. By using elliptic regularity, we have
\[
|I| \leq C_M \beta s \int \left| \nabla (v + w) \right|^2 \, dx \leq C_M \beta s \int \left( |\nabla v|^2 + |\nabla w|^2 \right) \, dx
\]
\[\leq C_M \beta s \int r^{-s+5} e^{2\beta \varphi} |\nabla u|^2 \, dx + C_M \beta^3 s^3 \int r^{-3s+3} e^{2\beta \varphi} |u|^2 \, dx
\]
\[+ C_M \beta s \int r^{-s+5} e^{2\beta \varphi} |f|^2 \, dx.
\]

(2.3)

On the other hand, we have
\[
II := 2 \text{Re} \int B'D \, dx
\]
\[= 2 \text{Re} \int a_{ij} a_{kl} x_j \left( \frac{m}{2} r^s + \beta s \right) q_{kl} x_j \partial_i (|v|^2) \, dx \]
\[+ 4 \text{Re} \int \left( \frac{m}{2} r^s + \beta s \right) a_{ij} x_j \partial_i w a_{kl} q_{kl} \, dx \]
\[= -2 \int |v|^2 \partial_i \left[ a_{ij} a_{kl} \left( \frac{m}{2} r^s + \beta s \right) q_{kl} x_j \right] \, dx \]
\[+ 4 \text{Re} \int \left( \frac{m}{2} r^s + \beta s \right) a_{ij} x_j \partial_i w a_{kl} q_{kl} \, dx \]

(2.3) \[\Rightarrow II' + II''.\]

When \( \beta \geq s \) is large enough, depending on \( \lambda \) and \( M \), the dominating term in \( II' \) is
\[
-2 \int |v|^2 \partial_i \left[ a_{ij} a_{kl} \left( \frac{m}{2} r^s + \beta s \right) \right] \beta^2 s^2 r^{-2s-4} x_k x_j \, dx
\]
\[\geq \beta^3 s^4 \int |v|^2 r^{-2s-6} a_{ij} a_{kl} x_i x_j x_k x_l \geq \lambda^2 \beta^3 s^4 \int r^{-2s-2} |v|^2 \, dx
\]
\[\geq \lambda^2 \beta^3 s^4 \int r^{-3s+3} e^{2\beta \varphi} |u|^2 \, dx.
\]

(2.4)

Next, using Cauchy-Schwarz and Lemma 2.2 below, \( II'' \) can be bounded by
\[
\frac{\lambda^2 \beta^3 s^4}{4} \int r^{-2s-2} |v|^2 \, dx + C_{\lambda, M} \beta^3 s^2 \int r^{-2s} |\nabla w|^2
\]
\[\leq \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta \varphi} |u|^2 \, dx + C_{\lambda, M} \beta^3 s^2 \int r^{-2s-\epsilon} |g|^2
\]
\[= \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta \varphi} |u|^2 \, dx + C_{\lambda, M} \beta^3 s^2 \int r^{-3s+5-\epsilon} e^{2\beta \varphi} |f|^2.
\]
From (2.24), (2.25), we get

\begin{equation}
(2.6) \quad \int r^\gamma e^{2\beta \varphi} |Pu + \text{div} f|^2 \, dx \geq \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta \varphi} |u|^2 \, dx \\
- C_{\lambda,M} \beta s \int r^{-s+5} e^{2\beta \varphi} |\nabla u|^2 \, dx - C_{\lambda,M} \beta^3 s^2 \int r^{-3s+5-\epsilon} e^{2\beta \varphi} |f|^2 \, dx.
\end{equation}

To get the $|\nabla u|^2$ term, we use integration by parts to derive

\begin{equation}
\int r^{-s+5} e^{2\beta \varphi} (Pu + \text{div} f) \tilde{u} \, dx = - \int r^{-s+5} e^{2\beta \varphi} a_{ij} \partial_i u \partial_j \tilde{u} \, dx \\
- \int \partial_j (a_{ij} r^{-s+5} e^{2\beta \varphi}) \partial_i u \tilde{u} \, dx - \int f : \nabla (r^{-s+5} e^{2\beta \varphi} \tilde{u}) \, dx.
\end{equation}

