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THE DIFFERENTIABLE SPHERE THEOREM FOR MANIFOLDS WITH POSITIVE RICCI CURVATURE

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Abstract. We prove that if M^n is a compact Riemannian n-manifold and if $Ric_{\min} > (n-1)\tau_n K_{\max}$, where $K_{\max}(x) := \max_{\pi\subset T_xM} K(\pi)$, $Ric_{\min}(x) := \min_{u\in U_xM} Ric(u)$, $K(\cdot)$ and $Ric(\cdot)$ are the sectional curvature and Ricci curvature of M respectively, and $\tau_n = 1 - \frac{6}{5(n-1)}$, then M is diffeomorphic to a spherical space form. In particular, if M is a compact simply connected manifold with $K \le 1$ and $Ric_M > (n-1)\tau_n$, then M is diffeomorphic to the standard n-sphere S^n. We also extend the differentiable sphere theorem above to submanifolds in Riemannian manifolds with codimension p.

1. Introduction and main results

In 1989, Shen [14] obtained the following important topological sphere theorem for manifolds of positive Ricci curvature (see also [16]).

Theorem A. *Let M^n be an n-dimensional complete and simply connected manifold. If the sectional curvature satisfies $K_M \le 1$, and the Ricci curvature satisfies $Ric_M \ge (n-1)\delta_n$, where*

$$\delta_n = \begin{cases} \frac{5}{8} - \frac{3}{8(n-1)} & \text{for even } n, \\ \frac{5}{8} - \frac{3}{4(n-1)} & \text{for odd } n, \end{cases}$$

then M is homeomorphic to the n-sphere S^n.

A natural question related to Shen's sphere theorems is as follows.

Question. *Can one prove a differentiable sphere theorem for manifolds satisfying similar pinching condition?*

In 1966, Calabi (unpublished) and Gromoll [8] first investigated the differentiable pinching problem for positive pinched compact manifolds. During the past four decades, there has been much progress on differentiable pinching problems for Riemannian manifolds and submanifolds [1, 2, 5, 16]. In 2007, Brendle and Schoen

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proved a remarkable differentiable sphere theorem [6, 7] for manifolds with quarter pinched curvature in the pointwise sense. Recently Brendle [3] obtained the following striking convergence theorem for the Ricci flow.

Theorem B. *Let (M, g_0) be a compact Riemannian manifold of dimension $n(\geq 4)$. Assume that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [0, 1]$. Then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)} + \frac{2}{n} r_{g(t)} g(t)$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

Let M^n be an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifolds N^{n+p}. Denote by H and S the mean curvature and the squared length of the second fundamental form of M, respectively. Let $\overline{K}(\pi)$ be the sectional curvature of N for the tangent 2-plane $\pi(\subset T_xN)$ at the point $x \in N$. Set $\overline{K}_{\max}(x) := \max_{\pi \subset T_xN} \overline{K}(\pi)$. Denote by $\overline{Ric}^{(\mathbf{k})}_{\min}(x)$ the minimum of the $\mathbf{k}$-th Ricci curvature of N at the point $x \in N$. The geometry and topology of the $\mathbf{k}$-th Ricci curvature were initiated by Hartman [10] in 1979 and developed by Wu [18], Shen [14, 15] and others.

In this paper, we investigate the differentiable pinching problem for compact submanifolds in Riemannian manifolds with codimension $p(\geq 0)$. Using Brendle's convergence theorem for the Ricci flow, we prove a differentiable sphere theorem for manifolds with lower bound for Ricci curvature and upper bound for sectional curvature, which is an answer to our question. We first prove the following differentiable sphere theorem for general submanifolds.

Theorem 1.1. *Let M^n be an n-dimensional, $n \geq 3$, complete submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p} with codimension p (≥ 0). If*

$$\sup_M \Big[S - \frac{n^2 H^2}{n-1} - \frac{5}{3}\Big(\overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k} - \frac{6}{5}) \overline{K}_{\max} \Big) \Big] < 0,$$

for some integer $\mathbf{k} \in [2, n+p-1]$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.

