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QUANTITATIVE UNIQUENESS ESTIMATE FOR THE MAXWELL SYSTEM WITH LIPSCHITZ ANISOTROPIC MEDIA

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ABSTRACT. We study quantitative uniqueness estimates for the time harmonic Maxwell system with Lipschitz anisotropic media. Our main results are a three-balls inequality and a minimal vanishing rate at a point of any nontrivial solution. The proof relies on a Carleman estimate with a divergence term.

1. INTRODUCTION

In this paper we study local properties of solutions of the time-harmonic Maxwell system with anisotropic media

$$\begin{cases} \mathrm{curl} H &= i\omega\varepsilon E \quad \text{in} \quad \Omega, \\ \mathrm{curl} E &= -i\omega\mu H \quad \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here Ω is an open subset of $\mathbb{R}^3$, $\omega \in \mathbb{C}\backslash\{0\}$ and $\varepsilon(x), \mu(x)$ are two real symmetric matrix-valued functions in Ω satisfying

$$\lambda|\xi|^2 \le \varepsilon(x)\xi\cdot\xi \le \lambda^{-1}|\xi|^2,\ \lambda|\xi|^2 \le \mu(x)\xi\cdot\xi \le \lambda^{-1}|\xi|^2,\ \forall\ x\in\Omega,\ \xi\in\mathbb{R}^3, \tag{1.2}$$

and

$$\|\varepsilon\|_{W^{1,\infty}(\Omega)} + \|\mu\|_{W^{1,\infty}(\Omega)} \le M, \tag{1.3}$$

where $0 < \lambda < 1$ and M are positive constants.

We prove a quantitative uniqueness estimate in the form of a three balls inequality for (E, H) satisfying (1.1). As a by-product, we derive a minimal vanishing rate at an arbitrary point of Ω for any nontrivial (E, H), which implies a unique continuation property for (1.1). When $\Omega = \mathbb{R}^3$, we also obtain a minimal decay rate of any nontrivial bounded (E, H) at infinity. We now state our main results. Let us denote $B_R(x_0) = \{|x - x_0| < R\}$.

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Theorem 1.1. *Assume that* (1.2) *and* (1.3) *hold. Then there exist* $\rho, s > 0$, *depending on* λ, M, *such that for any* $(E,H) \in (L^2_{loc}(\Omega))^6$ *solving* (1.1) *and* $r_0 < r_1 < r_2/2 < \rho$ *with* $B_{r_2}(x_0) \subset \Omega$, *we have*

$$\begin{aligned}(1.4)\qquad &\int_{B_{r_1}(x_0)} (|E|^2 + |H|^2)dx \\ &\le C\left(\int_{B_{r_0}(x_0)} (|E|^2+|H|^2)dx\right)^{\tau}\left(\int_{B_{r_2}(x_0)} (|E|^2+|H|^2)dx\right)^{1-\tau},\end{aligned}$$

where C *depends on* λ, M, r_1, r_2, s *and*

$$\tau = \frac{(2r_1)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}}.$$

In fact, the constant C *is explicitly given by*

$$C = \max\{C_{\lambda,M}, e^{2s[(r_1)^{-s}-(r_2/2)^{-s}]}\}$$

with $C_{\lambda,M}$ *depending on* λ *and* M.

Corollary 1.2. *Assume that the hypotheses in Theorem* 1.1 *hold. Let* r_0, r_1, r_2 *be described as above. Then for any nontrivial* $(E,H) \in (L^2_{loc}(\Omega))^6$ *satisfying* (1.1),

$$(1.5)\qquad \int_{B_{r_0}(x_0)} (|E|^2 + |H|^2)dx \ge \exp(-Cr_0^{-s}) \int_{B_{r_2}(x_0)} (|E|^2 + |H|^2)dx,$$

where C *depends on* λ, M, r_1, r_2, s *and*

$$\frac{\int_{B_{r_2}(x_0)} (|E|^2+|H|^2)dx}{\int_{B_{r_1}(x_0)} (|E|^2+|H|^2)dx}.$$

From Corollary 1.2, it immediately follows that if Ω is connected and for some $x_0 \in \Omega$, (E,H) satisfies

$$\int_{B_r(x_0)} (|E|^2 + |H|^2)dx \le C_N \exp(-Nr^{-s}), \quad \forall\, r \ll 1,\ \forall N \in \mathbb{N},$$

then $E = H \equiv 0$ in Ω. This is the unique continuation property with exponential vanishing rate for (1.1). Note that we do not impose any structural assumptions on the matrices ε and μ at x_0.

