ON THE MENGEL COVERING PROPERTY AND D-SPACES

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Abstract. The main results of this paper are:

• It is consistent that every subparacompact space \( X \) of size \( \omega_1 \) is a D-space.
• If there exists a Michael space, then all productively Lindelöf spaces have the Menger property and, therefore, are D-spaces.
• Every locally D-space which admits a \( \sigma \)-locally finite cover by Lindelöf spaces is a D-space.

1. Introduction

A neighbourhood assignment for a topological space \( X \) is a function \( N \) from \( X \) to the topology of \( X \) such that \( x \in N(x) \) for all \( x \). A topological space \( X \) is said to be a D-space \([6]\) if for every neighbourhood assignment \( N \) for \( X \) there exists a closed and discrete subset \( A \subset X \) such that \( N(A) = \bigcup_{x \in A} N(x) = X \).

It is unknown whether paracompact (even Lindelöf) spaces are D-spaces. Our first result in this note answers \([7\), Problem 3.8\] in the affirmative and may be thought of as a very partial solution to this problem.

Our second result shows that the affirmative answer to \([19\), Problem 2.6\], which asks whether all productively Lindelöf spaces are D-spaces, is consistent. It is worth mentioning that our premises (i.e., the existence of a Michael space) are not known to be inconsistent.

Our third result is a common generalization of two theorems from \([10]\).

Most of our proofs use either the recent important result of Aurichi \([2]\) asserting that every topological space with the Menger property is a D-space or the ideas from its proof. We consider only regular topological spaces. For the definitions of small cardinals \( d \) and \( \text{cov}(\mathcal{M}) \) used in this paper we refer the reader to \([22]\).

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2. Subparacompact spaces of size \( \omega_1 \)

Following [4] we say that a topological space \( X \) has the property \( E^*_\omega \) if for every sequence \( \langle u_n : n \in \omega \rangle \) of countable open covers of \( X \) there exists a sequence \( \langle u_n : n \in \omega \rangle \) such that \( v_n \in \{ u_n \}^{<\omega} \) and \( \bigcup_{n \in \omega} v_n = X \). In the realm of Lindelöf spaces the property \( E^*_\omega \) is usually called the Menger property or \( \bigcup_{n \in \omega} (\mathcal{O}, \mathcal{O}) \); see [21] and references therein.

We say that a topological space \( X \) has property \( D_\omega \) if for every neighbourhood assignment \( N \) there exists a countable collection \( \{ A_n : n \in \omega \} \) of closed discrete subsets of \( X \) such that \( X = \bigcup_{n \in \omega} N(A_n) \). Observe that the property \( D_\omega \) is inherited by all closed subsets.

The following theorem is the main result of this section.

**Theorem 2.1.** Suppose that a topological space \( X \) has properties \( D_\omega \) and \( E^*_\omega \). Then \( X \) is a \( D \)-space.

The proof of Theorem 2.1 is analogous to the proof of [3, Proposition 2.6]. In particular, it uses the following game of length \( \omega \) on a topological space \( X \): On the \( n \)th move player \( I \) chooses a countable open cover \( u_n = \{ U_{n,k} : k \in \omega \} \) such that \( U_{n,k} \subset U_{n,k+1} \) for all \( k \in \omega \), and player \( II \) responds by choosing a natural number \( n_k \). Player \( II \) wins the game if \( \bigcup_{n \in \omega} U_{n,k_n} = X \). Otherwise, player \( I \) wins. We shall call this game an \( E^*_\omega \)-game. In the realm of Lindelöf spaces this game is known under the name Menger game. It is well known that a Lindelöf space \( X \) has the property \( E^*_\omega \) if and only if the first player has no winning strategy in the \( E^*_\omega \)-game on \( X \); see [8, [14]. The proof of [14] Theorem 13] also works without any change for non-Lindelöf spaces.

**Proposition 2.2.** A topological space \( X \) has the property \( E^*_\omega \) if and only if the first player has no winning strategy in the \( E^*_\omega \)-game.

A strategy of the first player in the \( E^*_\omega \)-game may be thought of as a map \( \Upsilon : \omega^{<\omega} \to \mathcal{O}(X) \), where \( \mathcal{O}(X) \) stands for the collection of all countable open covers of \( X \). The strategy \( \Upsilon \) is winning if \( X \neq \bigcup_{n \in \omega} U_{\omega}(n) \) for all \( z \in \omega^{\omega} \), where \( \Upsilon(s) = \{ U_{s,k} : k \in \omega \} \in \mathcal{O}(X) \).