Moreover, we get the following differentiable sphere theorem.

Theorem 1.2. *Let M^n be an n-dimensional, $n \geq 3$, complete submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p} with codimension p (≥ 0). If*

$$\sup_M \Big[S - \frac{5\sqrt{2}}{3}\Big(\overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k} - \frac{6}{5}) \overline{K}_{\max} \Big) \Big] < 0,$$

for some integer $\mathbf{k} \in [2, n+p-1]$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.

Let $K(\pi)$ be the sectional curvature of M for the tangent 2-plane $\pi \subset T_xM$ at the point $x \in M$, and let $Ric(u)$ be the Ricci curvature of M for the unit tangent vector $u \in U_xM$ at $x \in M$. Set $K_{\max}(x) := \max_{\pi \subset T_xM} K(\pi)$, $Ric_{\min}(x) := \min_{u \in U_xM} Ric(u)$. We obtain the following differentiable sphere theorem for Riemannian manifolds with pinched curvatures in the pointwise sense.

Theorem 1.3. *Let M^n be an n-dimensional, $n \geq 3$, compact Riemannian manifold. If $Ric_{\min} > (n-1)\tau_n K_{\max}$, where $\tau_n = 1 - \frac{6}{5(n-1)}$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n .*

2. Notation and formulas

Let M^n be an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifolds N^{n+p}. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p;\ 1 \leq i, j, k, \ldots \leq n;$$
$$\text{if } p \geq 1,\ \ n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

For an arbitrary fixed point $x \in M \subset N$, we choose an orthonormal local frame field $\{e_A\}$ in N^{n+p} such that the e_i's are tangent to M. Denote by $\{\omega_A\}$ the dual frame field of $\{e_A\}$. Let

$$Rm = \sum_{i,j,k,l} R_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$
$$\overline{Rm} = \sum_{A,B,C,D} \overline{R}_{ABCD}\omega_A \otimes \omega_B \otimes \omega_C \otimes \omega_D$$

be the Riemannian curvature tensors of M and N, respectively. Denote by h the second fundamental form of M. When $p = 0$, h is identically equal to zero. When $p \geq 1$, we set $h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha$. The squared norm S of the second fundamental form and the mean curvature H of M are given by

$$S = \sum_{\alpha,i,j} (h^\alpha_{ij})^2,\ H = \Big|\frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha\Big|.$$

Then we have the Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} + \langle h(e_i, e_k), h(e_j, e_l)\rangle - \langle h(e_i, e_l), h(e_j, e_k)\rangle. \tag{2.1}$$

Denote by $Ric(\cdot)$ and $\overline{Ric}(\cdot)$ the Ricci curvature of M and N, respectively. Set

$$Ric(e_i) = \sum_j R_{ijij},\ \ Ric_{\min}(x) = \min_{u \in U_xM} Ric(u);$$
$$\overline{Ric}(e_A) = \sum_B \overline{R}_{ABAB},\ \ \overline{Ric}_{\min}(x) = \min_{u \in U_xN} \overline{Ric}(u).$$

For any unit tangent vector $u \in U_xM$ at the point $x \in M$, let $V^{\mathbf{k}}_x$ be a $\mathbf{k}$-dimensional subspace of T_xM satisfying $u \perp V^{\mathbf{k}}_x$. Choose an orthonormal basis $\{e_i\}$ in T_xM such that $e_{j_0} = u,\ span\{e_{j_1}, \ldots, e_{j_{\mathbf{k}}}\} = V^{\mathbf{k}}_x$, where the indices $1 \leq j_0, j_1, \ldots, j_{\mathbf{k}} \leq n$

are distinct from each other. We set

$$Ric^{(\mathbf{k})}(u;V_x^{\mathbf{k}}) = Ric^{(\mathbf{k})}(e_{j_0};e_{j_1},\dots,e_{j_{\mathbf{k}}}) = \sum_{\mathbf{s}=1}^{\mathbf{k}} R_{j_0 j_{\mathbf{s}} j_0 j_{\mathbf{s}}}, \tag{2.2}$$

