SMOOTH LIVŠIC REGULARITY
FOR PIECEWISE EXPANDING MAPS

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Abstract. We consider the regularity of measurable solutions $\chi$ to the cohomological equation

$$\phi = \chi \circ T - \chi,$$

where $(T, X, \mu)$ is a dynamical system and $\phi: X \to \mathbb{R}$ is a $C^k$ smooth real-valued cocycle in the setting in which $T: X \to X$ is a piecewise $C^k$ Gibbs-Markov map, an affine $\beta$-transformation of the unit interval or more generally a piecewise $C^k$ uniformly expanding map of an interval. We show that under mild assumptions, bounded solutions $\chi$ possess $C^k$ versions. In particular we show that if $(T, X, \mu)$ is a $\beta$-transformation, then $\chi$ has a $C^k$ version, thus improving a result of Pollicott and Yuri.

1. INTRODUCTION

In this paper we consider the regularity of solutions $\chi$ to the cohomological equation

$$\phi = \chi \circ T - \chi,$$

where $(T, X, \mu)$ is a dynamical system and $\phi: X \to \mathbb{R}$ is a $C^k$ smooth real-valued cocycle. Such solutions $\chi$ are called coboundaries for the observation $\phi$. In particular we are interested in the setting in which $T: X \to X$ is a piecewise $C^k$ Gibbs-Markov map, an affine $\beta$-transformation of the unit interval or more generally a piecewise $C^k$ uniformly expanding map of an interval. The question we consider is when a solution $\chi$ with a certain degree of regularity is forced by the dynamics to have a higher degree of regularity. Cohomological equations arise frequently in ergodic theory and dynamics and the existence and regularity of solutions determine, for example, whether observations $\phi$ have positive variance in the central limit theorem and also have implications for other distributional limits (for examples see [20, 2]). Related cohomological equations to Equation (1) decide on stable ergodicity and weak-mixing of compact group extensions of hyperbolic systems [11, 20, 19] and also play a role in determining whether two dynamical systems are (Hölder, smoothly) conjugate to each other.
Livšic [13, 14] gave seminal results on the regularity of measurable solutions to cohomological equations for Abelian group extensions of Anosov systems with an absolutely continuous invariant measure. Theorems which establish that a priori measurable solutions to cohomological equations must have a higher degree of regularity are often called measurable Livšic theorems in honor of his work.

We say that \( \chi: X \to \mathbb{R} \) has a \( C^k \) version (with respect to \( \mu \)) if there exists a \( C^k \) function \( h: X \to \mathbb{R} \) such that \( h(x) = \chi(x) \) for \( \mu \) a.e. \( x \in X \).

Pollicott and Yuri [23] prove Livšic theorems for Hölder \( \mathbb{R} \)-extensions of \( \beta \)-transformations \( (T: [0, 1) \to [0, 1), T(x) = \beta x \mod 1 \) where \( \beta > 1 \)) via transfer operator techniques. They show that any essentially bounded measurable solution \( \chi \) to Equation (1) is of bounded variation on \([0, T_1)\) for all \( T_1 > 0 \). In this paper we improve this result to show that measurable coboundaries \( \chi \) for \( C^k \mathbb{R} \)-valued cocycles \( \phi \) over \( \beta \)-transformations have \( C^k \) versions (see Theorem 2).

Jenkinson [10] proves that integrable measurable coboundaries \( \chi \) for \( \mathbb{R} \)-valued smooth cocycles \( \phi \) (i.e. again solutions to \( \phi = \chi \circ T - \chi \)) over smooth expanding Markov maps \( T \) of \( S^1 \) have versions which are smooth on each partition element.

