ON AN INCLUSION OF THE ESSENTIAL SPECTRUM OF LAPLACIANS UNDER NON-COMPACT CHANGE OF METRIC

JUN MASAMUNE

(Communicated by Matthew J. Gursky)

Abstract. The stability of essential self-adjointness and an inclusion of the essential spectra of Laplacians under the change of a Riemannian metric on a subset $K$ of $M$ are proved. The set $K$ may have infinite volume measured with the new metric, and its completion may contain a singular set such as the fractal set, to which the metric is not extendable.

1. Introduction

Let $(M, g)$ be a connected smooth Riemannian manifold without boundary. The Laplacian $\Delta$ of $g$ is called essentially self-adjoint if it has the unique self-adjoint extension $\overline{\Delta}$. In [4], Furutani showed that if $\Delta$ with the domain $C_0^\infty(M)$ is essentially self-adjoint and if $g$ is changed on a compact set $K \subset M$ to another smooth metric $g'$ on $M$, then the Laplacian $\Delta'$ of $g'$ with the domain $C_0^\infty(M)$ is essentially self-adjoint; and the essential spectra are stable under this change. In particular, the second result forms a strong contrast to the behavior of the eigenvalues, since eigenvalues change continuously with the perturbation of the metric in a certain way (see e.g. [1]).

 Needless to say, there are many important Riemannian manifolds with singularity, by which we mean that $g$ does not extend to the Cauchy boundary (the difference between the completion of $M$ and $M$), such as algebraic varieties, cone manifolds, edge manifolds, Riemannian orbifolds. In general, the analysis on such a singular space is complicated, and one of the methods to overcome the difficulties is to modify the singularity to a simpler one by the perturbation of the Riemannian metric. The crucial steps in this process is to study the stability of the essential self-adjointness of the Laplacian and to understand the behavior of its spectral structure under the perturbation.

Motivated by these facts, we extend Furutani’s theorem to more general $K$ so that $K$ is not compact and its completion $\overline{K}$ includes the singular set. In this
setting, the natural domain \(D(\Delta)\) for the Laplacian is the following:

\[
\begin{align*}
D(\nabla) &= \{u \in C^\infty \cap L^2 : \nabla u \in L^2\}, \\
D(\text{div}) &= \{X \in C^\infty \cap L^2 : \text{div} X \in L^2\}, \\
D(\Delta) &= \{u \in D(\nabla) : \nabla u \in D(\text{div})\}.
\end{align*}
\]

(We suppress \(M\) and the Riemann measure \(d\mu_g\) for the sake of simplicity.) Indeed, if the Cauchy boundary \(\partial_C M\) is almost polar, namely,

\[
\text{Cap}(\partial_C M) = 0
\]

(see Section 2 for the definition and see also e.g. [3]), then \(M\) has negligible boundary [9], and by the Gaffney theorem [6], \(\Delta\) is essentially self-adjoint. Throughout the article, we assume that the Laplacians have the domain defined in (1). The following is our main result:

**Theorem 1.** Let \(g\) and \(g'\) be Riemannian metrics on \(M\) such that \(g = g'\) outside a subset \(K\) of \(M\). If \(\Delta\) is essentially self-adjoint in \(L^2\) and the Cauchy boundary of \(K\) with respect to \(g'\) is almost polar, then \(\Delta'\) is essentially self-adjoint in \(L^2(M; d\mu_{g'})\). Additionally, if there is a function \(\chi\) on \(M\) satisfying

\[
\nabla \chi \in L^\infty, \quad \Delta \chi \in L^\infty, \quad \text{and} \quad \chi|_K = 1,
\]

where \(\nabla\) is the gradient of \(g\), and the inclusion

\[
H^1_0(N; d\mu_g) \subset L^2(N; d\mu_g)
\]

is compact for some \(N \supset N(\text{supp}(\chi); \epsilon)\) with some \(\epsilon > 0\), where \(N(\text{supp}(\chi); \epsilon)\) is the \(\epsilon\)-neighborhood of the support of \(\chi\), then

\[
\sigma_{\text{ess}}(\Delta) \subset \sigma_{\text{ess}}(\Delta').
\]

A special case of Theorem 1 is

**Corollary 1** (Furutani’s stability result [4]). Let \(g\) and \(g'\) be Riemannian metrics on \(M\) such that \(g = g'\) outside a compact subset \(K\) of \(M\). If \(\Delta\) is essentially self-adjoint in \(L^2\), then \(\Delta'\) is essentially self-adjoint in \(L^2(M; d\mu_{g'})\), and

\[
\sigma_{\text{ess}}(\Delta) = \sigma_{\text{ess}}(\Delta').
\]

A typical example of manifolds which satisfies the condition of Theorem 1 is given as follows:

**Corollary 2** (see Section 3). Let \(M\) be a complete manifold and let \(\Sigma \subset M\) be an almost polar compact subset. If \(\Sigma\) is almost polar with respect to a metric \(g'\) on \(M \setminus \Sigma\) and \(g = g'\) outside a compact set \(K \subset M\), then the same conclusion in the theorem holds true.