Similar to the result in [9], using Corollary 1.2, one can also study the minimal decay rate of any nontrivial bounded solution (E,H) to (1.1) with $\Omega = \mathbb{R}^3$. Denote

$$M_r(t) = \inf_{|x|=t} \int_{|y-x|<r} (|E|^2 + |H|^2)dy.$$

We have the following:

Corollary 1.3. *Assume that the hypotheses of Theorem* 1.1 *hold. Let* $(E,H) \in (L^2_{loc}(\mathbb{R}^3))^6$ *be a nontrivial solution to*

$$\begin{cases} \mathrm{curl} H &= i\omega\varepsilon E \quad \text{in } \mathbb{R}^3, \\ \mathrm{curl} E &= -i\omega\mu H \quad \text{in } \mathbb{R}^3. \end{cases}$$

Suppose that there exists $K > 0$ *such that*

$$\|E\|_{L^\infty(\mathbb{R}^3)} + \|H\|_{L^\infty(\mathbb{R}^3)} \le K.$$

Then there exist ρ_0 *depending on* λ, M, K, *and* $c > 1$ *depending on* λ, M, *and* L *depending on* $\int_{B_r(0)}(|E|^2 + |H|^2)dx$, λ, M, r, *such that for all* $0 < r \le \rho_0$,

$$M_r(t) \ge L^{c^{t/r}}. \tag{1.6}$$

We now mention some related results. When ε and μ are C^2 smooth, the unique continuation property was proved by Leis [6]. If ε and μ are C^1 smooth, the uniqueness of the Cauchy problem for (1.1) was established by Eller and Yamamoto [4]. Their result implies that if (E, H) vanishes in an open subset of Ω, then it vanishes identically in Ω. Our result is obviously an improvement of those in [6] and [4].

When both ε and μ are Lipschitz, the strong unique continuation property for (1.1) was proved by Ōkaji [10] and by Vogelsang [12]. A recent result by Colombini and Koch [3] also implies the strong unique continuation property for (1.1) when ε and μ are in the Gevrey class. However, it is important to point out that additional structural assumptions on ε and μ are required in [3], [10], and [12]. In view of the counterexamples by Alinhac [1], it seems that the strong unique continuation property for (1.1) may not hold without such assumptions.

As with previous unique continuation results for (1.1), our results are proved using a Carleman estimate, which we derive in Section 2. To handle the Lipschitz smoothness of coefficients, our Carleman estimate contains a divergence term. This type of Carleman estimate was first introduced for the Stokes equations in [5] and was also useful in treating the Lamé system with less regular coefficients [7], [8]. In Section 3, we first reduce the Maxwell system (1.1) to a weakly coupled second order elliptic system, following [6] and [2]. We then apply our Carleman estimate to this reduced system to obtain Theorem 1.1. The proof of Corollary 1.3 is given in Section 4.

2. Carleman estimate

Denote $\varphi(x) = |x|^{-s} = r^{-s}$. The proof of our main result relies on the following Carleman estimate.

Proposition 2.1. *Let* $P = \sum_{ij} a_{ij}(x)\partial_i\partial_j$ *be a second order elliptic operator with Lipschitz coefficients* $\{a_{ij}\}$ *satisfying*

$$\lambda\,|\xi|^2 \le \sum_{ij} a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}\,|\xi|^2 \quad \forall\ x \in \mathbb{R}^3,\ \xi \in \mathbb{R}^3.$$

Assume that there exists $M > 0$ *such that*

$$\|a\|_{L^\infty(\mathbb{R}^3)} + \|\nabla a\|_{L^\infty(\mathbb{R}^3)} \le M.$$

