PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 3, March 2012, Pages 765–777
S 0002-9939(2011)10950-X
Article electronically published on July 21, 2011

"DIVERGENT" RAMANUJAN-TYPE SUPERCONGRUENCES

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(Communicated by Matthew A. Papanikolas)

Abstract. "Divergent" Ramanujan-type series for $1/\pi$ and $1/\pi^2$ provide us with new nice examples of supercongruences of the same kind as those related to the convergent cases. In this paper we manage to prove three of the supercongruences by means of the Wilf–Zeilberger algorithmic technique.

1. Introduction

In the spirit of [19], the two supercongruences

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3}{n!^3}(3n+1)2^{2n}\equiv p \pmod{p^3}\quad\text{for } p>2, \tag{1}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^5}{n!^5}(10n^2+6n+1)(-1)^n2^{2n}\equiv p^2 \pmod{p^5}\quad\text{for } p>3, \tag{2}$$

correspond to divergent Ramanujan-type series for $1/\pi$ and $1/\pi^2$, respectively (cf. Section 4 below); the letter p is reserved for primes throughout the paper. Furthermore, we have more congruences of this kind:

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3}{(1)_n^3}(3n+1)(-1)^n2^{3n}\equiv\left(\frac{-1}{p}\right)p \pmod{p^3}\quad\text{for } p>2, \tag{3}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3}{(1)_n^3}(21n+8)2^{6n}\equiv 8p \pmod{p^3}\quad\text{for } p>2, \tag{4}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{(1)_n^3}(5n+1)(-1)^n\left(\frac43\right)^{2n}\overset{?}{\equiv}\left(\frac{-3}{p}\right)p \pmod{p^3}\quad\text{for } p>3, \tag{5}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{(1)_n^3}(35n+8)\left(\frac43\right)^{4n}\overset{?}{\equiv} 8p \pmod{p^3}\quad\text{for } p>3, \tag{6}$$

Received by the editors August 12, 2010 and, in revised form, December 14, 2010.
2010 *Mathematics Subject Classification.* Primary 11Y55, 33C20, 33F10; Secondary 11B65, 11D88, 11F33, 11F85, 11S80, 12H25, 40G99, 65-05, 65B10.
Key words and phrases. Congruence, hypergeometric series, Ramanujan-type identities for $1/\pi$, creative telescoping.
Work of the second author was supported by Australian Research Council grant DP110104419.

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$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac13)_n(\frac23)_n}{(1)_n^3}(11n+3)\left(\frac{27}{16}\right)^n \overset{?}{\equiv} 3p \pmod{p^3} \quad \text{for } p>2, \tag{7}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^5}{(1)_n^5}(205n^2+160n+32)(-1)^n2^{10n} \overset{?}{\equiv} 32p^2 \pmod{p^5} \quad \text{for } p>3, \tag{8}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac13)_n(\frac23)_n(\frac14)_n(\frac34)_n}{(1)_n^5} \times(172n^2+75n+9)(-1)^n\left(\frac{27}{16}\right)^n \overset{?}{\equiv} 9p^2 \pmod{p^5} \quad \text{for } p>2. \tag{9}$$

Here $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol and the Pochhammer notation $(a)_b$ is used for denoting $\Gamma(a+b)/\Gamma(b)$ also in the cases when b is not a non-negative integer. Of course, if $b=n\in\mathbb{Z}_{\ge0}$ we have, as usual, $(a)_n=\prod_{k=0}^{n-1}(a+k)$ with the convention that the empty product equals 1, while $1/(m)_{-n}=\Gamma(m)/\Gamma(m-n)$ vanishes for positive integers $m\le n$ — the fact we will use repeatedly in the text. The question mark indicates that the corresponding supercongruence remains conjectural. The non-questioned entries (1)–(3) are proved in this paper by extending the method of [19], while the supercongruence (4) (even in a more general form) is shown by Zhi-Wei Sun in his preprint [14].

Note that we can sum in (1), (2), (4), (3), and (8) up to $\frac{p-1}{2}$, since the p-adic order of $(\frac12)_n/n!$ is 1 for $n=\frac{p+1}{2},\dots,p-1$.

Main Theorem. *The following supercongruences take place*:

$$\sum_{n=0}^{(p-1)/2}\frac{(\frac12)_n^3}{n!^3}(3n+1)2^{2n}\equiv p \pmod{p^3} \quad \textit{for } p>2, \tag{10}$$

$$\sum_{n=0}^{(p-1)/2}\frac{(\frac12)_n^5}{n!^5}(10n^2+6n+1)(-1)^n2^{2n}\equiv p^2 \pmod{p^5} \quad \textit{for } p>3, \tag{11}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3}{(1)_n^3}(3n+1)(-1)^n2^{3n}\equiv(-1)^{(p-1)/2}p \pmod{p^3} \quad \textit{for } p>2. \tag{12}$$

We find it quite illogical that our strategy based on the creative Wilf–Zeilberger theory [11] of WZ-pairs allows us to do only three entries from the list (1)–(9); a very similar lack of luck was reported in [19]. Although we have WZ-pairs for (4)–(8) as well, they seem to be quite helpless for showing the corresponding congruences modulo the expected powers of p. Because of the clear relationship of such congruences with Ramanujan's formulae for $1/\pi$ and their generalizations (see [19] and Section 4), we do expect a more universal method for proving the Ramanujan-type supercongruences.