We may apply Theorem 1 for singular manifolds: we change \(g\) to \(g'\) on a bounded set \(K \supset \partial_C M\) so that \(g'\) can be extended to the almost polar Cauchy boundary with respect to \(g\) and conclude that \(\Delta\) is essentially self-adjoint in \(L^2(M; d\mu_g)\) and \(\sigma_{\text{ess}}(\Delta) \subset \sigma_{\text{ess}}(\Delta')\).

A sufficient condition for \(\partial_C M\) to be almost polar is that it has Minkowski co-dimension greater than 2 [7] (if the metric of \(g\) extends to \(\partial_C M\) and \(\partial_C M\) is a manifold, then it is almost polar if \(\partial_C M\) has co-dimension 2).

The idea for proving the inclusion of the essential spectrum in Theorem 1 is to apply Weyl’s criteria: a number \(\lambda\) belongs to \(\sigma_{\text{ess}}(\Delta)\) if and only if there is a
sequence $\phi_n$ of “limit-eigenfunctions” of $\Delta$ corresponding to $\lambda$ (see Proposition 2 for details). Indeed, we show that if $\chi$ satisfies (2) and (3), then there is a subsequence $\phi_{n(k)}$ such that $(1 - \chi)\phi_{n(k)}$ is a limit-eigenfunction of $\Delta'$.

Our results differ from Furutani’s original results on the following two points. In order to explain those differences, let us employ an example. Let $M = K \cup B(1)$, where $K = \{(x, y, z) \in S^2 : z \geq 0\} \setminus (0, 0, 1)$ and $B(r) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq r^2, z = 0\}$. Namely, $M$ is $S^2$ with flat bottom with deleted north point. (To be more precise, we need to smooth the intersection of $K$ and $B(1)$ so that $M$ is a smooth Riemannian manifold.) Since the Cauchy boundary of $M$ is the north point and it has null capacity, the Laplacian $\Delta$ is essentially self-adjoint. We modify $M$ by the stereographic projection so that $(M, g')$ is the 2-dimensional Euclidean space $\mathbb{R}^2$. Since $(M, g')$ is complete, the Cauchy boundary is empty, and the Laplacian $\Delta'$ is essentially self-adjoint. Next, we find the function $\chi$ which satisfies condition (2) as follows:

$$\chi \in C_0^\infty(M \setminus B(1/3))$$

and $\chi = 1$ on $K$.

By letting $\epsilon = 1/3$ and $N = M \setminus B(1/4)$, condition (3) is satisfied. Indeed, since the north point has null capacity, the spectrum of the Laplacian on $M$ consists only of the eigenvalues with finite multiplicity, whereas $\Delta'$ has only essential spectrum. This proves that the inclusion in Theorem 1 holds. This example also shows that the assumptions in Corollary 1 are sharp in the sense that we may not drop the assumption that $K$ must be compact in order to obtain (4), that is, Furutani’s stability result. Indeed, if we modify the metric of $\mathbb{R}^2$ to obtain $M$, then there is no subset $N$ of $\mathbb{R}^2$ which satisfies condition (3).

The second difference is that the essential self-adjointness of $\Delta$ does not need to imply that of $\Delta$ restricted to $C_0^\infty(M)$; for instance, $\Delta$ on $M$ is essentially self-adjoint, but $\Delta$ restricted to $C_0^\infty(M)$ has infinitely many self-adjoint extensions. In particular, it is not essentially self-adjoint (see e.g. [2]).

We organize the article in the following manner: in Section 2 we prove Theorem 1 and in Section 3 we present the examples.

2. Proofs

In this section we recall some definitions and prove Theorem 1 and Corollary 1. For the sake of simplicity, we often suppress the symbols $M$ and $d\mu_\gamma$.