$$Ric^{(\mathbf{k})}(u) = \min_{u\perp V_x^{\mathbf{k}}\subset T_xM} Ric^{(\mathbf{k})}(u;V_x^{\mathbf{k}}), \tag{2.3}$$

$$Ric^{(\mathbf{k})}_{\min}(x) = \min_{u\in U_xM} Ric^{(\mathbf{k})}(u) = \min_{u\in U_xM}\ \min_{u\perp V_x^{\mathbf{k}}\subset T_xM} Ric^{(\mathbf{k})}(u;V_x^{\mathbf{k}}). \tag{2.4}$$

Definition 2.1. We call $Ric^{(\mathbf{k})}(u;V_x^{\mathbf{k}})$ the $\mathbf{k}$-th Ricci curvature of M, and $Ric^{(\mathbf{k})}_{\min}(x)$ is called the minimum of the $\mathbf{k}$-th Ricci curvature of M at the point $x\in M$.

For any unit tangent vector $u\in U_xN$ at the point $x\in N$, let $V_x^{\mathbf{k}}$ be a $\mathbf{k}$-dimensional subspace of T_xN satisfying $u\perp V_x^{\mathbf{k}}$. Choose an orthonormal basis $\{e_A\}$ in T_xN such that $e_{A_0}=u$, $span\{e_{A_1},\dots,e_{A_{\mathbf{k}}}\}=V_x^{\mathbf{k}}$, where the indices $1\le A_0,A_1,\dots,A_{\mathbf{k}}\le n+p$ are distinct from each other. We define the $\mathbf{k}$-th Ricci curvature and the minimum of the $\mathbf{k}$-th Ricci curvature of N at the point $x\in N$ as follows:

$$\overline{Ric}^{(\mathbf{k})}(u;V_x^{\mathbf{k}}) = \sum_{\mathbf{s}=1}^{\mathbf{k}} \overline{R}_{A_0A_{\mathbf{s}}A_0A_{\mathbf{s}}}, \tag{2.5}$$

$$\overline{Ric}^{(\mathbf{k})}(u) = \min_{u\perp V_x^{\mathbf{k}}\subset T_xN} \overline{Ric}^{(\mathbf{k})}(u;V_x^{\mathbf{k}}), \tag{2.6}$$

$$\overline{Ric}^{(\mathbf{k})}_{\min}(x) = \min_{u\in U_xN}\ \min_{u\perp V_x^{\mathbf{k}}\subset T_xN} \overline{Ric}^{(\mathbf{k})}(u;V_x^{\mathbf{k}}). \tag{2.7}$$

Denote by $K(\pi)$ the sectional curvature of M for the tangent 2-plane $\pi(\subset T_xM)$ at the point $x\in M$ and by $\overline{K}(\pi)$ the sectional curvature of N for the tangent 2-plane $\pi(\subset T_xN)$ at the point $x\in N$. Set $K_{\min}(x)=\min_{\pi\subset T_xM}K(\pi)$, $K_{\max}(x)=\max_{\pi\subset T_xM}K(\pi)$, $\overline{K}_{\min}(x)=\min_{\pi\subset T_xN}\overline{K}(\pi)$, $\overline{K}_{\max}(x)=\max_{\pi\subset T_xN}\overline{K}(\pi)$. Then by Berger's inequality, we have

$$|R_{ijkl}| \le \frac{2}{3}(K_{\max}-K_{\min}) \tag{2.8}$$

for all distinct indices i, j, k, l, and

$$|\overline{R}_{ABCD}| \le \frac{2}{3}(\overline{K}_{\max}-\overline{K}_{\min}) \tag{2.9}$$

for all distinct indices A, B, C, D.

3. Proof of the theorems

Theorem 3.1. *Let M^n be an n-dimensional, $n\ge 4$, compact submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p} with codimension $p(\ge 0)$. If*

$$S < \frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min} - \Big(\mathbf{k}-\frac{6}{5}\Big)\overline{K}_{\max}\Big] + \frac{n^2H^2}{n-1},$$

for some integer $\mathbf{k}\in[2,n+p-1]$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.