In Section 2 we present auxiliary congruencesl; some of them are remarkable on their own. Section 3 contains the proofs of (10)–(12). The final section, Section 4, reviews the "divergent" Ramanujan-type series for $1/\pi$ and $1/\pi^2$ as our motivation to the the above list of supercongruences.

Throughout the paper, the record $a\equiv b \pmod{p^k}=c$ means $a\equiv b \pmod{p^k}$ and $b=c$ so that it signifies the congruence $a\equiv c \pmod{p^k}$ as a consequence. This is used in formulas (20), (28), (29), (30), (38), (43), and (45) below.

2. Precongruences

In this section we summarize our needs for proving the supercongruences of the Main Theorem.

Lemma 1. *The following congruences are valid*:

$$\sum_{n=1}^{(p-1)/2} \frac{\binom{2n}{n}}{n} \equiv 0 \pmod{p} \quad \textit{for } p > 3, \tag{13}$$

$$\sum_{n=1}^{(p-1)/2} \frac{(-1)^n\binom{2n}{n}}{n^2} \equiv 0 \pmod{p} \quad \textit{for } p > 5. \tag{14}$$

Proof. The first congruence follows from specialization $N = (p-1)/2$ of Staver's identity [13]:

$$\sum_{n=1}^{N} \binom{2n}{n}\frac{1}{n} = \frac{N+1}{3}\binom{2N+1}{N}\sum_{n=1}^{N}\frac{1}{n^2\binom{N}{n}^2}.$$

The second congruence is the modulo p reduction of Tauraso's congruence in [16, Theorem 4.2]. It is interesting to mention that the latter follows from the $N = p$ specialization of another combinatorial identity,

$$\sum_{n=1}^{N} \binom{2n}{n}\frac{n^2}{4N^4+n^4}\prod_{k=1}^{n-1}\frac{N^4-k^4}{4N^4+k^4} = \frac{2}{5N^2},$$

conjectured by Borwein and Bradley [3] and proved by Almkvist and Granville [1]. □

Denote $q(x) = q_p(x) = (x^{p-1}-1)/p$ as the Fermat quotient of $x \in \mathbb{Z}_p^*$.

Lemma 2. *The following congruence is true*:

$$\sum_{n=0}^{(p-3)/2} \frac{2^{-2n}\binom{2n}{n}}{2n+1} \equiv -(-1)^{(p-1)/2}q_p(2) \pmod{p} \quad \textit{for } p > 2. \tag{15}$$

Proof. Note that $(\frac{1}{2})_n \equiv (\frac{1}{2}-\frac{p}{2})_n \pmod{p}$ and write the left-hand side as

$$\sum_{n=0}^{(p-3)/2} \frac{(\frac{1}{2})_n}{n!(2n+1)} \equiv \sum_{n=0}^{(p-3)/2} \frac{(\frac{1}{2}-\frac{p}{2})_n}{n!(2n+1)} \pmod{p}. \tag{16}$$

The latter is nothing more than a terminating hypergeometric series with one term missing,

$$\sum_{n=0}^{N-1} \frac{(-N)_n(\frac{1}{2})_n}{n!(\frac{3}{2})_n} = -\frac{(-1)^N}{2N+1} + {}_2F_1\big(-N,\, \tfrac{1}{2};\, \tfrac{3}{2};\, 1\big), \quad \text{where } N = (p-1)/2.$$

It can be summed with the help of the Chu–Vandermonde theorem [12, Eq. (1.7.7)]:

$$\sum_{n=0}^{(p-3)/2} \frac{(\frac{1}{2}-\frac{p}{2})_n}{n!(2n+1)} = -\frac{(-1)^{(p-1)/2}}{p} + \frac{(1)_{(p-1)/2}}{p(\frac{1}{2})_{(p-1)/2}}. \tag{17}$$

Finally, recall Morley's congruence [10]:

$$\frac{(\frac{1}{2})_{(p-1)/2}}{(1)_{(p-1)/2}} = \binom{p-1}{\frac{p-1}{2}} 2^{-(p-1)} \equiv (-1)^{(p-1)/2} 2^{p-1} \pmod{p^2}. \tag{18}$$

Combining (16), (17), (18) and $2^{p-1} \equiv 1 \pmod{p}$ we arrive at (15). □

Lemma 3. *For a prime $p > 3$, let x be a rational number such that both x and $1-x$ do not involve p in their prime factorizations and $1-x$ is a quadratic residue modulo p. Take y such that $y^2 \equiv 1-x \pmod{p}$. Then*

(19)
$$\sum_{n=1}^{p-1} \binom{2n}{n} \left(\frac{x}{4}\right)^n \equiv \frac{px}{2(1-x)} \big(-q(x) + q(y+1)(y+1) - q(y-1)(y-1)\big) \pmod{p^2}.$$