We denote by $(\cdot, \cdot)$ and $(\cdot, \cdot)_1$ the inner product in $L^2$ and the Sobolev space $H^1$ of order $(1, 2)$, respectively. $H^1_0$ is the completion of the set $C_0^\infty$ of smooth functions with compact support with respect to the norm $\| \cdot \|_1 = \sqrt{(\cdot, \cdot)_1}$. Let $\mathcal{O}$ be the family of all open subsets of $\overline{M}$. For $A \in \mathcal{O}$ we define $\mathcal{L}_A = \{u \in H^1 : u \geq 1$ $\mu_\gamma$-a.e. on $M \cap A\}$,

$$\text{Cap}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \|u\|_1, & \mathcal{L}_A \neq \phi, \\ \infty, & \mathcal{L}_A = \phi, \end{cases}$$

and

$$\text{Cap}(\partial C M) = \inf_{A \in \mathcal{O}, \partial C \subset A} \text{Cap}(A).$$
We will use

**Proposition 1** (Lemma 2.1.1 of [3]). If $\mathcal{L}_A \neq \phi$ for $A \in \mathcal{O}$, there exists a unique element $e_A \in \mathcal{L}_A$ called the equilibrium potential of $A$ such that the following hold:

(i) $\|e_A\|^2 = \text{Cap}(A)$.
(ii) $0 \leq e_A \leq 1$ $\mu_g$-a.e. and $e_A = 1$ $\mu_g$-a.e. on $A \cap M$.
(iii) If $A, B \in \mathcal{O}$, $A \subset B$, then $e_A \leq e_B$ $\mu_g$-a.e.

We prove Theorem 1. We use the following characterization:

**Proof of the essential self-adjointness of $\Delta$.** For arbitrary $u \in H^1(M; d\mu_{g'})$, we have to find $\hat{u}_n \in H^1(M; d\mu_{g'})$ which converges to $u$ in $H^1(M; d\mu_{g'})$. Indeed, this implies that $(M, g')$ has negligible boundary, and hence $\Delta$ is essentially self-adjoint by Gaffney’s theorem [6].

Since $L^\infty(M) \cap H^1(M; d\mu_{g'})$ is dense in $H^1(M; d\mu_{g'})$, we may assume that $u \in L^\infty(M)$ without loss of generality. Let

$$\psi := (1 - r)_+,$$

where $r$ is the distance from $K$. The function $\psi \in L^\infty(M)$ enjoys the properties

$$\psi|_K = 1 \quad \text{and} \quad \|\nabla \psi\|_{L^\infty} \leq 1.$$

Since

$$|(1 - \psi)u(x)| \leq (1 + \|\psi\|_{L^\infty})|u(x)|$$

and

$$|\nabla((1 - \psi)u(x))| \leq (1 + \|\psi\|_{L^\infty})|\nabla u(x)| + |u(x)|,$$

for almost every $x \in M$, it follows that $(1 - \psi)u \in H^1$. Recalling that the essential self-adjointness of $\Delta$ implies $H_0^1 = H^1$ [3], we find $v_n \in C_0^\infty(M \setminus K)$ such that

$$v_n \rightarrow (1 - \psi)u \quad \text{as} \quad n \rightarrow \infty$$

in $H^1(M \setminus K)$. Because the Cauchy boundary $\partial_C M$ of $M$ associated to $g'$ is almost polar, there is a sequence of the equilibrium potentials $e_n$ of $O_n \supset \partial_C M$ such that $\bigcap_{n>1} O_n = \partial_C M$ and

$$\|e_n\|_{H^2(M; d\mu_{g'})} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Then $u_n := (1 - e_n)\psi u \in H_0^1(M; d\mu_{g'})$ satisfies

$$u_n \rightarrow \psi u \quad \text{as} \quad n \rightarrow \infty$$

in $H^1(M; d\mu_{g'})$, and we get $\hat{u}_n = u_n + v_n \in H^1_0(M; d\mu_{g'})$ such that

$$\hat{u}_n \rightarrow (1 - \psi)u + \psi u = u \quad \text{as} \quad n \rightarrow \infty$$

in $H^1(M; d\mu_{g'})$. 

Next, we prove the inclusion of the essential spectrum and complete the proof of Theorem 1. We use the following characterization:

**Proposition 2** (Weyl’s criterion). A number $\lambda$ belongs to the essential spectrum of $\Delta$ if and only if there is a sequence of orthonormal vectors $\{\phi_n\}$ of $L^2$ such that