Multiplying this with $\beta s^2$ and using Cauchy-Schwarz, we get

\begin{equation}
\lambda s^2 \int r^{-s+5} e^{2\beta \varphi} |\nabla u|^2 \, dx \leq \beta s^2 \int r^{-s+5} e^{2\beta \varphi} a_{ij} \partial_i u \partial_j \tilde{u} \, dx \\
\leq \int r^7 e^{2\beta \varphi} |Pu + \text{div} f|^2 \, dx + \frac{\lambda s^2}{2} \int r^{-s+5} e^{2\beta \varphi} |\nabla u|^2 \, dx \\
+ C_{\lambda,M} \beta^3 s^4 \int r^{-3s+3} e^{2\beta \varphi} |u|^2 \, dx + C_{\lambda,M} \beta s^2 \int r^{-s+5} e^{2\beta \varphi} |f|^2 \, dx,
\end{equation}

which implies that

\begin{equation}
(2.7) \quad \int r^7 e^{2\beta \varphi} |Pu + \text{div} f|^2 \, dx \geq \frac{\lambda s^2}{2} \int r^{-s+5} e^{2\beta \varphi} |\nabla u|^2 \, dx \\
- C_{\lambda,M} \beta^3 s^4 \int r^{-3s+3} e^{2\beta \varphi} |u|^2 \, dx - C_{\lambda,M} \beta s^2 \int r^{-s+5} e^{2\beta \varphi} |f|^2 \, dx.
\end{equation}

Multiplying (2.6) with $8\lambda^{-2} C_{\lambda,M}$, then adding (2.7), we get (2.1), provided $s > 4\lambda^{-1} C_{\lambda,M}$. □

Now we prove a technical lemma used in the proof above.

**Lemma 2.2.** Assume that the hypotheses in Proposition 2.1 hold. There exists $s_0 > 3$ such that for any $s > s_0$ and $\epsilon > 0$, there exists $R = R(s, \lambda, M, \epsilon) > 0$ such that if $w \in H^s_0(B_R \setminus \{0\})$, $g \in L^2_0(B_R \setminus \{0\})^3$ satisfying

\begin{equation}
\partial_i (a_{ij} \partial_j w) = \text{div} g,
\end{equation}

then

\begin{equation}
(2.8) \quad \int r^{-2s} |\nabla w|^2 \, dx \leq \int r^{-2s-\epsilon} |g|^2 \, dx.
\end{equation}

**Proof.** By an orthonormal change of coordinates, we can assume, without loss of generality, that $a_{ij}(0) = \alpha_i \delta_{ij}$, where $\lambda \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \lambda^{-1}$. Let

\begin{equation}
h_i(x) = -g_i(x) + [a_{ij}(x) - a_{ij}(0)] \partial_j w(x).
\end{equation}

Then we get

\begin{equation}
(2.9) \quad \partial_i (a_{ij}(0) \partial_j w) + \text{div} h = 0.
\end{equation}

Let $x = \Lambda y$, where $\Lambda = \text{diag}(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \sqrt{\alpha_3})$, $\tilde{w}(y) = w(\Lambda y)$, and $\tilde{h}(y) = h_i(\Lambda y)/\sqrt{\alpha_i}$. Then (2.10) is equivalent to

\begin{equation}
\Delta_y \tilde{w} + \text{div}_y \tilde{h} = 0,
\end{equation}

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which implies that
\[ |y|^2 \Delta_y \tilde{w} + |y| \text{div}_y(|y|\tilde{h}) = y \cdot \tilde{h}. \]