Nicol and Scott [15] have obtained measurable Livšic theorems for certain discontinuous hyperbolic systems, including \( \beta \)-transformations, Markov maps, mixing Lasota–Yorke maps, a simple class of toral-linked twist maps and Sinai dispersing billiards. They show that a measurable solution \( \chi \) to Equation (1) has a Lipschitz version for \( \beta \)-transformations and a simple class of toral-linked twist maps. For mixing Lasota–Yorke maps and Sinai dispersing billiards they show that such a \( \chi \) is Lipschitz on an open set. There is an error in [15, Theorem 1] in the setting of \( C^2 \) Markov maps; they only prove that measurable solutions \( \chi \) to equation (1) are Lipschitz on each element \( T\alpha \), \( \alpha \in \mathcal{P} \), where \( \mathcal{P} \) is the defining partition for the Markov map, and not that the solutions are Lipschitz on \( \alpha \), as Theorem 1 erroneously states. The error arose in the following way: if \( \chi \) is Lipschitz on \( \alpha \in \mathcal{P} \) it is possible to extend \( \chi \) as a Lipschitz function to \( T\alpha \) by defining \( \chi(Tx) = \phi(x) + \chi(x) \); however, extending \( \chi \) as a Lipschitz function from \( \alpha \) to \( T^2\alpha \) via the relation \( \chi(T^2x) = \phi(Tx) + \chi(Tx) \) may not be possible, as \( \phi \circ T \) may have discontinuities on \( T\alpha \). In this paper we give an example (see Section 3) which shows that for Markov maps this result cannot be improved on.

Gouëzel [7] has obtained similar results to Nicol and Scott [15] for cocycles into Abelian groups over one-dimensional Gibbs–Markov maps. In the setting of a Gibbs–Markov map with countable partition he proves that any measurable solution \( \chi \) to equation (1) is Lipschitz on each element \( T\alpha \), \( \alpha \in \mathcal{P} \), where \( \mathcal{P} \) is the defining partition for the Gibbs–Markov map.

In related work, Aaronson and Denker [1] Corollary 2.3] have shown that if \((T, X, \mu, \mathcal{P})\) is a mixing Gibbs–Markov map with countable Markov partition \( \mathcal{P} \) preserving a probability measure \( \mu \) and \( \phi: X \to \mathbb{R}^d \) is Lipschitz (with respect to a metric \( \rho \) on \( X \) derived from the symbolic dynamics), then any measurable solution \( \chi: X \to \mathbb{R}^d \) to \( \phi = \chi \circ T \) has a version \( \tilde{\chi} \) which is uniformly Lipschitz continuous on elements \( T(\alpha) \) with respect to \( \rho \); i.e., there exists \( C > 0 \) such that \( |\chi(x) - \chi(y)| \leq C \rho(x, y) \) for all \( x, y \in T(\alpha) \) and each \( \alpha \in \mathcal{P} \).

Bruin et al. [4] prove measurable Livšic theorems for dynamical systems modelled by Young towers and Hofbauer towers. Their regularity results apply to solutions of cohomological equations posed on Hénon-like mappings and a wide variety of non-uniformly hyperbolic systems. We note that Corollary 1 of [4, Theorem 1] is
not correct; the solution is Hölder only on $M_k$ and $TM_k$ rather than $T^j M_k$ for $j > 1$ as stated for reasons similar to those given above for the result in Nicol et al. \[15\].

2. Main results

We first describe one-dimensional Gibbs–Markov maps. Let $I \subset \mathbb{R}$ be a bounded interval and let $\mathcal{P}$ be a countable partition of $I$ into intervals. We let $m$ denote Lebesgue measure. Let $T: I \to I$ be a piecewise $C^k$, $k \geq 2$, expanding map such that $T$ is $C^k$ on the interior of each element of $\mathcal{P}$ with $|T'| > \lambda > 1$, and for each $\alpha \in \mathcal{P}$, $T\alpha$ is a union of elements in $\mathcal{P}$. Let $P_n := \bigvee_{j=0}^n T^{-j} \mathcal{P}$ and $J_T := \frac{d\mu_{T^n}}{dm}$.

We assume:

(i) (Big images property) There exists $C_1 > 0$ such that $m(T\alpha) > C_1$ for all $\alpha \in \mathcal{P}$.

(ii) There exists $0 < \gamma_1 < 1$ such that $m(\beta) < \gamma_1^n$ for all $\beta \in P_n$.

(iii) (Bounded distortion) There exists $0 < \gamma_2 < 1$ and $C_2 > 0$ such that

$$|1 - \frac{T(\chi(x))}{T(\chi(y))}| < C_2 \gamma_2^2$$

for all $x, y \in \beta$ if $\beta \in P_n$.