We are in a position now to present the proof of Theorem 2.1.

**Proof.** We shall define a strategy \( \Upsilon : X \to \mathcal{O}(X) \) of the player \( I \) in the \( E^*_\omega \)-game on \( X \) as follows. Set \( F_0 = X \). The property \( D_\omega \) yields an increasing sequence \( \langle A_{\emptyset,k} : k \in \omega \rangle \) of closed discrete subsets of \( F_0 \) such that \( X = \bigcup_{k \in \omega} N(A_{\emptyset,k}) \). Set \( \Upsilon(\emptyset) = u_0 = \{ N(A_{\emptyset,k}) : k \in \omega \} \).

Suppose that for some \( m \in \omega \) and all \( s \in \omega^{\leq m} \) we have already defined a closed subset \( F_s \) of \( X \), an increasing sequence \( \langle A_{s,k} : k \in \omega \rangle \) of closed discrete subsets of \( F_s \), and a countable open cover \( \Upsilon(s) = u_s \) of \( X \) such that \( u_s = \{ (X \setminus F_s) \cup N(A_{s,k}) : k \in \omega \} \).

Fix \( s \in \omega^{m+1} \). Since \( X \) has the property \( D_\omega \), so does its closed subspace \( F_s := X \setminus \bigcup_{i < m+1} N(A_{s,i,s(i)}) \), and hence there exists an increasing sequence \( \langle A_{s,k} : k \in \omega \rangle \) of closed discrete subsets of \( F_s \) such that \( F_s \subset \bigcup_{k \in \omega} N(A_{s,k}) \). Set \( \Upsilon(s) = u_s = \{ (X \setminus F_s) \cup N(A_{s,k}) : k \in \omega \} \). This completes the definition of \( \Upsilon \).

Since \( X \) has the property \( E^*_\omega \), \( \Upsilon \) is not winning. Thus there exists \( z \in \omega^\omega \) such that \( X = \bigcup_{n \in \omega} (X \setminus F_z(n)) \cup N(A_{z|n,z(n)}) \). By the inductive construction, \( X \setminus F_0 = \emptyset \) and \( X \setminus F_z(n) = \bigcup_{i < n} N(A_{z|n,z(i)}) \) for all \( n > 0 \). It follows from above that \( X = \bigcup_{n \in \omega} N(A_{z|n,z(n)}) \). In addition, \( A_{z|n,z(n)} \subset F_z(n) = X \setminus \bigcup_{i < n} N(A_{z|n,z(i)}) \).
for all $n > 0$, which implies that $A := \bigcup_{n \in \omega} A_{z|n,z(n)}$ is a closed discrete subset of $X$. It suffices to note that $N(A) = X$. \hfill \Box

We recall from [5] that a topological space $X$ is called \textit{subparacompact} if every open cover of $X$ has a $\sigma$-locally finite closed refinement.

\textbf{Lemma 2.3.} Suppose that $X$ is a subparacompact topological space which can be covered by $\omega_1$-many of its Lindelöf subspaces. Then $X$ has the property $D_\omega$.\footnote{By the methods of [15] the submetalindelöfness is sufficient here.}

In particular, every subparacompact space of size $\omega_1$ has the property $D_\omega$.

\textbf{Proof.} Let $\mathcal{L} = \{L_\xi : \xi < \omega_1\}$ be an increasing cover of $X$ by Lindelöf subspaces, let $\tau$ be the topology of $X$, and let $N : X \to \tau$ be a neighbourhood assignment. Construct by induction a sequence $(C_\alpha : \alpha < \omega_1)$ of (possibly empty) countable subspaces of $X$ such that

(i) $L_0 \subset N(C_0)$;
(ii) $C_\alpha \cap N(\bigcup_{\xi<\alpha} C_\xi) = \emptyset$ for all $\alpha < \omega_1$; and
(iii) $L_\alpha \setminus N(\bigcup_{\xi<\alpha} C_\xi) \subset N(C_\alpha)$ for all $\alpha < \omega_1$.