Using the Carleman inequality with a divergence term in [8, Lemma 2.1] (taking \( n = 3 \) there), we obtain
\[
s \int |y|^{-2s+1} |\log |y||^{-4s+2} |\nabla_y \tilde{w}|^2 dy \\
\leq C \int |y|^{-2s+1} \left( |y|^2 \Delta_y \tilde{w} + |y| \text{div}_y(|y|\tilde{h}) \right)^2 dy + C s^2 \int |y|^{-2s-1} |y|\tilde{h}|^2 dy \\
\leq C s^2 \int |y|^{-2s+1} |\tilde{h}|^2 dy.
\]

Here \( C \) is an absolute constant. Undoing the change of variables and using (2.9) we get
\[
\int |x|^{-2s+1} \left( |\log |x|| \right)^{-4s+2} |\nabla_x w|^2 \\
\leq \frac{Cs}{\lambda^{2s+3}} \int |x|^{-2s+1} \left( |g|^2 + M^2 |x|^2 |\nabla_x w|^2 \right).
\]

Note that (2.11) holds true as long as \( s \) is sufficiently large and the supports of \( w \) and \( h \) are contained in \( B_{\sqrt{\lambda}} \setminus \{0\} \). Choose \( R = R(s, \lambda, \epsilon) < \sqrt{\lambda} \) small enough so that
\[ \left| \log \frac{|x|}{\sqrt{\lambda}} \right|^{-4s+2} \geq \frac{2Cs}{\lambda^{2s+3}} |x|^{\epsilon} \text{ if } |x| \leq R. \]

Then
\[
2 \int |x|^{-2s+1+\epsilon} |\nabla_x w|^2 \leq \int |x|^{-2s+1} \left( |g|^2 + M^2 |x|^2 |\nabla_x w|^2 \right).
\]

If furthermore, \( M^2 R^{2-\epsilon} < 1 \), then the second terms of the right-hand side will be absorbed by the left-hand side, and we obtain an equivalent form of (2.8).

3. **Three-ball inequality**

In this section, we prove the main result: the three-ball inequality in Theorem 1.1. We first reduce the Maxwell system (1.1) to a weakly coupled second order elliptic system, following [2, Lemma 1] and [6, page 168].

Denote
\[
\gamma_{jl}^k = \begin{cases} 
1 & \text{if } (k, j, l) \text{ is an even permutation of } (1, 2, 3), \\
-1 & \text{if } (k, j, l) \text{ is an odd permutation of } (1, 2, 3), \\
0 & \text{otherwise},
\end{cases}
\]
so that by (1.1),
\[ \partial_k H = \nabla H_k + i\omega \gamma^k \varepsilon E \quad \text{and} \quad \partial_k E = \nabla E_k - i\omega \gamma^k \mu H. \]

Taking the divergence of (1.1), we get
\[ \text{div}(\varepsilon E) = \text{div}(\mu H) = 0; \]

hence for \( k = 1, 2, 3, \)
\[ 0 = \partial_k \text{div}(\mu H) = \text{div}(\mu \partial_k H) + \text{div}(\partial_k \mu H) \]
\[ = \text{div}(\mu \nabla H_k) + \text{div}(\partial_k \mu H + i\omega \gamma^k \varepsilon E). \]
Similarly, we have

\[ 0 = \text{div}(\varepsilon \nabla E_k) + \text{div}(\partial_k \varepsilon E - i \omega \varepsilon \chi H). \]  

Together, (3.1) and (3.2) constitute our weakly coupled system.

Without loss of generality, we can assume \( x_0 = 0 \). Let \( 0 \leq \chi \leq 1 \) be a cutoff function satisfying

- \( \chi(x) = 1 \) if \( 2r_0/3 < |x| < r_2/2 \),
- \( \chi(x) = 0 \) if \( |x| \leq r_0/2 \) or \( |x| \geq 2r_2/3 \), and
- \( |\partial^\alpha \chi(x)| \leq 10 |x|^{-|\alpha|} \forall x \), for \( |\alpha| = 1, 2 \).