Under these assumptions $T$ has an invariant absolutely continuous probability measure $\mu$ and the density of $\mu$, $h = \frac{d\mu}{dm}$, is bounded above and below by a constant $0 < C^{-1} \leq h(x) \leq C$ for $m$ a.e. $x \in I$.

Note that a Markov map satisfies (i), (ii) and (iii) for finite partitions $\mathcal{P}$.

It is proved in \[15\] for the case of a Markov map (this means that $\mathcal{P}$ is finite), and in \[7\] for the case of a Gibbs–Markov map ($\mathcal{P}$ is countable) that if $\phi: I \to \mathbb{R}$ is Hölder continuous or Lipschitz continuous, and $\phi = \chi \circ T - \chi$ for some measurable function $\chi: I \to \mathbb{R}$, then there exists a function $\chi_0: I \to \mathbb{R}$ that is Hölder or Lipschitz on each of the elements of $\mathcal{P}$ respectively, and $\chi_0 = \chi$ holds $\mu$- (or $m$-) a.e. A related result to \[7\] is given in \[4\] Theorem 7, where $T$ is the base map of a Young Tower, which has a Gibbs–Markov structure.

Fried \[6\] has shown that the transfer operator of a graph directed Markov system with $C^{k, \alpha}$-contractions, acting on a space of $C^{k, \alpha}$-functions, has a spectral gap. If we apply his result to our setting, letting the contractions be the inverse branches of a Gibbs–Markov map we can conclude that the transfer operator of a Gibbs–Markov map acting on $C^k$-functions has a spectral gap. As in Jenkinson’s paper \[10\] and with the same proof, this gives us immediately the following proposition, which is implied by the results of Fried and Jenkinson:

**Proposition 1.** Let $T: T \to I$ be a mixing Gibbs–Markov map such that $T$ is $C^k$ on each partition element and $T^{-1}: T(\alpha) \to \alpha$ is $C^k$ on each partition element $\alpha \in \mathcal{P}$. Let $\phi: I \to \mathbb{R}$ be uniformly $C^k$ on each of the partition elements $\alpha \in \mathcal{P}$. Suppose $\chi: I \to \mathbb{R}$ is a measurable function such that $\phi = \chi \circ T - \chi$. Then there exists a function $\chi_0: I \to \mathbb{R}$ such that $\chi_0$ is uniformly $C^k$ on $T\alpha$ for each partition element $\alpha \in \mathcal{P}$, and $\chi_0 = \chi$ almost everywhere.

3. A counterexample

We remark that, in general, if $\phi = \chi \circ T - \chi$, one cannot expect $\chi$ to be continuous on $I$ if $\phi$ is $C^k$ on $I$. We give an example of a Markov map $T$ with Markov partition $\mathcal{P}$, a function $\phi$ that is $C^k$ on $I$, and a function $\chi$ that is $C^k$ on each element $\alpha$ of $\mathcal{P}$ such that $\phi = \chi \circ T - \chi$, yet $\chi$ has no version that is continuous on $I$. 

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Let $0 < c < \frac{1}{4}$. Put $d = 2 - 4c$. Define $T : [0, 1] \to [0, 1]$ by

$$T(x) = \begin{cases} 
2x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{4}, \\
d(x - \frac{1}{2}) + \frac{1}{2} & \text{if } \frac{1}{4} < x < \frac{3}{4}, \\
2x - \frac{3}{2} & \text{if } \frac{3}{4} \leq x \leq 1.
\end{cases}$$