Proof. It is well known that

$$\begin{aligned} \sum_{n=1}^{p-1} \frac{x^n}{n} &= \sum_{n=1}^{p-1} \frac{(1)_{n-1} x^n}{(1)_n} \equiv \sum_{n=1}^{p-1} \frac{(1-p)_{n-1} x^n}{(1)_n} \pmod{p} \\ &= -\frac{1}{p} \sum_{n=1}^{p-1} \frac{(-p)_n x^n}{(1)_n} = \frac{1}{p}\left(1 - x^p - \sum_{n=0}^{p} \binom{p}{n} (-x)^n\right) \\ &= \frac{1}{p}\big(1 - x^p - (1-x)^p\big) \end{aligned} \tag{20}$$

and, by replacing x with $-x$ and taking the appropriate linear combination of the two expressions,

$$\sum_{k=1}^{(p-1)/2} \frac{x^{2k-1}}{2k-1} \equiv \frac{1}{2p}\big((1+x)^p - (1-x)^p - 2x^p\big) \pmod{p}. \tag{21}$$

Consider

$$F(n,k) = \frac{(\frac{1}{2}-k)_n}{(1)_n} \frac{x^n}{(1-x)^k} \quad \text{and} \quad G(n,k) = -\frac{(\frac{3}{2}-k)_{n-1}}{(1)_{n-1}} \frac{x^n}{(1-x)^k}.$$

These functions satisfy

$$\sum_{n=1}^{p-1} \binom{2n}{n} \left(\frac{x}{4}\right)^n = \sum_{n=1}^{p-1} F(n,0) \tag{22}$$

and

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k). \tag{23}$$

The latter relation signifies the fact that $F(n,k)$ and $G(n,k)$ form a *WZ-pair*.

Summing (23) over $n = 0, 1, \dots, p-1$ we obtain

$$\sum_{n=0}^{p-1} F(n,k-1) - \sum_{n=0}^{p-1} F(n,k) = G(p,k) - G(0,k) = G(p,k); \tag{24}$$

here $G(0,k) = 0$ because its expression involves $(1)_{-1}$ in the denominator. Summing the result (24) over $k = 1, \dots, \frac{p+1}{2}$ we arrive at

$$\sum_{n=1}^{p-1} F(n,0) = -F(0,0) + \sum_{n=0}^{p-1} F(n, \tfrac{p+1}{2}) + \sum_{k=1}^{(p+1)/2} G(p,k). \tag{25}$$

For $n = 1, 2, \dots, p-1$,

$$F(n, \tfrac{p+1}{2}) = \frac{(-\frac{p}{2})_n}{(1)_n} \frac{x^n}{(1-x)^{(p+1)/2}} = -\frac{p}{2} \frac{(1-\frac{p}{2})_{n-1}}{(1)_n} \frac{x^n}{(1-x)^{(p+1)/2}},$$

and since $(1-\frac{p}{2})_{n-1} \equiv (1)_{n-1} \pmod{p}$, we obtain

$$\sum_{n=1}^{p-1} F(n, \tfrac{p+1}{2}) \equiv -\frac{p}{2(1-x)^{(p+1)/2}} \sum_{n=1}^{p-1} \frac{x^n}{n} \equiv -\frac{1-x^p-(1-x)^p}{2(1-x)^{(p+1)/2}} \pmod{p^2} \tag{26}$$

by (20). In addition,

$$F(0,0) = 1 \quad \text{and} \quad F(0, \tfrac{p+1}{2}) = \frac{1}{(1-x)^{(p+1)/2}}. \tag{27}$$

Furthermore, we have

$$G(p,k) = -\frac{(\frac{1}{2}-k)_p}{(\frac{1}{2}-k)(p-1)!} \frac{x^p}{(1-x)^k}.$$

Note that the multiples $\frac{1}{2} - k + j$, $j = 0, 1, \dots, p-1$, runs through the complete residue system modulo p, with exactly one of them, $\frac{p}{2}$ for $j = \frac{p-1}{2} + k$, divisible by p; therefore $(\frac{1}{2}-k)_p \equiv \frac{1}{2}p! \pmod{p^2}$ for $k = 1, \dots, \frac{p-1}{2}$. Since $\frac{1}{2} - k$ is coprime with p for those k, we have

$$G(p,k) \equiv \frac{p}{2k-1} \frac{x^p}{(1-x)^k} \pmod{p^2}.$$

If $k = \frac{p+1}{2}$, then

$$\frac{(\frac{1}{2}-k)_p}{\frac{1}{2}-k} = (1-\tfrac{p}{2})_{p-1} \equiv (p-1)! \pmod{p^2},$$

because $(1-\varepsilon)_{p-1} = (1)_{p-1} \cdot (1 - \varepsilon H_{p-1} + O(\varepsilon^2))$ and the harmonic number $H_{p-1} = 1 + \frac{1}{2} + \dots + \frac{1}{p-1} \equiv 0 \pmod{p}$ (see, for example, (20) with $x = 1$). Thus, we have