Let \( U = (E, H) \in (L^2_{\text{loc}}(\Omega))^6 \), \( \tilde{E} = \chi E \), \( \tilde{H} = \chi H \), \( \tilde{U} = \chi U \). Then from (3.1) and (3.2) we obtain

\[
|\text{div}(\varepsilon \nabla \tilde{E}_k) + \text{div}(\partial_k \varepsilon \tilde{E})| \leq C_M |x|^{-2}(|U| + |\nabla U|),
\]

\[
|\text{div}(\mu \nabla \tilde{H}_k) + \text{div}(\partial_k \mu \tilde{H})| \leq C_M |x|^{-2}(|U| + |\nabla U|).
\]

Let \( \epsilon = 1 \) and \( R(s, \lambda, M, 1) \) be the constant given in Proposition 2.1. Note that when \( \varepsilon \) and \( \mu \) are Lipschitz and \( (E, H) \in (L^2_{\text{loc}}(\Omega))^6 \), by the regularity theorem of [13], we have \( (E, H) \in (H^1_{\text{loc}}(\Omega))^6 \). Hence, if \( \rho \leq R(s, \lambda, M, 1) \), we can apply the Carleman estimate (2.1) to \( \tilde{E} \) and \( \tilde{H} \) to obtain

\[
\beta^3 s^4 \int |x|^{-3s+3} \varepsilon^2 \beta \phi |\tilde{U}|^2 dx + \beta s^2 \int |x|^{-s+5} \varepsilon^2 \beta \phi |\nabla \tilde{U}|^2 dx
\]

\[
\leq C_{\lambda, M} \int_{2r_0/3 < |x| < r_2/2} |x|^3 \varepsilon^2 \beta \phi \left( |U|^2 + |\nabla U|^2 \right) dx
\]

\[
+ C_{\lambda, M} \int_{(r_0/2 < |x| < 2r_0/3) \cup (r_2/2 < |x| < 2r_2/3)} |x|^3 \varepsilon^2 \beta \phi \left( |U|^2 + |\nabla U|^2 \right) dx
\]

\[
+ C_{\lambda, M} \beta^3 s^2 \int |x|^{-3s+4} \varepsilon^2 \beta \phi |\tilde{U}|^2 dx.
\]

Choose \( s = \max\{C_{\lambda, M}, s_0\} \). Then the first term of the right-hand side of (3.3) is absorbed by its left-hand side. For sufficiently small \( \rho \), the third term of the right-hand side of (3.3) is also absorbed by the left-hand side. Consequently, we obtain

\[
\int_{2r_0/3 < |x| < r_1} \varepsilon^2 \beta \phi |U|^2 dx \leq \int_{2r_0/3 < |x| < r_2/2} |x|^{-3s+3} \varepsilon^2 \beta \phi |U|^2 dx
\]

\[
\leq \int_{(r_0/2 < |x| < 2r_0/3) \cup (r_2/2 < |x| < 2r_2/3)} |x|^3 \varepsilon^2 \beta \phi \left( |U|^2 + |\nabla U|^2 \right) dx.
\]

It follows that

\[
\varepsilon^2 \beta \phi s^4 \int_{2r_0/3 < |x| < r_1} |U|^2 dx
\]

\[
\leq (r_0/2)^3 \varepsilon^2 \beta (r_0/2)^{-s} \int_{r_0/2 < |x| < 2r_0/3} \left( |U|^2 + |\nabla U|^2 \right) dx
\]

\[
+ (r_2/2)^3 \varepsilon^2 \beta (r_2/2)^{-s} \int_{r_2/2 < |x| < 2r_2/3} \left( |U|^2 + |\nabla U|^2 \right) dx.
\]
Adding $e^{2\beta r_1^2} \int_{|x|<2r_0/3} |U|^2 dx$ to both sides of (3.5) and using a Caccioppoli-type inequality, we obtain

$$\int_{|x|<r_1} |U|^2 dx \leq C \lambda M e^{2\beta [(r_0/2)^{s-r}]-r_1^{-s}} \int_{|x|<r_0} |U|^2 dx + C \lambda M e^{2\beta [(r_2/2)^{s-r}]-r_2^{-s}} \int_{|x|<r_2} |U|^2 dx,$$

(3.6)

for all $\beta \geq s$.