If $c = \frac{1}{8}$, then the partition

$$\mathcal{P} = \left\{ \left[0, \frac{1}{8}\right], \left[\frac{1}{8}, \frac{1}{4}\right], \left[\frac{1}{4} - \frac{1}{4d}, \frac{1}{4} - \frac{1}{4d}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}\right], \left[\frac{1}{2} + \frac{1}{4d}, \frac{3}{8}\right], \left[\frac{3}{8}, \frac{7}{8}\right], \left[\frac{7}{8}, 1\right] \right\}$$

is a Markov partition for $T$. Define $\chi$ such that $\chi$ is 0 on $[\frac{1}{2} - \frac{1}{4d}, \frac{1}{2}]$ and 1 on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}]$. On $[0, \frac{1}{4})$ we define $\chi$ so that $\chi(0) = 1$ and $\lim_{x \to \frac{1}{4}} \chi(x) = 0$, and on $(\frac{3}{4}, 1]$ we define $\chi$ so that $\chi(1) = 0$ and $\lim_{x \to \frac{3}{4}} \chi(x) = 1$. For any natural number $k$, this can be done so that $\chi$ is $C^k$ except at the point $\frac{1}{2}$ where it has a jump. One easily checks that $\phi$ defined by $\phi = \chi \circ T - \chi$ is $C^k$. This is illustrated in Figures 1–4.
4. Livšic Theorems for Piecewise Expanding Maps of an Interval

Let \( I = [0, 1) \) and let \( m \) denote Lebesgue measure on \( I \). We consider piecewise expanding maps \( T: I \to I \), satisfying the following assumptions:

(i) There is a number \( \lambda > 1 \), and a finite partition \( \mathcal{P} \) of \( I \) into intervals, such that the restriction of \( T \) to any interval in \( \mathcal{P} \) can be extended to a \( C^2 \)-function on the closure, and \( |T'| > \lambda \) on this interval.

(ii) \( T \) has an absolutely continuous invariant measure \( \mu \) with respect to which \( T \) is mixing.

(iii) \( T \) has the property of being weakly covering, as defined by Liverani in [12], namely that there exists an \( n_0 \) such that for any element \( \alpha \in \mathcal{P} \),

\[
\bigcup_{j=0}^{n_0} T^j(\alpha) = I.
\]

For any \( n \geq 0 \) we define the partition \( \mathcal{P}_n = \mathcal{P} \lor \cdots \lor T^{-n+1} \mathcal{P} \). The partition elements of \( \mathcal{P}_n \) are called \( n \)-cylinders, and \( \mathcal{P}_n \) is called the partition of \( I \) into \( n \)-cylinders.

We prove the following two theorems.

**Theorem 1.** Let \((T,I,\mu)\) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). Let \( \phi: I \to \mathbb{R} \) be a function of bounded variation such that \( \phi = \chi \circ T - \chi \) for some measurable function \( \chi \), with \( e^{-\chi} \in L_1(m) \). Then there exists a function \( \chi_0 \) such that \( \chi_0 \) has bounded variation and \( \chi_0 = \chi \) almost everywhere.

For the next theorem we need some more definitions. Let \( A \) be a set, and denote by \( \text{int} A \) the interior of the set \( A \). We assume that the open sets \( T(\text{int} \alpha) \), where \( \alpha \) is an element in \( \mathcal{P} \), cover \( \text{int} I \).

We will now define a new partition \( \mathcal{Q} \). For a point \( x \) in the interior of some element of \( \mathcal{P} \), we let \( Q(x) \) be the largest open set such that for any \( x_2 \in Q(x) \), and any \( n \)-cylinder \( C_m \), there is a natural number \( n \), and two sequences of points \((y_{1,k}, y_{2,k})_{k=1}^n\), such that \( y_{1,k} \) and \( y_{2,k} \) are in the same element of \( \mathcal{P} \), \( T(y_{1,k+1}) = y_{1,k} \), \( T(y_{1,1}) = x \), \( T(y_{2,1}) = x_2 \), and \( y_{1,n}, y_{2,n} \in C_m \). (This forces \( n \geq m \).)

Note that if \( Q(x) \cap Q(y) \neq \emptyset \), then for \( z \in Q(x) \cap Q(y) \) we have \( Q(z) = Q(x) \cup Q(y) \). We let \( \mathcal{Q} \) be the coarsest collection of connected sets such that any element of \( \mathcal{Q} \) can be represented as a union of sets \( Q(x) \).