$$G(p, \tfrac{p+1}{2}) \equiv -\frac{x^p}{(1-x)^{(p+1)/2}} \pmod{p^2}$$

so that

$$\begin{aligned}
\sum_{k=1}^{(p+1)/2} G(p,k) &\equiv p x^p (1-x)^{-1/2} \sum_{k=1}^{(p-1)/2} \frac{\big((1-x)^{-1/2}\big)^{2k-1}}{2k-1} - \frac{x^p}{(1-x)^{(p+1)/2}} \\
&\equiv x^p \cdot \frac{\big(1+(1-x)^{-1/2}\big)^p - \big(1-(1-x)^{-1/2}\big)^p - 2(1-x)^{-p/2}}{2(1-x)^{1/2}} \\
&\qquad - \frac{x^p}{(1-x)^{(p+1)/2}} \pmod{p^2} \\
&= x^p \cdot \frac{\frac{1}{2}\big((1-x)^{1/2}+1\big)^p - \frac{1}{2}\big((1-x)^{1/2}-1\big)^p - 2}{(1-x)^{(p+1)/2}}
\end{aligned} \tag{28}$$

by (21).

Substituting (26), (27) and (28) into (25) we obtain, modulo p^2,

(29)
$$\begin{aligned}\sum_{n=1}^{p-1} F(n,0) &\equiv -1 + \frac{1+(1-x)^p - 3x^p + x^p\big((1-x)^{1/2}+1\big)^p - x^p\big((1-x)^{1/2}-1\big)^p}{2(1-x)^{(p+1)/2}}\\ &= \frac{\big((1-x)^{(p-1)/2}-1\big)^2}{2(1-x)^{(p-1)/2}}\\ &\quad + \frac{x}{2(1-x)}\,\frac{1-3x^{p-1}+x^{p-1}\big((1-x)^{1/2}+1\big)^p - x^{p-1}\big((1-x)^{1/2}-1\big)^p}{(1-x)^{(p-1)/2}}\\ &= \frac{(y^{p-1}-1)^2}{2y^{p-1}} + \frac{x}{2(1-x)}\,\frac{1-3x^{p-1}+x^{p-1}(y+1)^p - x^{p-1}(y-1)^p}{y^{p-1}}.\end{aligned}$$

Noting that $(y^{p-1}-1)^2 = p^2 q(y)^2 \equiv 0 \pmod{p^2}$,

(30)
$$\begin{aligned}&\frac{1-3x^{p-1}+x^{p-1}(y+1)^p - x^{p-1}(y-1)^p}{p}\\ &\quad = -\frac{x^{p-1}-1}{p}(3-(y+1)^p+(y-1)^p)\\ &\qquad + \frac{(y+1)^{p-1}-1}{p}(y+1) - \frac{(y-1)^{p-1}-1}{p}(y-1)\\ &\quad \equiv -q(x)(3-(y+1)+(y-1)) + q(y+1)(y+1) - q(y-1)(y-1) \pmod{p}\\ &\quad = -q(x) + q(y+1)(y+1) - q(y-1)(y-1),\end{aligned}$$

and $y^{p-1} \equiv 1 \pmod{p}$, we obtain the required congruence (19) from (22) and (29). □

Lemma 4. *The following congruences are valid*:

$$\sum_{n=1}^{p-1} \frac{(-1)^n 2^n \binom{2n}{n}}{n} \equiv -4q_p(2) \pmod{p} \quad \textit{for } p>2, \tag{31}$$

$$3\sum_{n=1}^{p-1} (-1)^n 2^n \binom{2n}{n} \equiv -4pq_p(2) \pmod{p^2} \quad \textit{for } p>2. \tag{32}$$

Proof. It is shown in [15, Theorem 1.2] that for m in $\mathbb{Z}_p^*$,

$$\sum_{n=1}^{p-1} \frac{(-1)^n \binom{2n}{n}}{nm^n} \equiv \frac{2}{m}\cdot\frac{m^p - V_p(m)}{p} \pmod{p}, \tag{33}$$

where the sequence $V_k(x)$ is defined by $V_0(x)=2$, $V_1(x)=x$, and $V_k(x) = x(V_{k-1}(x)+V_{k-2}(x))$ for $k\ge 2$. Although the theorem is stated for $m\in\mathbb{Z}$ only, the proof does not make use of this integrality; we can apply it for $m=1/2$ as well. In this case $V_k(1/2) = 1+(-1)^k/2^k$ so that the right-hand side of (33) becomes (31) if we additionally use $2^{p-1}\equiv 1 \pmod{p}$.