Then there exist constants $s_0 = s_0(\lambda, M)$ *such that if* $\beta \ge s \ge s_0$, $\epsilon > 0$ *we can find* $R = R(s, \lambda, M, \epsilon) \in (0, 1)$ *so that for any* $u \in H_0^1(B_R\backslash\{0\})$, $f \in L_0^2(B_R\backslash\{0\})^3$ *with* $Pu + \mathrm{div} f \in L^2(B_R\backslash\{0\})$, *the following inequality holds:*

$$\begin{aligned}\beta^3 s^4 \int r^{-3s+3} e^{2\beta\varphi}|u|^2 dx + \beta s^2 \int r^{-s+5} e^{2\beta\varphi}\,|\nabla u|^2\,dx \\ \le C_{\lambda,M}\int r^7 e^{2\beta\varphi}\,|Pu + \mathrm{div} f|^2\,dx + C_{\lambda,M}\beta^3 s^2 \int r^{-3s+5-\epsilon} e^{2\beta\varphi}\,|f|^2\,dx.\end{aligned} \tag{2.1}$$

Remark. It is clear that this estimate also holds when P is of divergence form, i.e., $P = \sum_{ij}\partial_i(a_{ij}\partial_j)$.

Proof. Using standard approximation arguments, it is easy to see that we can assume that $u \in C_0^\infty(B_R\backslash\{0\})$ and $\text{div} f \in L_0^2(B_R\backslash\{0\})$. Let $m = s-5$ and

$$v = r^{-m/2}e^{\beta\varphi}u, \quad g = r^{-m/2}e^{\beta\varphi}f.$$

Then we have

$$\begin{aligned} & r^{\frac{7}{2}}e^{\beta\varphi}(Pu + \text{div} f) \\ & = A + B + D + r^{\frac{m+7}{2}}\text{div} g + r^{\frac{m+7}{2}}\left(\frac{m}{2}r^{-2} + \beta s r^{-s-2}\right) x\cdot g, \end{aligned}$$

where

$$\begin{aligned} A &= r^{\frac{m+7}{2}}a_{ij}\partial_{ij}v, \\ B &= 2r^{\frac{m+7}{2}}\left(\frac{m}{2}r^{-2} + \beta s r^{-s-2}\right)a_{ij}\partial_i v x_j, \\ D &= r^{\frac{m+7}{2}}v a_{ij}q_{ij}, \\ q_{ij} &= x_i x_j\left(\beta^2 s^2 r^{-2s-4} - 7\beta s r^{-s-4} + \frac{m(m-4)}{4}r^{-s}\right) \\ &\quad + \delta_{ij}\left(\frac{m}{2}r^{-2} + \beta s r^{-s-2}\right). \end{aligned}$$

Let $w \in H_0^1(B_R\backslash\{0\})$ be a solution of

$$\partial_i(a_{ij}\partial_j w) = \text{div} g$$

and let

$$\begin{aligned} A' &= r^{\frac{m+7}{2}}a_{ij}\partial_{ij}(v+w), \\ B' &= 2r^{\frac{m+7}{2}}\left(\frac{m}{2}r^{-2} + \beta s r^{-s-2}\right)a_{ij}x_j\partial_i(v+w). \end{aligned}$$

Then we have

$$\begin{aligned} & r^{\frac{7}{2}}e^{\beta\varphi}(Pu + \text{div} f) - (A' + B' + D) \\ & \quad = r^{\frac{m+7}{2}}\partial_i a_{ij}\partial_j w - 2r^{\frac{m+7}{2}}\left(\frac{m}{2}r^{-2} + \beta s r^{-s-2}\right)a_{ij}x_j\partial_i w \\ & \qquad + r^{\frac{7}{2}}e^{\beta\varphi}\left(\frac{m}{2}r^{-2} + \beta s r^{-s-2}\right)x\cdot f. \end{aligned}$$

Using Lemma 2.2 below, it follows that

$$\begin{aligned} & \int \left| r^{\frac{7}{2}}e^{\beta\varphi}(Pu + \text{div} f) - (A' + B' + D)\right|^2 dx \\ & \le \beta^2 s^2 \int r^{-2s+5}e^{2\beta\varphi}|f|^2\,dx + C_M\beta^2 s^2\int r^{-s}|\nabla w|^2\,dx \\ & \le \beta^2 s^2 \int r^{-2s+5}e^{2\beta\varphi}|f|^2\,dx + C_{\lambda,M}\beta^2\int r^{-2s-\epsilon}|g|^2\,dx \\ & \le C_{\lambda,M}\beta^2\int r^{-3s+5-\epsilon}e^{2\beta\varphi}|f|^2\,dx. \end{aligned} \tag{2.2}$$