By standard arguments, we can then deduce that

$$\int_{|x|<r_1} |U|^2 dx \leq C \left( \int_{|x|<r_0} |U|^2 dx \right)^\tau \left( \int_{|x|<r_2} |U|^2 dx \right)^{1-\tau},$$

where

$$C = \max\{C\lambda M, e^{2\beta [(r_1)^{s-r}-(r_2/2)^{s-r}]}\} \quad \text{and} \quad \tau = \frac{(2r_1)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}}.$$

Corollary 1.2 is an easy consequence of the three-ball inequality (1.4). The arguments can be found in [7].

4. Minimal decay rate at infinity

In this section, we prove Corollary 1.3. Choose $\rho_0 < \rho/5$ small depending on $K$ so that for any $x_0$,

$$\int_{B_{3\rho_0}(x_0)} (|E|^2 + |H|^2) dx \leq 1.$$

Then from the three-ball inequality (1.4) with $r_0 = r$, $r_1 = 2r$, $r_2 = 5r$ where $r \leq \rho_0$ we get

$$\int_{B_2(x_0)} (|E|^2 + |H|^2) dx \leq C \left( \int_{B_r(x_0)} (|E|^2 + |H|^2) dx \right)^\tau,$$

(4.1)

where $C$ and $\tau$ are defined as in Theorem 1.1.

If $|x_0 - x_1| \leq r$ so that $B_r(x_1) \subset B_2(x_0)$, then we obtain from (4.1),

$$C^{-1/\tau} \left( \int_{B_r(x_1)} (|E|^2 + |H|^2) dx \right)^{1/\tau} \leq \int_{B_r(x_0)} (|E|^2 + |H|^2) dx.$$

(4.2)

Now if $|x_0| = t$, we can find a sequence of points $x_1, \cdots, x_N = 0$ such that $|x_j - x_{j+1}| \leq r$ for $j = 0, \cdots, N - 1$, where $N \leq 1 + t/r$. Applying (4.2) repeatedly with
$x_j$ and $x_{j+1}$ in place of $x_0$ and $x_1$, we get

$$
\int_{B_r(x_0)} (|E|^2 + |H|^2) \, dx \geq C^{-1/\tau} \left( \int_{B_r(x_1)} (|E|^2 + |H|^2) \, dx \right)^{1/\tau}
$$

$$
\geq C^{-1/\tau - 1/\tau^2} \left( \int_{B_r(x_2)} (|E|^2 + |H|^2) \, dx \right)^{1/\tau^2}
$$

$$
\geq \ldots \geq C^{-1/\tau + \frac{1-N}{1-\tau^2}} \left( \int_{B_r(0)} (|E|^2 + |H|^2) \, dx \right)^{1/\tau^N}
$$

$$
\geq \left( C^{-\frac{1}{\tau}} \int_{B_r(0)} (|E|^2 + |H|^2) \, dx \right)^{1/\tau^N}.
$$

Noting that

$$
\tau = \frac{(2r_1)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}} = \frac{4^{-s} - 5^{-s}}{1 - 5^{-s}}
$$

is independent of $r$ and $N \leq 1 + t/r$, we obtain (1.6) with $c = 1/\tau$ and

$$
L = \left( C^{-\frac{1}{\tau}} \int_{B_r(0)} (|E|^2 + |H|^2) \, dx \right)^{1/\tau}.
$$

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