**Theorem 2.** Let \((T,I,\mu)\) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). If \( \phi: I \to \mathbb{R} \) is a continuously differentiable function such that \( \phi = \chi \circ T - \chi \) for some measurable function \( \chi \), with \( e^{-\chi} \in L_1(m) \), then there exists a function \( \chi_0 \) such that \( \chi_0 \) is continuously differentiable on each element of \( \mathcal{Q} \) and \( \chi_0 = \chi \) almost everywhere. If \( T' \) is constant on the elements of \( \mathcal{P} \), then \( \chi_0 \) is piecewise \( C^k \) on \( \mathcal{Q} \) if \( \phi \) is in \( C^k \). If for each \( r \), \( \frac{1}{T'} \) is in \( C^{k-1} \) with derivatives up to order \( k - 1 \) uniformly bounded, then \( \chi_0 \) is piecewise \( C^k \) on \( \mathcal{Q} \) if \( \phi \) is in \( C^k \).

It is not always clear how big the elements in the partition \( \mathcal{Q} \) are. The following lemma gives a lower bound on the diameter of the elements in \( \mathcal{Q} \).
Lemma 1. Assume that the sets \( \{ T(\text{int } \alpha) : \alpha \in \mathcal{P} \} \) cover \((0,1)\). Let \( \delta \) be the Lebesgue number of the cover. Then the diameter of \( Q(x) \) is at least \( \delta/2 \) for all \( x \).

Proof. Let \( C_m \) be a cylinder of generation \( m \). We need to show that for some \( n \geq m \) there are sequences \( \{y_{1,k}\}_{k=1}^n \) and \( \{y_{2,k}\}_{k=1}^n \) as in the definition of \( Q \) above.

Take \( n_0 \) such that \( \mu(T^{n_0}(C_m)) = 1 \). Write \( C_m \) as a finite union of cylinders of generation \( n_0 \), \( C_m = \bigcup_i D_i \). Then \( R := [0,1] \setminus T^{n_1}(\bigcup_i \text{int } D_i) \) consists of finitely many points. Let \( \epsilon \) be the smallest distance between two of these points.

Let \( I_\delta \) be an open interval of diameter \( \delta \). Let \( n_1 \) be such that \( \delta \lambda^{-n_1} < \epsilon \). Consider the collection of intervals \( H \) such that \( I_\delta \) is a continuous one-to-one image of \( H \) under \( T^{n_1} \). By the definition of \( \delta \), there is at least one such pre-image \( H \), and any such pre-image is of diameter less than \( \epsilon \). Hence any pre-image contains at most one point from \( R \).

If the pre-image \( H \) does not contain any point of \( R \), then \( I_\delta \) is contained in some element of \( Q \) and we are done. Assume that there is a point \( z \) in \( I_\delta \) corresponding to the point of \( R \) in the pre-image of \( I_\delta \). Assume that \( z \) is in the right half of \( I_\delta \). The case when \( z \) is in the left part is treated in a similar way. Take a new open interval \( J_\delta \) of length \( \delta \) such that the left half of \( J_\delta \) coincides with the right half of \( I_\delta \).

Arguing in the same way as for \( I_\delta \), we find that a pre-image of \( J_\delta \) contains at most one point of \( R \). If there is no such point, or the corresponding point \( z_j \in J_\delta \) is not equal to \( z \), then \( I_\delta \cup J_\delta \) is contained in an element in \( Q \) and we are done.

It remains to consider the case \( z = z_j \). Let \( I_\delta = (a,b) \) and \( J_\delta = (c,d) \). Then the intervals \((a,z)\) and \((z,d)\) are both of length at least \( \delta/2 \), and both are contained in some element of \( Q \). This finishes the proof. \( \square \)

Corollary 1. If \( \beta > 1 \) and \( T : x \mapsto \beta x \pmod{1} \) is a \( \beta \)-transformation, then clearly \( T \) is weakly covering and \( Q = \{(0,1)\} \), so in this case Theorem 2 and Theorem 1 of [15] imply that if \( \chi \) is measurable and \( e^{-\chi} \in L_1(m) \), then \( \chi_0 \) is in \( C^k \) if \( \phi \) is in \( C^k \).