Proof. When $p=0$, it is easy to see from (2.2) that

$$Ric^{(\mathbf{k})}_{\min} \le K_{\min} + (\mathbf{k}-1)K_{\max}.$$

Then we have

$$K_{\min} \geq Ric^{(\mathbf{k})}_{\min} - (\mathbf{k}-1)K_{\max}. \tag{3.1}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda \in \mathbb{R}$. From (2.2), (2.8) and (3.1) we obtain

$$\begin{aligned} & R_{1313} + R_{2323} - |R_{1234}| \\ & \geq Ric^{(\mathbf{k})}_{\min} - \sum_{i=3}^{\mathbf{k}+1} R_{i3i3} - \frac{2}{3}(K_{\max} - K_{\min}) \\ & \geq Ric^{(\mathbf{k})}_{\min} - (\mathbf{k}-2)K_{\max} - \frac{2}{3}(\mathbf{k}K_{\max} - Ric^{(\mathbf{k})}_{\min}) \\ & \geq \frac{5}{3}\Big[Ric^{(\mathbf{k})}_{\min} - \Big(\mathbf{k} - \frac{6}{5}\Big)K_{\max}\Big]. \end{aligned} \tag{3.2}$$

Similarly we get

$$R_{1414} + R_{2424} - |R_{1234}| \geq \frac{5}{3}\Big[Ric^{(\mathbf{k})}_{\min} - \Big(\mathbf{k} - \frac{6}{5}\Big)K_{\max}\Big]. \tag{3.3}$$

From (3.2), (3.3) and the assumption we obtain

$$\begin{aligned} & R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ & \geq R_{1313} + R_{2323} - |R_{1234}| + \lambda^2(R_{1414} + R_{2424} - |R_{1234}|) \\ & \geq \frac{5(1+\lambda^2)}{3}\Big[Ric^{(\mathbf{k})}_{\min} - \Big(\mathbf{k} - \frac{6}{5}\Big)K_{\max}\Big] \\ & > 0. \end{aligned} \tag{3.4}$$

Hence the assertion follows from Theorem B.

When $p \geq 1$, it is easy to see from (2.5) that

$$\overline{Ric}^{(\mathbf{k})}_{\min} \leq \overline{K}_{\min} + (\mathbf{k}-1)\overline{K}_{\max}.$$

This implies

$$\overline{K}_{\min} \geq \overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k}-1)\overline{K}_{\max}. \tag{3.5}$$

Setting $S_\alpha := \sum_{i,j=1}^n (h^\alpha_{ij})^2$, we have

$$\Big(\sum_{i=1}^n h^\alpha_{ii}\Big)^2 = (n-1)\Big[\sum_{i=1}^n (h^\alpha_{ii})^2 + \sum_{i\neq j}(h^\alpha_{ij})^2 + \frac{(\sum_{i=1}^n h^\alpha_{ii})^2}{n-1} - S_\alpha\Big]. \tag{3.6}$$

Note that for $t \neq s$

$$\begin{aligned} \Big(\sum_{i=1}^n h^\alpha_{ii}\Big)^2 &\leq (n-1)\Big[(h^\alpha_{tt} + h^\alpha_{ss})^2 + \sum_{i\neq t,s}(h^\alpha_{ii})^2\Big] \\ &= (n-1)\Big[\sum_{i=1}^n (h^\alpha_{ii})^2 + 2h^\alpha_{tt}h^\alpha_{ss}\Big]. \end{aligned}$$