We have $\int |A' + B' + D|^2 dx \ge 2\mathrm{Re} \int (A'\overline{B'} + B'\overline{D}) dx$. First consider

$$\begin{aligned} I &:= 2\mathrm{Re} \int A'\overline{B'} dx \\ &= 4\mathrm{Re} \int a_{ij} a_{kl} x_j \left(\frac{m}{2} r^s + \beta s \right) \partial_i (v+w) \partial_{kl} (\overline{v+w}) dx \\ &= -4\mathrm{Re} \int \partial_l \left[\left(\frac{m}{2} r^s + \beta s \right) a_{kl} a_{ij} x_j \right] \partial_k (v+w) \partial_i (\overline{v+w}) dx \\ &\quad + 2\mathrm{Re} \int \partial_i \left[\left(\frac{m}{2} r^s + \beta s \right) a_{kl} a_{ij} x_j \right] \partial_k (v+w) \partial_l (\overline{v+w}) dx. \end{aligned}$$

Here in the last identity we have used integration by parts three times. By using elliptic regularity, we have

$$\begin{aligned} |I| &\le C_M \beta s \int |\nabla (v+w)|^2 dx \le C_M \beta s \int \left(|\nabla v|^2 + |\nabla w|^2 \right) dx \\ &\le C_M \beta s \int r^{-s+5} e^{2\beta\varphi} |\nabla u|^2 dx + C_M \beta^3 s^3 \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx \\ &\quad + C_M \beta s \int r^{-s+5} e^{2\beta\varphi} |f|^2 dx. \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\begin{aligned} II &:= 2\mathrm{Re} \int B'\overline{D} dx \\ &= 2\mathrm{Re} \int a_{ij} a_{kl} \left(\frac{m}{2} r^s + \beta s \right) q_{kl} x_j \partial_i \left(|v|^2 \right) dx \\ &\quad + 4\mathrm{Re} \int \left(\frac{m}{2} r^s + \beta s \right) a_{ij} x_j \partial_i w a_{kl} q_{kl} \overline{v} dx \\ &= -2 \int |v|^2 \partial_i \left[a_{ij} a_{kl} \left(\frac{m}{2} r^s + \beta s \right) q_{kl} x_j \right] dx \\ &\quad + 4\mathrm{Re} \int \left(\frac{m}{2} r^s + \beta s \right) a_{ij} x_j \partial_i w a_{kl} q_{kl} \overline{v} dx \\ &=: II' + II''. \end{aligned}$$

When $\beta \ge s$ is large enough, depending on λ and M, the dominating term in II' is

$$\begin{aligned} &- 2 \int |v|^2 \partial_i \left[a_{ij} a_{kl} \left(\frac{m}{2} r^s + \beta s \right) \beta^2 s^2 r^{-2s-4} x_k x_l x_j \right] dx \\ &\ge \beta^3 s^4 \int |v|^2 r^{-2s-6} a_{ij} a_{kl} x_i x_j x_k x_l \ge \lambda^2 \beta^3 s^4 \int r^{-2s-2} |v|^2 dx \\ &= \lambda^2 \beta^3 s^4 \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx. \end{aligned} \tag{2.4}$$

Next, using Cauchy-Schwarz and Lemma 2.2 below, II'' can be bounded by

$$\begin{aligned} &\frac{\lambda^2 \beta^3 s^4}{4} \int r^{-2s-2} |v|^2 dx + C_{\lambda,M} \beta^3 s^2 \int r^{-2s} |\nabla w|^2 \\ &\le \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx + C_{\lambda,M} \beta^3 s^2 \int r^{-2s-\epsilon} |g|^2 \\ &= \frac{\lambda^2 \beta^3 s^4}{4} \int r^{-3s+3} e^{2\beta\varphi} |u|^2 dx + C_{\lambda,M} \beta^3 s^2 \int r^{-3s+5-\epsilon} e^{2\beta\varphi} |f|^2 . \end{aligned} \tag{2.5}$$