The congruence (32) is clear for $p=3$, while for $p>3$ it follows from specialization $x=-8$, $y=3$ of (19) and noting that

$$q_p(-8)=\frac{2^{3(p-1)}-1}{p}=\frac{2^{p-1}-1}{p}\,(2^{2(p-1)}+2^{p-1}+1)\equiv 3q_p(2)\pmod p,$$
$$q_p(4)=\frac{2^{2(p-1)}-1}{p}=\frac{2^{p-1}-1}{p}\,(2^{p-1}+1)\equiv 2q_p(2)\pmod p. \qquad \square$$

3. Proofs of the supercongruences

Proof of (10). Take

$$F(n,k)=(3n+2k+1)\frac{(\frac12)_n(\frac12+k)_n^2}{(1)_n^3}2^{2n}\quad\text{and}\quad G(n,k)=-\frac{(\frac12)_n(\frac12+k)_{n-1}^2}{(1)_{n-1}^3}2^{2n}.$$

Then we have

$$\sum_{n=0}^{(p-1)/2}\frac{(\frac12)_n^3}{n!^3}(3n+1)2^{2n}=\sum_{n=0}^{(p-1)/2}F(n,0) \tag{34}$$

and (23), so that $F(n,k)$ and $G(n,k)$ form a WZ-pair. Summing (23) over $n=0,1,\dots,\frac{p-1}{2}$, we obtain

$$\sum_{n=0}^{(p-1)/2}F(n,k-1)-\sum_{n=0}^{(p-1)/2}F(n,k)=G(\tfrac{p+1}{2},k)-G(0,k)=G(\tfrac{p+1}{2},k). \tag{35}$$

Furthermore, for $k=1,2,\dots,\frac{p-1}{2}$ we have

$$G(\tfrac{p+1}{2},k)=-\frac{(\frac12)_{(p+1)/2}(\frac12+k)_{(p-1)/2}^2}{(1)_{(p-1)/2}^3}2^{p+1}\equiv 0\pmod{p^3},$$

because each of the three Pochhammer products in the numerator is divisible by p while the denominator, $(\frac{p-1}{2})!^3$, is coprime with p. Comparing this result with (35), as in the proof of Theorem 1 in [19], we see that

$$\sum_{n=0}^{(p-1)/2}F(n,0)\equiv\sum_{n=0}^{(p-1)/2}F(n,1)\equiv\sum_{n=0}^{(p-1)/2}F(n,2)\equiv\cdots\equiv\sum_{n=0}^{(p-1)/2}F(n,\tfrac{p-1}{2})\pmod{p^3}.$$

Hence we can replace, modulo p^3, our sum (34) by

$$\begin{aligned}\sum_{n=0}^{(p-1)/2}F(n,\tfrac{p-1}{2})&=\sum_{n=0}^{(p-1)/2}(3n+p)\frac{(\frac12)_n(\frac p2)_n^2}{(1)_n^3}2^{2n}\\&=p+p\sum_{n=1}^{(p-1)/2}\frac{(\frac12)_n(\frac p2)_n^2}{(1)_n^3}2^{2n}+3\sum_{n=1}^{(p-1)/2}n\frac{(\frac12)_n(\frac p2)_n^2}{(1)_n^3}2^{2n}\\&=p+\frac{p^3}{4}\sum_{n=1}^{(p-1)/2}\frac{(\frac12)_n(1+\frac p2)_{n-1}^2}{(1)_n^3}2^{2n}\\&\quad+\frac{3p^2}{4}\sum_{n=1}^{(p-1)/2}n\frac{(\frac12)_n(1+\frac p2)_{n-1}^2}{(1)_n^3}2^{2n}.\end{aligned} \tag{36}$$

(Note that, in contrast with the proofs in [19], the newer sum is not reduced to a single term.) Comparing the resulted expression (36) for (34) we see that (10) is equivalent to

$$\sum_{n=1}^{(p-1)/2} n\frac{(\frac12)_n(1+\frac p2)_{n-1}^2}{(1)_n^3}2^{2n} \equiv 0 \pmod p \quad \text{for } p>3. \tag{37}$$

On noting that $(1+\frac p2)_{n-1}\equiv(1)_{n-1} \pmod p$, we reduce (37) to its equivalent,

$$\sum_{n=1}^{(p-1)/2} \frac{(\frac12)_n}{n(1)_n}2^{2n} = \sum_{n=1}^{(p-1)/2} \frac{\binom{2n}{n}}{n} \equiv 0 \pmod p \quad \text{for } p>3,$$

which is exactly (13). □

Proof of (11). The proof is very similar. This time we take

$$F(n,k) = (10n^2+12nk+4k^2+6n+4k+1)\frac{(\frac12)_n(\frac12+k)_n^4}{(1)_n^5}(-1)^n2^{2n}$$

and

$$G(n,k) = (n+2k-1)\frac{(\frac12)_n(\frac12+k)_{n-1}^4}{(1)_{n-1}^5}(-1)^n2^{2n+1}$$

with the motive

$$\sum_{n=0}^{(p-1)/2} \frac{(\frac12)_n^5}{n!^5}(10n^2+6n+1)(-1)^n2^{2n} = \sum_{n=0}^{(p-1)/2} F(n,0)$$