Remark 1. If \( T : x \mapsto \beta x + \alpha \pmod{1} \) is an affine \( \beta \)-transformation, then \( Q = \{(0,1)\} \), and hence if \( e^{-\chi} \) is in \( L_1(m) \), then \( \chi \) has a \( C^k \) version.

5. Proof of Theorem 1

We continue to assume that \( (T,I,\mu) \) is a piecewise expanding map satisfying assumptions (i), (ii) and (iii). For a function \( \psi : I \rightarrow \mathbb{R} \) we define the weighted transfer operator \( \mathcal{L}_\psi \) by

\[
\mathcal{L}_\psi f(x) = \sum_{T(y) = x} e^{\psi(y)} \frac{1}{|d_y T|} f(y).
\]

The proof is based on the following two facts, which can be found in Hofbauer and Keller’s papers [8][9]. The first fact is

(2) There is a function \( h \geq 0 \) of bounded variation such that if \( f \in L^1 \) with \( f \geq 0 \) and \( f \neq 0 \), then \( \mathcal{L}_\psi^n f \) converges to \( \int f \, d\mu \) in \( L^1 \).
The second fact is

Let \( f \in L^1 \) with \( f \geq 0 \) and \( f \neq 0 \) be fixed. There is a function \( w \geq 0 \) with bounded variation, a measure \( \nu \), and a number \( a > 0 \), depending on \( \phi \), such that

\[
a^n \mathcal{L}_\phi^n f \to w \int f \, d\nu
\]

in \( L^1 \).

For the second fact it is important that \( \phi \) is of bounded variation.

For \( f \) of bounded variation, these facts are proved as follows. Theorem 1 of \([8]\) gives us the desired spectral decomposition for the transfer operator acting on functions of bounded variation. Proposition 3.6 of Baladi’s book \([3]\) gives us that \( \phi \) is analogous to the argument used by Pollicott and Yuri in \([23]\) for \( \beta \)-expansions.

Let us now see how Theorem 1 follows from these facts. The following argument is analogous to the argument used by Pollicott and Yuri in \([23]\) for \( \beta \)-expansions.

We first note that it is sufficient to prove that \( \phi = \chi \circ f - \chi \) implies that

\[
\mathcal{L}_\phi^n 1(x) = \sum_{T^n(y) = x} e^{S_n \phi (y)} \frac{1}{|d_y T^n|} = \sum_{T^n(y) = x} e^{\chi(T^n y) - \chi(y)} \frac{1}{|d_y T^n|} = e^{\chi(x)} \sum_{T^n(y) = x} e^{-\chi(y)} \frac{1}{|d_y T^n|} = e^{\chi(x)} \mathcal{L}_0^n e^{-\chi(x)}.
\]

Since \( a^n \mathcal{L}_\phi^n 1 \to w \) and \( e^{-\chi} \mathcal{L}_\phi^n 1 = \mathcal{L}_0^n e^{-\chi} \to h \int e^{-\chi} \, d\nu \) we have that \( a^n \mathcal{L}_\phi^n 1 \) converges to \( w \) in \( L^1 \) and \( \mathcal{L}_\phi^n 1 \) converges to \( h e^{\chi} \int e^{-\chi} \, d\nu \) in \( L^1 \). By taking a subsequence, we can achieve that the convergences are a.e. Therefore, we must have \( a = 1 \) and

\[
w(x) = e^{\chi(x)} h(x) \int e^{-\chi} \, d\nu, \quad \text{a.e.}
\]

It follows that

\[
\chi(x) = \log w(x) - \log \int e^{-\chi} \, d\nu - \log h(x),
\]

almost everywhere. Since \( h \) and \( w \) are bounded away from zero, their logarithms are of bounded variation. This proves the theorem.

### 6. Proof of Theorem 2

We first note that it is sufficient to prove that \( \chi_0 \) is continuously differentiable on elements of the form \( Q(x) \).

Let \( x \) and \( y \) satisfy \( T(y) = x \). Then by \( \phi = \chi \circ T - \chi \) we have \( \chi(x) = \phi(y) + \chi(y) \).