This together with (3.6) implies

$$2h^\alpha_{tt}h^\alpha_{ss} \geq \sum_{i\neq j}(h^\alpha_{ij})^2 + \frac{(\sum_{i=1}^n h^\alpha_{ii})^2}{n-1} - S_\alpha, \tag{3.7}$$

where $t \neq s$. Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda \in \mathbb{R}$. From (2.1), (2.5), (2.9), (3.5) and (3.7) we get

$$
\begin{aligned}
&R_{1313} + R_{2323} - |R_{1234}| \\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min} - \sum_{A=3}^{\mathbf{k}+1} \overline{R}_{A3A3} - \frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min}) + \sum_{\alpha} \Big[h^{\alpha}_{11} h^{\alpha}_{33} + h^{\alpha}_{22} h^{\alpha}_{33} \\
&- \frac{3}{2}(h^{\alpha}_{13})^2 - \frac{3}{2}(h^{\alpha}_{23})^2 - \frac{1}{2}(h^{\alpha}_{24})^2 - \frac{1}{2}(h^{\alpha}_{14})^2 \Big] \\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k}-2)\overline{K}_{\max} - \frac{2}{3}[\mathbf{k}\overline{K}_{\max} - \overline{Ric}^{(\mathbf{k})}_{\min}] + \sum_{\alpha} \Big[\sum_{i \neq j} (h^{\alpha}_{ij})^2 \\
&+ \frac{(\sum_{i=1}^{n} h^{\alpha}_{ii})^2}{n-1} - S_{\alpha} - \frac{3}{2}(h^{\alpha}_{13})^2 - \frac{3}{2}(h^{\alpha}_{23})^2 - \frac{1}{2}(h^{\alpha}_{24})^2 - \frac{1}{2}(h^{\alpha}_{14})^2 \Big] \\
&\geq \frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min} - \Big(\mathbf{k} - \frac{6}{5}\Big)\overline{K}_{\max}\Big] + \frac{n^2 H^2}{n-1} - S.
\end{aligned} \tag{3.8}
$$

Similarly we obtain

$$
R_{1414} + R_{2424} - |R_{1234}| \geq \frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min} - \Big(\mathbf{k} - \frac{6}{5}\Big)\overline{K}_{\max}\Big] + \frac{n^2 H^2}{n-1} - S. \tag{3.9}
$$

It follows from (3.8), (3.9) and the assumption that

$$
\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\
&\geq R_{1313} + R_{2323} - |R_{1234}| + \lambda^2 (R_{1414} + R_{2424} - |R_{1234}|) \\
&\geq (1+\lambda^2)\Big\{\frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min} - \Big(\mathbf{k} - \frac{6}{5}\Big)\overline{K}_{\max}\Big] + \frac{n^2 H^2}{n-1} - S\Big\} \\
&> 0.
\end{aligned} \tag{3.10}
$$

This together with Theorem B implies that M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let M be a 3-dimensional complete submanifold in a Riemannian manifold N^{3+p} with codimension $p(\geq 0)$. If*

$$
S < \overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k}-2)\overline{K}_{\max} + \frac{9}{2}H^2
$$

for some integer $\mathbf{k} \in [2, 2+p]$, then M is diffeomorphic to a spherical space form or $\mathbb{R}^3$.

Proof. When $p = 0$, the assertion follows from the theorems due to Hamilton [9] and Schoen and Yau [13].

When $p \geq 1$, for any unit tangent vector $u \in U_x M$ at $x \in M$, we choose an orthonormal three-frame $\{e_1, e_2, e_3\}$ such that $e_3 = u$. From (2.1), (2.5) and (3.7)

we obtain

$$
\begin{aligned}
Ric(u) &= R_{1313} + R_{2323} \\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min} - \sum_{A=3}^{\mathbf{k}+1} \overline{R}_{A3A3} + \sum_{\alpha} \Big[h^{\alpha}_{11} h^{\alpha}_{33} + h^{\alpha}_{22} h^{\alpha}_{33} - (h^{\alpha}_{13})^2 - (h^{\alpha}_{23})^2 \Big] \\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k}-2)\overline{K}_{\max} \\
&\quad + \sum_{\alpha} \Big[\sum_{i \neq j} (h^{\alpha}_{ij})^2 + \frac{(\sum_{i=1}^{3} h^{\alpha}_{ii})^2}{2} - S_{\alpha} - (h^{\alpha}_{13})^2 - (h^{\alpha}_{23})^2 \Big] \\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k}-2)\overline{K}_{\max} + \frac{9}{2} H^2 - S.
\end{aligned} \tag{3.11}
$$