From (2.2)-(2.5), we get

$$\int r^7 e^{2\beta\varphi}\left|Pu+\mathrm{div} f\right|^2 dx \geq \frac{\lambda^2\beta^3 s^4}{4}\int r^{-3s+3}e^{2\beta\varphi}|u|^2 dx \\ - C_{\lambda,M}\beta s\int r^{-s+5}e^{2\beta\varphi}\left|\nabla u\right|^2 dx - C_{\lambda,M}\beta^3 s^2\int r^{-3s+5-\epsilon}e^{2\beta\varphi}\left|f\right|^2 dx. \tag{2.6}$$

To get the $|\nabla u|^2$ term, we use integration by parts to derive

$$\int r^{-s+5}e^{2\beta\varphi}(Pu+\mathrm{div} f)\overline{u}dx = -\int r^{-s+5}e^{2\beta\varphi}a_{ij}\partial_i u\partial_j\overline{u}dx \\ -\int \partial_j\left(a_{ij}r^{-s+5}e^{2\beta\varphi}\right)\partial_i u\overline{u}dx - \int f\cdot\nabla\left(r^{-s+5}e^{2\beta\varphi}\overline{u}\right)dx.$$

Multiplying this with βs^2 and using Cauchy-Schwarz, we get

$$\lambda\beta s^2\int r^{-s+5}e^{2\beta\varphi}\left|\nabla u\right|^2 dx \leq \beta s^2\int r^{-s+5}e^{2\beta\varphi}a_{ij}\partial_i u\partial_j\overline{u}dx \\ \leq \int r^7 e^{2\beta\varphi}\left|Pu+\mathrm{div} f\right|^2 dx + \frac{\lambda\beta s^2}{2}\int r^{-s+5}e^{2\beta\varphi}\left|\nabla u\right|^2 dx \\ + C_{\lambda,M}\beta^3 s^4\int r^{-3s+3}e^{2\beta\varphi}|u|^2 dx + C_{\lambda,M}\beta s^2\int r^{-s+5}e^{2\beta\varphi}\left|f\right|^2 dx,$$

which implies that

$$\int r^7 e^{2\beta\varphi}\left|Pu+\mathrm{div} f\right|^2 dx \geq \frac{\lambda\beta s^2}{2}\int r^{-s+5}e^{2\beta\varphi}\left|\nabla u\right|^2 dx \\ - C_{\lambda,M}\beta^3 s^4\int r^{-3s+3}e^{2\beta\varphi}|u|^2 dx - C_{\lambda,M}\beta s^2\int r^{-s+5}e^{2\beta\varphi}\left|f\right|^2 dx. \tag{2.7}$$

Multiplying (2.6) with $8\lambda^{-2}C_{\lambda,M}$, then adding (2.7), we get (2.1), provided $s > 4\lambda^{-1}C_{\lambda,M}$. □

Now we prove a technical lemma used in the proof above.

Lemma 2.2. *Assume that the hypotheses in Proposition* 2.1 *hold. There exists* $s_0 > 3$ *such that for any* $s > s_0$ *and* $\epsilon > 0$, *there exists* $R = R(s,\lambda,M,\epsilon) > 0$ *such that if* $w \in H_0^1(B_R\backslash\{0\})$, $g \in L_0^2(B_R\backslash\{0\})^3$ *satisfying*

$$\partial_i(a_{ij}\partial_j w) = \mathrm{div} g,$$

then

$$\int r^{-2s}\left|\nabla w\right|^2 dx \leq \int r^{-2s-\epsilon}\left|g\right|^2 dx. \tag{2.8}$$

Proof. By an orthonormal change of coordinates, we can assume, without loss of generality, that $a_{ij}(0) = \alpha_i\delta_{ij}$, where $\lambda \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \lambda^{-1}$. Let

$$h_i(x) = -g_i(x) + \left[a_{ij}(x) - a_{ij}(0)\right]\partial_j w(x). \tag{2.9}$$