Set $C = \bigcup_{\alpha < \omega_1} C_\alpha$. The subparacompactness of $X$ yields a closed cover $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$ of $X$ which refines $\mathcal{U} = \{N(x) : x \in C\}$ and such that each $\mathcal{F}_n$ is locally finite. Since every element of $\mathcal{U}$ contains at most countably many elements of $C$, so do elements of $\mathcal{F}$. Therefore for every $F \in \mathcal{F}_n$ such that $C \cap F \neq \emptyset$ we can write this intersection in the form $\{x_{n,F,m} : m \in \omega\}$. Now it is easy to see that $A_{n,m} := \{x_{n,F,m} : F \in \mathcal{F}_n, C \cap F \neq \emptyset\}$ is a closed discrete subset of $X$ and $\bigcup_{n,m \in \omega} A_{n,m} = C$. \hfill \Box

\textbf{Remark 2.4.} What we have actually used in the proof of Lemma 2.3 is the following weakening of subparacompactness: every open cover $\mathcal{U}$ which is closed under unions of its countable subsets admits a $\sigma$-locally finite closed refinement. We do not know whether this property is strictly weaker than subparacompactness.

\textbf{Corollary 2.5.} Let $X$ be a countably tight paracompact topological space of density $\omega_1$. Then $X$ has the property $D_\omega$.

\textbf{Proof.} Let $\{x_\alpha : \alpha < \omega_1\}$ be a dense subspace of $X$. Since $X$ has countable tightness, $X = \bigcup_{\alpha < \omega_1} \{x_\xi : \xi < \alpha\}$. It suffices to note that the closure of any countable subspace of a paracompact space is Lindelöf. \hfill \Box

It is well known [9, Theorem 4.4] (and it easily follows from corresponding definitions) that any Lindelöf space of size $\aleph \emptyset$ has the Menger property. The same argument shows that every topological space of size $\aleph \emptyset$ has the property $E^*_\omega$. Combining this with Theorem 2.4 and Lemma 2.3 we get the following corollary, which implies the first of the results mentioned in our abstract.

\textbf{Corollary 2.6.} Suppose that $X$ is a subparacompact topological space of size $|X| < \aleph \emptyset$ which can be covered by $\omega_1$-many of its Lindelöf subspaces. Then $X$ is a $D$-space.

3. CONCERNING THE EXISTENCE OF A MICHAEL SPACE

A topological space $X$ is said to be \textit{productively Lindelöf} if $X \times Y$ is Lindelöf for all Lindelöf spaces $Y$. It was asked in [19] whether productively Lindelöf spaces are $D$-spaces. The positive answer to the above question has been proved consistent,
and in a stream of recent papers (see the list of references in [19]) several sufficient set-theoretical conditions were established. The following statement gives a uniform proof for some of these results. In particular, it implies [16] Theorems 5 and 7 and [1] Corollary 4.5 and answers [17] Question 15 in the affirmative.

A Lindelöf space \( Y \) is called a Michael space if \( \omega^\omega \times Y \) is not Lindelöf.

**Proposition 3.1.** If there exists a Michael space, then every productively Lindelöf space has the Menger property.

We refer the reader to [11], where the existence of a Michael space was reformulated in a combinatorial language and a number of set-theoretical conditions guaranteeing the existence of Michael spaces were established.

In the proof of Proposition 3.1 we shall use set-valued maps; see [13]. By a **set-valued map** \( \Phi \) from a set \( X \) into a set \( Y \) we understand a map from \( X \) into \( \mathcal{P}(Y) \) and write \( \Phi : X \to Y \) (here \( \mathcal{P}(Y) \) denotes the set of all subsets of \( Y \)). For a subset \( A \) of \( X \) we set \( \Phi(A) = \bigcup_{x \in A} \Phi(x) \subset Y \). A set-valued map \( \Phi \) from a topological spaces \( X \) to a topological space \( Y \) is said to be

- **compact-valued** if \( \Phi(x) \) is compact for every \( x \in X \);
- **upper semicontinuous** if for every open subset \( V \) of \( Y \) the set \( \Phi^{-1}(V) = \{ x \in X : \Phi(x) \subset V \} \) is open in \( X \).

The proof of the following claim is straightforward.

**Claim 3.2.**

1. Suppose that \( X, Y \) are topological spaces, \( X \) is Lindelöf, and \( \Phi : X \to Y \) is a compact-valued upper semicontinuous map such that \( Y = \Phi(X) \). Then \( Y \) is Lindelöf.
2. If \( \Phi_0 : X_0 \to Y_0 \) and \( \Phi_1 : X_1 \to Y_1 \) are compact-valued upper semicontinuous, then so is the map \( \Phi : X_0 \times X_1 \to Y_0 \times Y_1 \) assigning to each \( (x_0, x_1) \in X_0 \times X_1 \) the product \( \Phi_0(x_0) \times \Phi_1(x_1) \).

**Proof of Proposition 3.1.** Suppose, contrary to our claim, that \( X \) is a productively Lindelöf space which does not have the Menger property and \( Y \) is a Michael space. It suffices to show that \( X \times Y \) is not Lindelöf.

