EQUIVARIANT $K$-THEORY AND THE CHERN CHARACTER
FOR DISCRETE GROUPS

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Abstract. Let $X$ be a compact Hausdorff space, let $\Gamma$ be a discrete group that acts continuously on $X$ from the right, define $\tilde{X} = \{(x, \gamma) \in X \times \Gamma : x \cdot \gamma = x\}$, and let $\Gamma$ act on $\tilde{X}$ via the formula $(x, \gamma) \cdot \alpha = (x \cdot \alpha, \alpha^{-1} \gamma \alpha)$. Results of P. Baum and A. Connes, along with facts about the Chern character, imply that $K^i_\Gamma(X)$ and $K^i(\tilde{X}/\Gamma)$ are isomorphic up to torsion for $i = 0, 1$. In this paper, we present an example where the groups $K^i_\Gamma(X)$ and $K^i(\tilde{X}/\Gamma)$ are not isomorphic.

Let $\Gamma$ be a finite discrete group acting continuously on a compact Hausdorff space $X$ from the right. Define $\tilde{X} = \{(x, \gamma) \in X \times \Gamma : x \cdot \gamma = x\}$ and endow $\tilde{X}$ with the subspace topology that it inherits from $X \times \Gamma$. The group $\Gamma$ acts on $\tilde{X}$ via the formula $(x, \gamma) \cdot \alpha = (x \cdot \alpha, \alpha^{-1} \gamma \alpha)$, and we can consider the orbit space $\tilde{X}/\Gamma$.

Theorem 1.19 in [1] states that there exist isomorphisms

$$K^0_\Gamma(X) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j}(\tilde{X}/\Gamma; \mathbb{C}),$$

$$K^1_\Gamma(X) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j+1}(\tilde{X}/\Gamma; \mathbb{C}),$$

where $H^*(\tilde{X}/\Gamma; \mathbb{C})$ denotes the Čech cohomology of $\tilde{X}/\Gamma$ with complex coefficients. We also have the Chern character isomorphisms

$$K^0(\tilde{X}/\Gamma) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j}(\tilde{X}/\Gamma; \mathbb{C}),$$

$$K^1(\tilde{X}/\Gamma) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j+1}(\tilde{X}/\Gamma; \mathbb{C}).$$

Therefore $K^j_\Gamma(X) \otimes \mathbb{C} \cong K^i(\tilde{X}/\Gamma) \otimes \mathbb{C}$ for $i = 0, 1$. In fact, a careful reading of [1] shows that we can replace $\mathbb{C}$ by $\mathbb{Q}(\omega)$, where $\omega$ is the order of $\Gamma$ and where $\omega = \exp(2\pi i/n)$. In any event, the groups $K^i_\Gamma(X)$ and $K^i(\tilde{X}/\Gamma)$ are isomorphic up to torsion.

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to torsion. In this paper, we present an example where these groups do not have isomorphic torsion subgroups.

Our example is Example B in [1]. Consider the unit 3-sphere $S^3$ in $\mathbb{R}^4$ and define $\alpha : S^3 \rightarrow S^3$ by $\alpha(x, y, z, t) = (-x, -y, -z, t)$. The map $\alpha$ defines a $\mathbb{Z}_2$ action on $S^3$. From [2], we know that the $\Gamma$-equivariant $K$-theory groups of a compact Hausdorff space $X$ are isomorphic to the operator algebra $K$-theory groups of $C(X) \rtimes \Gamma$. Combining this fact with the computations in [1] we have $K^0_{\mathbb{Z}_2}(S^3) \cong \mathbb{Z}$ and $K^1_{\mathbb{Z}_2}(S^3) \cong 0$.

Observe that $S^3$ is homeomorphic to the disjoint union of $S^3$ and the fixed point set $F$ of the action of $\alpha$, and hence $S^3/\mathbb{Z}_2$ is homeomorphic to the disjoint union of $S^3/\mathbb{Z}_2$ and $F$. In our example, $F$ is the two-point set $\{(0, 0, 0, 1), (0, 0, 0, -1)\}$, so $K^0(F) \cong \mathbb{Z}^2$ and $K^1(F) \cong 0$.

To compute the $K$-theory of $S^3/\mathbb{Z}_2$, define closed sets

\[
A = \{(x, y, z, t) \in S^3 : t \geq 0\},
\]

\[
B = \{(x, y, z, t) \in S^3 : t \leq 0\}.
\]

Then $(A/\mathbb{Z}_2) \cup (B/\mathbb{Z}_2) = S^3/\mathbb{Z}_2$ and $(A/\mathbb{Z}_2) \cap (B/\mathbb{Z}_2) \cong S^2/\mathbb{Z}_2 \cong \mathbb{R}P^2$. Applying the Mayer-Vietoris sequence for reduced $K$-theory ([1], Exercise 3.2), we have the six-term exact sequence

\[
\begin{array}{cccccc}
\tilde{K}^0(S^3/\mathbb{Z}_2) & \longrightarrow & \tilde{K}^0(A/\mathbb{Z}_2) \oplus \tilde{K}^0(B/\mathbb{Z}_2) & \longrightarrow & \tilde{K}^0(\mathbb{R}P^2) \\
\uparrow & & & & \downarrow \\
\tilde{K}^1(\mathbb{R}P^2) & \longleftarrow & \tilde{K}^1(A/\mathbb{Z}_2) \oplus \tilde{K}^1(B/\mathbb{Z}_2) & \longleftarrow & \tilde{K}^1(S^3/\mathbb{Z}_2).
\end{array}
\]

Both $A/\mathbb{Z}_2$ and $B/\mathbb{Z}_2$ are homeomorphic to the cone over $\mathbb{R}P^2$, which has trivial reduced $K$-theory groups, and so the vertical maps in the six-term exact sequence are isomorphisms. Therefore

\[
K^0(S^3/\mathbb{Z}_2) \cong \tilde{K}^0(S^3/\mathbb{Z}_2) \oplus \mathbb{Z} \cong \tilde{K}^1(\mathbb{R}P^2) \oplus \mathbb{Z} \cong \mathbb{Z},
\]

\[
K^1(S^3/\mathbb{Z}_2) \cong \tilde{K}^1(S^3/\mathbb{Z}_2) \cong \tilde{K}^0(\mathbb{R}P^2) \cong \mathbb{Z}_2,
\]

which yield

\[
K^0(\tilde{S^3}/\mathbb{Z}_2) \cong K^0(S^3/\mathbb{Z}_2) \oplus K^0(F) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},
\]

\[
K^1(\tilde{S^3}/\mathbb{Z}_2) \cong K^1(S^3/\mathbb{Z}_2) \oplus K^1(F) \cong \mathbb{Z}_2.
\]

Thus $K^1_{\mathbb{Z}_2}(S^3)$ and $K^1(\tilde{S^3}/\mathbb{Z}_2)$ are isomorphic up to torsion, but the groups $K^1_{\mathbb{Z}_2}(S^3)$ and $K^1(S^3/\mathbb{Z}_2)$ are not isomorphic.

References


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