Then we get

$$\partial_i(a_{ij}(0)\partial_j w) + \mathrm{div} h = 0. \tag{2.10}$$

Let $x = \Lambda y$, where $\Lambda = \mathrm{diag}(\sqrt{\alpha_1},\sqrt{\alpha_2},\sqrt{\alpha_3})$, $\widehat{w}(y) = w(\Lambda y)$, and $\widehat{h}_i(y) = h_i(\Lambda y)/\sqrt{\alpha_i}$. Then (2.10) is equivalent to

$$\Delta_y\widehat{w} + \mathrm{div}_y\widehat{h} = 0,$$

which implies that

$$|y|^2\Delta_y\widehat{w}+|y|\mathrm{div}_y(|y|\widehat{h})=y\cdot\widehat{h}.$$

Using the Carleman inequality with a divergence term in [8, Lemma 2.1] (taking $n=3$ there), we obtain

$$\begin{aligned}&s\int|y|^{-2s+1}|\log|y||^{-4s+2}\,|\nabla_y\widehat{w}|^2\,dy\\&\le C\int|y|^{-2s+1}\left||y|^2\Delta_y\widehat{w}+|y|\mathrm{div}_y(|y|\widehat{h})\right|^2dy+Cs^2\int|y|^{-2s-1}\left||y|\widehat{h}\right|^2dy\\&\le Cs^2\int|y|^{-2s+1}|\widehat{h}|^2dy.\end{aligned}$$

Here C is an absolute constant. Undoing the change of variables and using (2.9) we get

$$\begin{aligned}&\int|x|^{-2s+1}\left|\log\frac{|x|}{\sqrt{\lambda}}\right|^{-4s+2}|\nabla_x w|^2\\&\le\frac{Cs}{\lambda^{2s+3}}\int|x|^{-2s+1}\left[|g|^2+M^2|x|^2|\nabla_x w|^2\right].\end{aligned}\tag{2.11}$$

Note that (2.11) holds true as long as s is sufficiently large and the supports of w and h are contained in $B_{\sqrt{\lambda}}\backslash\{0\}$. Choose $R=R(s,\lambda,\epsilon)<\sqrt{\lambda}$ small enough so that

$$\left|\log\frac{|x|}{\sqrt{\lambda}}\right|^{-4s+2}\ge\frac{2Cs}{\lambda^{2s+3}}|x|^{\epsilon}\quad\text{if}\quad|x|\le R.$$

Then

$$2\int|x|^{-2s+1+\epsilon}|\nabla_x w|^2\le\int|x|^{-2s+1}\left[|g|^2+M^2|x|^2|\nabla_x w|^2\right].$$

If furthermore, $M^2R^{2-\epsilon}<1$, then the second terms of the right-hand side will be absorbed by the left-hand side, and we obtain an equivalent form of (2.8). □

3. Three-ball inequality

In this section, we prove the main result: the three-ball inequality in Theorem 1.1. We first reduce the Maxwell system (1.1) to a weakly coupled second order elliptic system, following [2, Lemma 1] and [6, page 168].

Denote

$$\gamma^k_{jl}=\begin{cases}1&\text{if }(k,j,l)\text{ is an even permutation of }(1,2,3),\\-1&\text{if }(k,j,l)\text{ is an odd permutation of }(1,2,3),\\0&\text{otherwise},\end{cases}$$

so that by (1.1),

$$\partial_kH=\nabla H_k+\mathrm{i}\omega\gamma^k\varepsilon E\quad\text{and}\quad\partial_kE=\nabla E_k-\mathrm{i}\omega\gamma^k\mu H.$$

Taking the divergence of (1.1), we get

$$\mathrm{div}(\varepsilon E)=\mathrm{div}(\mu H)=0;$$

hence for $k=1,2,3$,

$$\begin{aligned}0&=\partial_k\mathrm{div}(\mu H)=\mathrm{div}(\mu\partial_kH)+\mathrm{div}(\partial_k\mu H)\\&=\mathrm{div}(\mu\nabla H_k)+\mathrm{div}(\partial_k\mu H+\mathrm{i}\omega\mu\gamma^k\varepsilon E).\end{aligned}\tag{3.1}$$

Similarly, we have

$$0 = \operatorname{div}(\varepsilon\nabla E_k) + \operatorname{div}(\partial_k \varepsilon E - \mathrm{i}\omega\varepsilon\gamma^k \mu H). \tag{3.2}$$

Together, (3.1) and (3.2) constitute our weakly coupled system.