Indeed, by [23] Theorem 8 there exists a compact-valued upper semicontinuous map \( \Phi : X \to \omega^\omega \) such that \( \Phi(X) = \omega^\omega \). By Claim 3.2(1) the product \( \omega^\omega \times Y \) is the image of \( X \times Y \) under a compact-valued upper semicontinuous map. By the definition of a Michael space, \( \omega^\omega \times Y \) is not Lindelöf. By applying Claim 3.2(2) we can conclude that \( X \times Y \) is not Lindelöf either.

By a result of Tall [16] the existence of a Michael space implies that all productively Lindelöf analytic metrizable spaces are \( \sigma \)-compact. Combining recent results obtained in [1] and [12], we can consistently extend this result to all \( \Sigma^1_2 \) definable subsets of \( 2^\omega \).

**Theorem 3.3.** Suppose that \( \text{cov}(\mathcal{M}) > \omega_1 \) and that there exists a Michael space. Then every productively Lindelöf \( \Sigma^1_2 \) definable subset of \( 2^\omega \) is \( \sigma \)-compact.

**Proof.** Let \( X \) be a productively Lindelöf \( \Sigma^1_2 \) definable subset of \( 2^\omega \).

If \( X \) cannot be written as a union of \( \omega_1 \)-many of its compact subspaces, then it contains a closed copy of \( \omega^\omega \) [12], and hence the existence of the Michael space implies that \( X \) is not productively Lindelöf, a contradiction.

Thus \( X \) can be written as a union of \( \omega_1 \)-many of its compact subspaces, and therefore it is \( \sigma \)-compact by [1] Corollary 4.15.]
We do not know whether the assumption \( \text{cov}(M) > \omega_1 \) can be dropped from Theorem 3.3.

**Question 3.4.** Suppose that there exists a Michael space. Is every coanalytic productively Lindelöf space \( \sigma \)-compact?

By [18, Proposition 31] the affirmative answer to the question above follows from the Axiom of Projective Determinacy.

## 4. Locally finite unions

**Theorem 4.1.** Suppose that \( X \) is a locally \( D \)-space which admits a \( \sigma \)-locally finite cover by Lindelöf spaces. Then \( X \) is a \( D \)-space.

**Proof.** Let \( \mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n \) be a cover of \( X \) by Lindelöf subspaces such that \( \mathcal{F}_n \) is locally finite. Fix \( F \in \mathcal{F}_n \). For every \( x \in F \) there exists an open neighbourhood \( U_x \) of \( x \) such that \( U_x \) is a \( D \)-space. Let \( C_F \) be a countable subset of \( F \) such that \( F \subseteq \bigcup_{x \in C_F} U_x \). Then \( \mathcal{Z}_F = \{ F \cap U_x : x \in C_F \} \) is a countable cover of \( F \) consisting of closed \( D \)-subspaces of \( X \) such that \( F \cap Z \) is dense in \( Z \) for all \( Z \in \mathcal{Z}_F \). It follows from the above that \( X \) admits a \( \sigma \)-locally finite cover consisting of closed \( D \)-subspaces. Since a union of a locally finite family of closed \( D \)-subspaces is easily seen to be a closed \( D \)-subspace, \( X \) is a union of an increasing sequence of its closed \( D \)-subspaces. Therefore it is a \( D \)-space by results of [3]. □

**Corollary 4.2.** If a topological space \( X \) admits a \( \sigma \)-locally finite locally countable cover by topological spaces with the Menger property, then it is a \( D \)-space. In particular, a locally Lindelöf space admitting a \( \sigma \)-locally finite cover by topological spaces with the Menger property is a \( D \)-space.

**Proof.** The second part is a direct consequence of the first one since every \( \sigma \)-locally countable family of subspaces of a locally Lindelöf space is locally countable.

To prove the first assertion, note that by local countability every point \( x \in X \) has a closed neighbourhood which is a countable union of its subspaces with the Menger property, and hence it has the Menger property itself. Therefore \( X \) is a locally \( D \)-space. It now suffices to apply Theorem 4.1. □

It is known that every Lindelöf \( C \)-scattered space is \( C \)-like and that \( C \)-like spaces have the Menger property; see [20, p. 247] and references therein. Thus Corollary 4.2 implies Theorems 2.2 and 3.1 from [10].

## References

17. Tall, F.D., Lindel¨of spaces which are Menger, Hurewicz, Alster, productive, or $D$, Topology Appl., to appear.