Let \( x_i \) be a point in an element of \( Q \) and take \( x_2 \in Q(x_1) \). We choose pre-images \( y_{1,i} \) and \( y_{2,i} \) of \( x_1 \) and \( x_2 \) such that \( T(y_{i,1}) = x_i \) and \( T(y_{i,j}) = y_{i,j-1} \), for \( i = 1, 2 \).
We then have
\[ \chi(x_1) - \chi(x_2) = \sum_{j=1}^{n} (\phi(y_{1,j}) - \phi(y_{2,j})) + \chi(y_{1,n}) - \chi(y_{2,n}). \]

We would like to let \( n \to \infty \) and conclude that \( \chi(y_{1,n}) - \chi(y_{2,n}) \to 0 \). By Theorem 11 we know that \( \chi \) has bounded variation. Assume for a contradiction that no matter how we choose \( y_{1,j} \) and \( y_{2,j} \) we cannot make \( |\chi(y_{1,n}) - \chi(y_{2,n})| \) smaller than some \( \varepsilon > 0 \). Let \( m \) be large and consider the cylinders of generation \( m \). For any such cylinder \( C_m \), we can choose \( y_{1,j} \) and \( y_{2,j} \) such that \( y_{1,n} \) and \( y_{2,n} \) are both in \( C_m \). Since \( |\chi(y_{1,n}) - \chi(y_{2,n})| \geq \varepsilon \), the variation of \( \chi \) on \( C_m \) is at least \( \varepsilon \). Summing over all cylinders of generation \( m \), we conclude that the variation of \( \chi \) on \( I \) is at least \( N(m)\varepsilon \). Since \( m \) is arbitrary and \( N(m) \to \infty \) as \( m \to \infty \), we get a contradiction to the fact that \( \chi \) is of bounded variation.

Hence we can make \( |\chi(y_{1,n}) - \chi(y_{2,n})| \) smaller than any \( \varepsilon > 0 \) by choosing \( y_{1,j} \) and \( y_{2,j} \) in an appropriate way. We conclude that
\[ \chi(x_1) - \chi(x_2) = \sum_{j=1}^{\infty} (\phi(y_{1,j}) - \phi(y_{2,j})). \]

If \( x_1 \neq x_2 \), then \( y_{1,j} \neq y_{2,j} \) for all \( j \), and we have
\[ \frac{\chi(x_1) - \chi(x_2)}{x_1 - x_2} = \sum_{j=1}^{\infty} \frac{\phi(y_{1,j}) - \phi(y_{2,j})}{y_{1,j} - y_{2,j}} \frac{y_{1,j} - y_{2,j}}{x_1 - x_2}. \]

Clearly, the limit of the right-hand side exists as \( x_2 \to x_1 \) and is
\[ \sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}. \]

The series converges since \( |(T^j)'| > \lambda^j \). This shows that \( \chi'(x_1) \) exists and satisfies
\[ \chi'(x_1) = \sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}. \]

If \( T^r \) is constant on the elements of \( P \), then (4) implies that \( \chi \) is in \( C^k \) provided that \( \phi \) is in \( C^k \).

Let us now assume that \( \frac{1}{(T^r)'} \) is in \( C^{k-1} \) with derivatives up to order \( k-1 \) uniformly bounded in \( r \). We proceed by induction. Let \( g_n = \frac{1}{(T^r)'} \). Assume that
\[ \chi^{(m)}(x) = \sum_{n=1}^{\infty} \psi_{n,m}(y_n) g_n(y_n), \]
where \( (\psi_{n,m})_{n=1}^{\infty} \) is in \( C^{k-m} \) with derivatives up to order \( k-m \) uniformly bounded. Then
\[ \chi^{(m+1)}(x) = \sum_{n=1}^{\infty} (\psi_{n,m}(y_n) g_n(y_n) + \psi_{n,m}(y_n) g'_n(y_n)) g_n(y_n) = \sum_{n=1}^{\infty} \psi_{n,m+1} g_n(y_n). \]

This proves that there are uniformly bounded functions \( \psi_{n,m} \) such that (5) holds for \( 1 \leq m \leq k \). The series in (5) converges uniformly since \( g_n \) decays with exponential speed. This proves that \( \chi \) is in \( C^k \). \( \square \)
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