and (23). Then, as above, we find that

$$\begin{aligned}\sum_{n=0}^{(p-1)/2} F(n,0) &\equiv \sum_{n=0}^{(p-1)/2} F(n,\tfrac{p-1}{2}) \pmod{p^5} \\ &= \sum_{n=0}^{(p-1)/2} (10n^2+6np+p^2)\frac{(\frac12)_n(\frac p2)_n^4}{(1)_n^5}(-1)^n2^{2n} \\ &= p^2 + \frac{p^6}{16}\sum_{n=1}^{(p-1)/2} \frac{(\frac12)_n(1+\frac p2)_{n-1}^4}{(1)_n^5}(-1)^n2^{2n} \\ &\quad + \frac{3p^5}{8}\sum_{n=1}^{(p-1)/2} n\frac{(\frac12)_n(1+\frac p2)_{n-1}^4}{(1)_n^5}(-1)^n2^{2n} \\ &\quad + \frac{5p^4}{8}\sum_{n=1}^{(p-1)/2} n^2\frac{(\frac12)_n(1+\frac p2)_{n-1}^4}{(1)_n^5}(-1)^n2^{2n},\end{aligned} \tag{38}$$

and our task is to show that

$$\sum_{n=1}^{(p-1)/2} n^2\frac{(\frac12)_n(1+\frac p2)_{n-1}^4}{(1)_n^5}(-1)^n2^{2n} \equiv 0 \pmod p \quad \text{for } p>5.$$

Using $(1+\frac{p}{2})_{n-1} \equiv (1)_{n-1} \pmod{p}$ the latter reduces to

$$\sum_{n=1}^{(p-1)/2} \frac{(\frac{1}{2})_n}{n^2(1)_n}(-1)^n 2^{2n} = \sum_{n=1}^{(p-1)/2} \frac{(-1)^n\binom{2n}{n}}{n^2} \equiv 0 \pmod{p} \quad \text{for } p>5,$$

which is (14). □

Proof of (12). Take the WZ-pair

$$F(n,k) = (3n+2k+1)\frac{(\frac{1}{2})_n(\frac{1}{2}+k)_n^2(\frac{1}{2})_k}{(1)_n^2(1+2k)_n(1)_k}(-1)^n 2^{3n}$$

and

$$G(n,k) = \frac{(\frac{1}{2})_n(\frac{1}{2}+k)_{n-1}^2(\frac{1}{2})_k}{(1)_{n-1}^2(1+2k)_{n-1}(1)_k}(-1)^n 2^{3n-2}.$$

Summing (23) over $n = 0, 1, \ldots, p-1$ we get

$$\sum_{n=0}^{p-1} F(n,k-1) - \sum_{n=0}^{p-1} F(n,k) = G(p,k) - G(0,k) = G(p,k). \tag{39}$$

Summing (39) further over $k = 1, 2, \ldots, \frac{p-1}{2}$ we obtain

$$\sum_{n=0}^{p-1} F(n,0) = \sum_{n=0}^{p-1} F(n,\tfrac{p-1}{2}) + \sum_{k=1}^{(p-1)/2} G(p,k). \tag{40}$$

Note that

$$\begin{aligned} G(p,k) &= \frac{(\frac{1}{2}+p)_{k-1}^2(1)_{2k}}{(\frac{1}{2})_k(1+p)_{2k-1}(1)_k} \cdot \frac{(\frac{1}{2})_p^3}{p(1)_{p-1}^3}(-1)^p 2^{3p-2} \\ &= -\frac{(\frac{1}{2}+p)_{k-1}^2 \cdot 2^{2k}}{(1+p)_{2k-1}} \cdot \frac{(\frac{1}{2})_p^3}{p(p-1)!^3} 2^{3p-2}. \end{aligned}$$

For $k = 1, 2, \ldots, \frac{p-1}{2}$ neither $(\frac{1}{2}+p)_{k-1}$ nor $(1+p)_{2k-1}$ is divisible by p. In addition, we have $(\frac{1}{2}+p)_{k-1} \equiv (\frac{1}{2})_{k-1} \pmod{p}$ and $(1+p)_{2k-1} \equiv (1)_{2k-1} \pmod{p}$. Since the factor in $G(p,k)$ independent of k is divisible by p^2, we conclude that

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} G(p,k) &\equiv -\frac{(\frac{1}{2})_p^3}{p(p-1)!^3} 2^{3p-2} \sum_{k=1}^{(p-1)/2} \frac{(\frac{1}{2})_{k-1}^2}{(2k-1)!} 2^{2k} \\ &\equiv -p^2 \binom{2p-1}{p-1}^3 2^{-3p+3} \sum_{k=1}^{(p-1)/2} \frac{2^{-2(k-1)}\binom{2k-2}{k-1}}{2k-1} \pmod{p^3}. \end{aligned}$$

Finally, we use $\binom{2p-1}{p-1} \equiv 1 \pmod{p}$, $2^{p-1} \equiv 1 \pmod{p}$ and the congruence (15) to get

$$\sum_{k=1}^{(p-1)/2} G(p,k) \equiv (-1)^{(p-1)/2} p(2^{p-1}-1) \pmod{p^3}. \tag{41}$$