This together with the assumption implies that $Ric_M > 0$. It follows from the theorems of Hamilton [9] and Schoen and Yau [13] that M is diffeomorphic to a spherical space form or $\mathbb{R}^3$. This proves Theorem 3.2. □

Proof of Theorem 1.1. From (3.4), (3.10), (3.11) and the assumption, it follows that there exists an $\epsilon > 0$ such that $Ric_{\min} \geq \frac{n-1}{2} Ric^{(2)}_{\min} \geq \epsilon$. By the classical Myers theorem, we know that M is compact. This together with Theorems 3.1 and 3.2 implies M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. When $p = 0$, the assertion follows from Theorem 1.1.

When $p \geq 1$, setting $S_{\alpha} := \sum_{i,j=1}^{n} (h^{\alpha}_{ij})^2$, we have

$$
\Big(\sqrt{2} h^{\alpha}_{tt} + \sum_{i \neq t} h^{\alpha}_{ii} \Big)^2 \geq (n-1) \Big[\sum_{i=1}^{n} (h^{\alpha}_{ii})^2 + \sum_{i \neq j} (h^{\alpha}_{ij})^2 - S_{\alpha} \Big]. \tag{3.12}
$$

Note that for distinct t, s, l

$$
\begin{aligned}
\Big(\sqrt{2} h^{\alpha}_{tt} + \sum_{i \neq t} h^{\alpha}_{ii} \Big)^2 &\leq (n-1) \Big[\Big(\frac{h^{\alpha}_{tt}}{\sqrt{2}} + h^{\alpha}_{ss} \Big)^2 + \Big(\frac{h^{\alpha}_{tt}}{\sqrt{2}} + h^{\alpha}_{ll} \Big)^2 + \sum_{i \neq t,s,l} (h^{\alpha}_{ii})^2 \Big] \\
&= (n-1) \Big[\sum_{i=1}^{n} (h^{\alpha}_{ii})^2 + \sqrt{2} (h^{\alpha}_{tt} h^{\alpha}_{ss} + h^{\alpha}_{tt} h^{\alpha}_{ll}) \Big].
\end{aligned}
$$

This together with (3.12) implies

$$
h^{\alpha}_{tt} h^{\alpha}_{ss} + h^{\alpha}_{tt} h^{\alpha}_{ll} \geq \frac{1}{\sqrt{2}} \Big[\sum_{i \neq j} (h^{\alpha}_{ij})^2 - S_{\alpha} \Big], \tag{3.13}
$$

for distinct t, s, l.

When $n=3$, for any unit vector $u\in U_xM$ at $x\in M$, we choose an orthonormal three-frame $\{e_1,e_2,e_3\}$ such that $e_3=u$. From (2.1), (2.5) and (3.13) we obtain

(3.14)
$$\begin{aligned}
Ric(u) &= R_{1313}+R_{2323}\\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min}-\sum_{A=3}^{\mathbf{k}+1}\overline{R}_{A3A3}+\sum_{\alpha}\Big[h^{\alpha}_{11}h^{\alpha}_{33}+h^{\alpha}_{22}h^{\alpha}_{33}-(h^{\alpha}_{13})^2-(h^{\alpha}_{23})^2\Big]\\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min}-(\mathbf{k}-2)\overline{K}_{\max}+\sum_{\alpha}\Big[\frac{\sum_{i\neq j}(h^{\alpha}_{ij})^2-S_{\alpha}}{\sqrt{2}}-(h^{\alpha}_{13})^2-(h^{\alpha}_{23})^2\Big]\\
&\geq \frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min}-\Big(\mathbf{k}-\frac{6}{5}\Big)\overline{K}_{\max}\Big]-\frac{S}{\sqrt{2}}.
\end{aligned}$$