Without loss of generality, we can assume $x_0 = 0$. Let $0 \le \chi \le 1$ be a cutoff function satisfying

- $\chi(x) = 1$ if $2r_0/3 < |x| < r_2/2$,
- $\chi(x) = 0$ if $|x| \le r_0/2$ or $|x| \ge 2r_2/3$, and
- $|\partial^\alpha \chi(x)| \le 10\,|x|^{-|\alpha|}\ \forall x$, for $|\alpha| = 1, 2$.

Let $U = (E, H) \in (L^2_{loc}(\Omega))^6$, $\widetilde{E} = \chi E$, $\widetilde{H} = \chi H$, $\widetilde{U} = \chi U$. Then from (3.1) and (3.2) we obtain

$$\begin{aligned}
&|\operatorname{div}(\varepsilon\nabla \widetilde{E}_k) + \operatorname{div}(\partial_k \varepsilon \widetilde{E})| \le C_M |x|^{-2}(|U| + |\nabla U|),\\
&|\operatorname{div}(\mu\nabla \widetilde{H}_k) + \operatorname{div}(\partial_k \mu \widetilde{H})| \le C_M |x|^{-2}(|U| + |\nabla U|).
\end{aligned}$$

Let $\epsilon = 1$ and $R(s, \lambda, M, 1)$ be the constant given in Proposition 2.1. Note that when ε and μ are Lipschitz and $(E, H) \in (L^2_{loc}(\Omega))^6$, by the regularity theorem of [13], we have $(E, H) \in (H^1_{loc}(\Omega))^6$. Hence, if $\rho \le R(s, \lambda, M, 1)$, we can apply the Carleman estimate (2.1) to $\widetilde{E}$ and $\widetilde{H}$ to obtain

$$\begin{aligned}
&\beta^3 s^4 \int |x|^{-3s+3} e^{2\beta\varphi} |\widetilde{U}|^2 dx + \beta s^2 \int |x|^{-s+5} e^{2\beta\varphi} |\nabla \widetilde{U}|^2 dx\\
&\le C_{\lambda,M} \int_{2r_0/3<|x|<r_2/2} |x|^3 e^{2\beta\varphi} \left(|U|^2 + |\nabla U|^2\right) dx\\
&\quad + C_{\lambda,M} \int_{\{r_0/2<|x|<2r_0/3\}\cup\{r_2/2<|x|<2r_2/3\}} |x|^3 e^{2\beta\varphi} \left(|U|^2 + |\nabla U|^2\right) dx\\
&\quad + C_{\lambda,M} \beta^3 s^2 \int |x|^{-3s+4} e^{2\beta\varphi} |\widetilde{U}|^2 dx.
\end{aligned} \tag{3.3}$$

Choose $s = \max\{C_{\lambda,M}, s_0\}$. Then the first term of the right-hand side of (3.3) is absorbed by its left-hand side. For sufficiently small ρ, the third term of the right-hand side of (3.3) is also absorbed by the left-hand side. Consequently, we obtain

$$\begin{aligned}
\int_{2r_0/3<|x|<r_1} e^{2\beta\varphi} |U|^2 dx &\le \int_{2r_0/3<|x|<r_2/2} |x|^{-3s+3} e^{2\beta\varphi} |U|^2 dx\\
&\le \int_{\{r_0/2<|x|<2r_0/3\}\cup\{r_2/2<|x|<2r_2/3\}} |x|^3 e^{2\beta\varphi} \left(|U|^2 + |\nabla U|^2\right) dx.
\end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned}
&e^{2\beta r_1^{-s}} \int_{2r_0/3<|x|<r_1} |U|^2 dx\\
&\le (r_0/2)^3 e^{2\beta(r_0/2)^{-s}} \int_{r_0/2<|x|<2r_0/3} (|U|^2 + |\nabla U|^2) dx\\
&\quad + (r_2/2)^3 e^{2\beta(r_2/2)^{-s}} \int_{r_2/2<|x|<2r_2/3} (|U|^2 + |\nabla U|^2) dx.
\end{aligned} \tag{3.5}$$