Furthermore,

$$F(0,\tfrac{p-1}{2}) = p\frac{(\frac{1}{2})_{(p-1)/2}}{(1)_{(p-1)/2}} \equiv (-1)^{(p-1)/2} p 2^{p-1} \pmod{p^3} \tag{42}$$

by (18) and

$$\begin{aligned} F(n,\tfrac{p-1}{2}) &= (3n+p)\frac{(\frac12)_n(\frac p2)_n^2(\frac12)_{(p-1)/2}}{(1)_n^2(p)_n(1)_{(p-1)/2}}(-1)^n2^{3n} \\ &\equiv (3n+p)\frac{(\frac12)_n(1+\frac p2)_{n-1}^2}{(1)_n^2(1+p)_{n-1}}(-1)^n2^{3n-2}\cdot(-1)^{(p-1)/2}p2^{p-1} \pmod{p^3}. \end{aligned}$$

Using $(1+\varepsilon)_{n-1}=(1)_{n-1}\cdot(1+\varepsilon H_{n-1}+O(\varepsilon^2))$, where $H_{n-1}=1+\frac12+\dots+\frac{1}{n-1}$, with $\varepsilon=\frac p2$ and p we find that

$$\frac{(1+\frac p2)_{n-1}^2}{(1+p)_{n-1}}=\frac{(1)_{n-1}^2\cdot(1+\frac p2H_{n-1}+O(p^2))^2}{(1)_{n-1}\cdot(1+pH_{n-1}+O(p^2))}=(1)_{n-1}\cdot(1+O(p^2))\equiv(1)_{n-1}\pmod{p^2}.$$

Hence

$$\begin{aligned} F(n,\tfrac{p-1}{2}) &\equiv (3n+p)\frac{(\frac12)_n}{(1)_n n}(-1)^n2^{3n-2}\cdot(-1)^{(p-1)/2}p2^{p-1} \pmod{p^3} \\ &= \frac14(3n+p)\frac{(-1)^n2^n\binom{2n}{n}}{n}\cdot(-1)^{(p-1)/2}p2^{p-1} \end{aligned} \tag{43}$$

and

$$\begin{aligned} \sum_{n=1}^{p-1}F(n,\tfrac{p-1}{2}) &\equiv (-1)^{(p-1)/2}p2^{p-1}\biggl(\frac34\sum_{n=1}^{p-1}(-1)^n2^n\binom{2n}{n}+\frac p4\sum_{n=1}^{p-1}\frac{(-1)^n2^n\binom{2n}{n}}{n}\biggr) \\ &\equiv (-1)^{(p-1)/2}p2^{p-1}\cdot2(1-2^{p-1}) \\ &\equiv (-1)^{(p-1)/2}p\cdot2(1-2^{p-1}) \pmod{p^3} \end{aligned} \tag{44}$$

by (31), (32) and $2^{p-1}\equiv1\pmod p$.

Substituting (41), (42) and (44) into (40) we obtain

$$\begin{aligned} \sum_{n=0}^{p-1}F(n,0) &\equiv (-1)^{(p-1)/2}p\bigl(2^{p-1}+2(1-2^{p-1})+(2^{p-1}-1)\bigr) \pmod{p^3} \\ &= (-1)^{(p-1)/2}p, \end{aligned} \tag{45}$$

the required congruence. □

4. "Divergent" Ramanujan-type series

In [19], the second-named author generalized an observation of L. Van Hamme about Ramanujan-type identities for $1/\pi$ and $1/\pi^2$. The idea is to associate with each such identity

$$\sum_{n=0}^{\infty}A_n(a+bn)z^n=\frac{r\sqrt d}{\pi}\quad\text{or}\quad\sum_{n=0}^{\infty}A_n(a+bn+cn^2)z^n=\frac{r\sqrt d}{\pi^2},$$

where a, b, c, z, and r are rational and A_n is a related Pochhammer ratio (or, more generally, an Apéry-like sequence; cf. [19]), the supercongruence

$$\sum_{n=0}^{p-1}A_n(a+bn)z^n\overset{?}{\equiv}a\biggl(\frac{-d}{p}\biggr)p\pmod{p^3}$$

or

$$\sum_{n=0}^{p-1} A_n(a+bn+cn^2)z^n \overset{?}{\equiv} a\left(\frac{d}{p}\right)p^2 \pmod{p^5}, \tag{46}$$

respectively, for all $p \geq p_0$. Recently [9], the first-named author went even further and considerably extended the pattern; however this remains an unproven observation.

The general machinery for proving Ramanujan-like series for $1/\pi$ [2, 4, 18] produces, in several cases, *divergent instances* such as

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3}(3n+1)2^{2n} \text{ “=” } \frac{-2i}{\pi}, \qquad \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3}(3n+1)(-1)^n 2^{3n} \text{ “=” } \frac{1}{\pi}. \tag{47}$$

The summations in (47) have to be understood as the analytic continuation of the corresponding ${}_3F_2$-hypergeometric series; for example, the second formula in (47) can be given by

$$\frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} \frac{(\frac{1}{2})_s^3}{(1)_s^2}\,\Gamma(-s)(3s+1)2^{3s}\,\mathrm{d}s = \frac{1}{\pi}.$$

The first appearance of divergent series for $1/\pi$ is [2, p. 371]. In view of the observation from [19], the formulae in (47) motivate our supercongruences (1) and (3), respectively.