When $n\geq 4$, suppose $\{e_1,e_2,e_3,e_4\}$ is an orthonormal four-frame and $\lambda\in\mathbb{R}$. From (2.1), (2.5), (2.9), (3.5) and (3.13) we get

$$\begin{aligned}
&R_{1313}+R_{2323}-|R_{1234}|\\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min}-\sum_{A=3}^{\mathbf{k}+1}\overline{R}_{A3A3}-\frac{2}{3}(\overline{K}_{\max}-\overline{K}_{\min})+\sum_{\alpha}\Big[h^{\alpha}_{11}h^{\alpha}_{33}+h^{\alpha}_{22}h^{\alpha}_{33}\\
&-\frac{5}{4}(h^{\alpha}_{13})^2-\frac{5}{4}(h^{\alpha}_{23})^2-(h^{\alpha}_{24})^2-(h^{\alpha}_{14})^2\Big]\\
&\geq \overline{Ric}^{(\mathbf{k})}_{\min}-(\mathbf{k}-2)\overline{K}_{\max}-\frac{2}{3}[\mathbf{k}\overline{K}_{\max}-\overline{Ric}^{(\mathbf{k})}_{\min}]\\
&+\sum_{\alpha}\Big[\frac{\sum_{i\neq j}(h^{\alpha}_{ij})^2-S_{\alpha}}{\sqrt{2}}-\frac{5}{4}(h^{\alpha}_{13})^2-\frac{5}{4}(h^{\alpha}_{23})^2-(h^{\alpha}_{24})^2-(h^{\alpha}_{14})^2\Big]\\
&\geq \frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min}-\Big(\mathbf{k}-\frac{6}{5}\Big)\overline{K}_{\max}\Big]-\frac{S}{\sqrt{2}}.
\end{aligned}\tag{3.15}$$

Similarly we get

$$R_{1414}+R_{2424}-|R_{1234}|\geq \frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min}-\Big(\mathbf{k}-\frac{6}{5}\Big)\overline{K}_{\max}\Big]-\frac{S}{\sqrt{2}}.\tag{3.16}$$

From (3.15) and (3.16) we obtain

$$\begin{aligned}
&R_{1313}+\lambda^2R_{1414}+R_{2323}+\lambda^2R_{2424}-2\lambda R_{1234}\\
&\geq R_{1313}+R_{2323}-|R_{1234}|+\lambda^2(R_{1414}+R_{2424}-|R_{1234}|)\\
&\geq (1+\lambda^2)\Big\{\frac{5}{3}\Big[\overline{Ric}^{(\mathbf{k})}_{\min}-\Big(\mathbf{k}-\frac{6}{5}\Big)\overline{K}_{\max}\Big]-\frac{S}{\sqrt{2}}\Big\}.
\end{aligned}\tag{3.17}$$

From (3.14), (3.17) and the assumption, it follows that there exists an $\epsilon>0$ such that $Ric_{\min}\geq\frac{n-1}{2}Ric^{(2)}_{\min}\geq\epsilon$. By the classical Myers theorem, we know that M is compact. Moreover, we have $Ric_M>0$ when $n=3$ and $R_{1313}+\lambda^2R_{1414}+R_{2323}+\lambda^2R_{2424}-2\lambda R_{1234}>0$ when $n\geq 4$. This together with Hamilton's theorem [9] and Theorem B implies M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n. This proves Theorem 1.2. □

The following sphere theorem for 3-dimensional submanifolds follows from the proof of Theorem 1.2 and Schoen-Yau's theorem [13].

Theorem 3.3. *Let M be a 3-dimensional complete submanifold in a $(3+p)$-dimensional Riemannian manifold N^{3+p} with codimension $p(\geq 0)$. If*

$$S < \sqrt{2}[\overline{Ric}^{(\mathbf{k})}_{\min} - (\mathbf{k}-2)\overline{K}_{\max}]$$

for some integer $\mathbf{k} \in [2, 2+p]$, then M is diffeomorphic to a spherical space form or $\mathbb{R}^3$.

Proof of Theorem 1.3. When $n = 3$, the assertion follows from the assumption and Hamilton's theorem [9].