Adding $e^{2\beta r_1^{-s}}\int_{|x|<2r_0/3}|U|^2dx$ to both sides of (3.5) and using a Caccioppolli-type inequality, we obtain

$$\begin{aligned}\int_{|x|<r_1}|U|^2dx &\le C_{\lambda,M}e^{2\beta[(r_0/2)^{-s}-r_1^{-s}]}\int_{|x|<r_0}|U|^2\,dx\\ &+\ C_{\lambda,M}e^{2\beta[(r_2/2)^{-s}-r_1^{-s}]}\int_{|x|<r_2}|U|^2\,dx,\end{aligned}\tag{3.6}$$

for all $\beta\ge s$.

By standard arguments, we can then deduce that

$$\int_{|x|<r_1}|U|^2dx\le C\left(\int_{|x|<r_0}|U|^2\,dx\right)^{\tau}\left(\int_{|x|<r_2}|U|^2\,dx\right)^{1-\tau},$$

where

$$C=\max\{C_{\lambda,M},e^{2s[(r_1)^{-s}-(r_2/2)^{-s}]}\}\quad\text{and}\quad\tau=\frac{(2r_1)^{-s}-r_2^{-s}}{r_0^{-s}-r_2^{-s}}.$$

Corollary 1.2 is an easy consequence of the three-ball inequality (1.4). The arguments can be found in [7].

4. Minimal decay rate at infinity

In this section, we prove Corollary 1.3. Choose $\rho_0<\rho/5$ small depending on K so that for any x_0,

$$\int_{B_{5\rho_o}(x_0)}(|E|^2+|H|^2)dx\le 1.$$

Then from the three-ball inequality (1.4) with $r_0=r$, $r_1=2r$, $r_2=5r$ where $r\le\rho_0$ we get

$$\int_{B_{2r}(x_0)}(|E|^2+|H|^2)dx\le C\left(\int_{B_r(x_0)}(|E|^2+|H|^2)dx\right)^{\tau},\tag{4.1}$$

where C and τ are defined as in Theorem 1.1.

If $|x_0-x_1|\le r$ so that $B_r(x_1)\subset B_{2r}(x_0)$, then we obtain from (4.1),

$$C^{-1/\tau}\left(\int_{B_r(x_1)}(|E|^2+|H|^2)dx\right)^{1/\tau}\le\int_{B_r(x_0)}(|E|^2+|H|^2)dx.\tag{4.2}$$

Now if $|x_0|=t$, we can find a sequence of points $x_1,\cdots,x_N=0$ such that $|x_{j+1}-x_j|\le r$ for $j=0,\cdots,N-1$, where $N\le 1+t/r$. Applying (4.2) repeatedly with

x_j and x_{j+1} in place of x_0 and x_1, we get

$$\begin{aligned}&\int_{B_r(x_0)}(|E|^2+|H|^2)dx \geq C^{-1/\tau}\left(\int_{B_r(x_1)}(|E|^2+|H|^2)dx\right)^{1/\tau}\\&\geq C^{-1/\tau-1/\tau^2}\left(\int_{B_r(x_2)}(|E|^2+|H|^2)dx\right)^{1/\tau^2}\\&\geq \ldots \geq C^{-\frac{1-\tau^N}{\tau^N(1-\tau)}}\left(\int_{B_r(0)}(|E|^2+|H|^2)dx\right)^{1/\tau^N}\\&\geq \left(C^{-\frac{1}{1-\tau}}\int_{B_r(0)}(|E|^2+|H|^2)dx\right)^{1/\tau^N}.\end{aligned}$$

Noting that

$$\tau = \frac{(2r_1)^{-s}-r_2^{-s}}{r_0^{-s}-r_2^{-s}} = \frac{4^{-s}-5^{-s}}{1-5^{-s}}$$

is independent of r and $N \leq 1+t/r$, we obtain (1.6) with $c = 1/\tau$ and

$$L = \left(C^{-\frac{1}{1-\tau}}\int_{B_r(0)}(|E|^2+|H|^2)dx\right)^{1/\tau}.$$

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