Curiously, our study of the first identity in (47) led us to "complex" convergent Ramanujan-type formulae for $1/\pi$ such as

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3}\left(\frac{105-21\sqrt{-7}}{32}n + \frac{49-13\sqrt{-7}}{64}\right)\cdot\left(\frac{47+45\sqrt{-7}}{128}\right)^n = \frac{\sqrt{7}}{\pi}. \tag{48}$$

As far as we know, such formulae do not exist in the literature. Note that application of the quadratic transformation

$${}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2}\\ 1,\,1\end{matrix}\,\right|\, z\right) = (1-z)^{-1/2}\cdot {}_3F_2\left(\left.\begin{matrix}\frac{1}{4},\,\frac{1}{2},\,\frac{3}{4}\\ 1,\,1\end{matrix}\,\right|\, \frac{-4z}{(1-z)^2}\right)$$

(the method used in [5] and [17]) translates (48) into the identity

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(1)_n^3}(35n+8)\left(\frac{4}{3}\right)^{4n} \text{ “=” } -\frac{18i}{\pi}, \tag{49}$$

which has to be understood as the analytic continuation of the hypergeometric series on the left-hand side to $\mathbb{C}\setminus(-\infty,0]$ and which serves as the prototype of (6). All identities for $1/\pi$, including the divergent and complex instances (47)–(49) and others, can be proven by the modular argument. We plan to address these issues in another project. In [8], the first-named author gives proofs of several divergent hypergeometric formulae for $1/\pi$ and $1/\pi^2$ using a version of the Wilf–Zeilberger algorithm.

The theory developed in [7] allows us to obtain numerically the parameters of the divergent Ramanujan-like series for $1/\pi^2$ as well. For example, the expansion

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{n+x}^5}{(1)_{n+x}^5}(a+b(n+x)+c(n+x)^2)(-1)^n z^{n+x} = \frac{1}{\pi^2} - \frac{k}{2}x^2 + O(x^4) \quad \text{as } x \to 0,$$

where $z > 0$, corresponds to the case $s = t = 1/2$ and $u = -1$ of [7, Exp. 1.2], and defines z, a, b and c as functions of k. All these quantities admit a natural parametrization by means of τ, where $\tau^2 = c^2/(1+z)$; see [7, Eq. (3.47)] for details. For this case, the first-named author discovered experimentally that the relation $z(\tau_1)z(\tau_2) = 1$ implies

$$(k_1+1)\tau_2 = (k_2+1)\tau_1, \quad (k_1+1)(k_2+1)+8 = 4\tau_1\tau_2, \tag{50}$$

and also

$$c(\tau_2) = \frac{\tau_2}{\tau_1}\cdot\frac{c}{\sqrt{z}}(\tau_1), \quad b(\tau_2) = \frac{\tau_2}{\tau_1}\cdot\frac{c-b}{\sqrt{z}}(\tau_1), \quad a(\tau_2) = \frac{\tau_2}{\tau_1}\cdot\frac{c-2b+4a}{4\sqrt{z}}(\tau_1). \tag{51}$$

The choice $\tau_1 = \sqrt{5}$ corresponds to the Ramanujan-like series

$$\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^5}{(1)_n^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}} = \frac{8}{\pi^2}$$

proven in [6], in which case $k = 1$ and $z(\tau_1) = 1/4$. This series suggests the existence of a "divergent" companion with $z(\tau_2) = 4$, $\tau_2 = \sqrt{5}/2$, $k = 0$, $c = 5/2$, $b = 3/2$ and $a = 1/4$; the values τ and k are found from (50) and c, b and a from (51). The resulting set corresponds to the series

$$\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^5}{(1)_n^5}(10n^2+6n+1)(-1)^n 2^{2n} \text{ “=” } \frac{4}{\pi^2},$$

with the left-hand side understood as the analytic continuation of the participating hypergeometric series to $\mathbb{C}\setminus[1,+\infty)$. A similar duality for the ${}_3F_2$-evaluations of $1/\pi$ can be explained by the modular origin of the corresponding hypergeometric series, such as the one we give for (47). The duality mechanism for the ${}_5F_4$-examples remains a mystery.

As already pointed out in [19], *all* Ramanujan-type series for $1/\pi$ and their generalizations possess unexpectedly strong arithmetic properties. In particular, these are reflected by the supercongruences for truncated sums; it is probably not surprising to see the examples (1)–(9). What is more remarkable, the origin of the p-analogues makes no difference, whether they come from convergent or divergent formulae. This kind of democracy as well as an apparent simplicity of the supercongruences make them an attractive object for further investigation.

Acknowledgments

After posting the preprint online we were informed by Zhi-Wei Sun that he had experimentally and independently discovered the congruences (1)–(8), but also proved in [14] a more general supercongruence than in (4). We thank him for bringing our attention to his work. We are indebted to the anonymous referee for a careful reading and helpful suggestions on improving the presentation.

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