When $n \geq 4$, we take $p = 0$ and $\mathbf{k} = n-1$ in Theorem 3.1 and obtain that M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n. This completes the proof of Theorem 1.3. □

Corollary 3.4. *Let M^n be an n-dimensional, $n \geq 3$, compact Riemannian manifold. If the sectional curvature satisfies $K_M \leq 1$ and the Ricci curvature satisfies $Ric_M > n - \frac{11}{5}$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n .*

Proof. From $K_M \leq 1$ and $Ric_M > n - \frac{11}{5}$, we obtain $Ric_{\min} > \left(n - \frac{11}{5}\right)K_{\max}$, which together with Theorem 1.3 completes the proof. □

Example 3.5. W. Ziller [21] constructed a family of metrics g_s on $M^n = S^{2m+1} = U(m+1)/U(m)$. This family of metrics g_s has

$$K_{\max} = \frac{(m+1)s^2}{2m}, \qquad K_{\min} = 4 - \frac{3(m+1)}{2m}s^2$$

for $s \geq \sqrt{\frac{2m}{m+1}}$. By a calculation of the Ricci curvature due to Shen [14], we have

$$Ric_{\min} = (2 - \frac{s^2}{m})(m+1) \quad \text{for} \quad s \geq \sqrt{\frac{2m}{m+1}}.$$

Note that g_s is the standard metric on S^{2m+1} for $s = \sqrt{\frac{2m}{m+1}}$. Then

$$\tau_1(s) := \frac{Ric_{\min}}{2mK_{\max}} = \frac{2}{s^2} - \frac{1}{m},$$
$$\tau_2(s) := \frac{K_{\min}}{K_{\max}} = \frac{8m}{(m+1)s^2} - 3.$$

When $\sqrt{\frac{2m}{m+1}} \leq s < \sqrt{\frac{2m}{1+m\tau_n}}$, we see that $\tau_1(s) > \tau_n$. This means that g_s satisfies the pinching condition in Theorem 1.3. In particular, taking $n = 5$, i.e., $m = 2$, and s close to $\sqrt{\frac{2m}{1+m\tau_n}}$, we get that $\tau_2(s)$ is close to $\frac{4(1+m\tau_n)}{m+1} - 3 = \frac{1}{5}$. In this case, g_s does not posses $\frac{1}{4}$-pinched curvature in the pointwise sense. Therefore, Theorem 1.3 is nontrivial.

By a direct computation, we have the normalized Ricci curvatures of the compact rank one symmetric spaces (CROSS) with standard metrics:

$$Ric_0(\mathbb{C}P^m) = \frac{m+1}{4m-2}, \quad \dim_{\mathbb{R}}(\mathbb{C}P^m) = 2m, \qquad m \geq 2;$$
$$Ric_0(\mathbb{H}P^m) = \frac{m+2}{4m-1}, \quad \dim_{\mathbb{R}}(\mathbb{H}P^m) = 4m, \qquad m \geq 2;$$
$$Ric_0(\mathbb{O}P^2) = \frac{3}{5}, \quad \dim_{\mathbb{R}}(\mathbb{O}P^2) = 16.$$

Motivated by Theorem 3 and the computation above, we would like to propose the following conjecture.

Conjecture A. *Let $M^n (n \geq 4)$ be a compact Riemannian manifold. If $Ric_{\min} > \frac{3}{5}(n-1)K_{\max}$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.*

When $n = 4$, Theorem 1.3 provides an affirmative answer to Conjecture A. We would also like to propose the following:

Conjecture B. *Let $M^n (n \geq 4)$ be an even dimensional complete and simply connected Riemannian manifold such that $K_M \leq 1$, $Ric_M \geq (n-1)\sigma_n$, where*

$$\sigma_n = \begin{cases} \frac{n+2}{4(n-1)} & \text{for } n = 4 \text{ or } 4k+2, \quad k \geq 2, \\ \frac{n+8}{4(n-1)} & \text{for } n = 4k \geq 8, \quad k \neq 4, \\ \frac{3}{5} & \text{for } n = 16. \end{cases}$$

Then M is either diffeomorphic to S^n or isometric to a compact rank one symmetric space.

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