$$\|((\Delta - \lambda)\phi_n)\|_{L^2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$
Proof of the inclusion of the essential spectrum. We assume (2) and (3) to prove the inclusion of the essential spectrum. Hereafter, we denote $\Delta = \Delta$ and $\Delta' = \Delta'$ because of their essential self-adjointness. Let $\lambda \in \sigma_{\text{ess}}(\Delta)$ and $\phi_n \in D(\Delta)$ such that
\[
(\phi_i, \phi_j) = \delta_{ij},
\]
\[
\| (\Delta - \lambda) \phi_n \| \to 0 \text{ as } n \to \infty,
\]
where $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. Let $\chi$ be the function satisfying (2) and let $\phi$ be the function defined as
\[
\phi = (1 - \hat{r}/\epsilon)_+,
\]
where $\hat{r}$ is the distance from the support of $\chi$. Clearly, we have
\[
\text{(6)} \quad \sup_{n > 0} \| \phi \phi_n \| < \infty.
\]
Moreover, taking into account that $\phi_n \in D(\Delta)$ implies
\[
\| \nabla \phi_n \|^2 = -(\phi_n, \Delta \phi_n),
\]
it follows that
\[
\limsup_{n \to \infty} \| \nabla (\phi \phi_n) \| \leq \| \nabla \phi \|_{L^\infty} + \limsup_{n \to \infty} \| \nabla \phi_n \| \leq \| \nabla \phi \|_{L^\infty} + \limsup_{n \to \infty} (\| \Delta \phi_n \| \| \phi_n \|^{1/2} \leq \infty.
\]
Hence,
\[
\text{(7)} \quad \limsup_{n \to \infty} \| \phi \phi_n \|_1 < \infty.
\]
Now, specifying $\epsilon > 0$ as in the statement, by (6) and the fact that the embedding $H^1_0(N) \subset L^2(N)$ is compact, there exists a subsequence $\phi_{n(k)}$ of $\phi_n$ and $\phi' \in L^2$ such that
\[
\phi_{n(k)} \to \phi' \text{ strongly in } L^2 \text{ as } k \to \infty.
\]
However, if $f \in L^2$, then $f \phi \in L^2$ and
\[
(f, \phi_n) = (f \phi, \phi_n) \to 0 \text{ as } n \to \infty;
\]
hence, $\phi \phi_{n(k)} \to 0$ weakly in $L^2$ as $k \to \infty$. Because of the uniqueness of the weak-limits, it follows that $\phi' = 0$, and we may assume
\[
\text{(8)} \quad \| \phi \phi_n \| \to 0 \text{ as } n \to \infty
\]
without loss of generality. Since $\phi = 1$ on supp($\chi$),
\[
\| (\Delta - \lambda)(\chi \phi_n) \| \leq \| (\Delta \chi) \phi_n \| + 2\| (\nabla \chi, \nabla \phi_n) \| + \| \chi (\Delta - \lambda) \phi_n \| \leq \| \Delta \chi \|_{L^\infty} \| \phi_n \| + 2\| \nabla \chi \|_{L^\infty} \| \nabla \phi_n \|_{L^2(\text{supp}(\chi))} + \| \chi \|_{L^\infty} \| (\Delta - \lambda) \phi_n \|.
\]
The first and third terms in the last line tend to 0 as $n \to \infty$ because of (5) and (6). The second term can be estimated as
\[
\| \nabla \chi \|_{L^\infty} \| \nabla (\phi \phi_n) \| \leq \| \nabla \chi \|_{L^\infty} \| \Delta \phi_n \| \| \phi \phi_n \| \to 0 \text{ as } n \to \infty.
\]
Thus, since $1 - \chi = 0$ and $g = g'$ on $M \setminus K$,
\[
\| (\Delta' - \lambda)((1 - \chi) \phi_n) \|_{L^2(M; g_{\mu_r})} = \| (\Delta - \lambda)((1 - \chi) \phi_n) \| \to 0 \text{ as } n \to \infty.
\]
On the other hand,
\[ \|(1 - \chi)\varphi_n\|_{L^2(M; d\mu_g)} \geq \|\varphi_n\| - \|\chi\varphi_n\| \geq 1 - \|\varphi\varphi_n\| \to 1 \text{ as } n \to \infty, \]
and we conclude that \( \lambda \in \sigma_{\text{ess}}(\Delta') \) by Weyl’s criterion. \qed

Finally, we assume that \( K \) is compact to prove the essential self-adjointness of the Laplacian and the stability of the essential spectrum, namely, Corollary 1.

**Proof of Corollary 1.** We will show that the Laplacian \( \Delta' \) is essentially self-adjoint and that there exist the function \( \chi \) and the subset \( N \) of \( M \) which satisfy conditions (2) and (3) for each metric \( g \) and \( g' \). This will imply \( \sigma_{\text{ess}}(\Delta) = \sigma_{\text{ess}}(\Delta') \) by Theorem 1.

Recall that if \( K \) is compact, then its Cauchy boundary is empty so that the Laplacian \( \Delta' \) is essentially self-adjoint.

Since \( K \) is compact and \( g \) is smooth, there exists \( \epsilon > 0 \) such that for any \( 0 < \epsilon < \epsilon_0 \), the metric \( g \) and its higher order (up to 2nd) derivatives are bounded on \( N = N(K; 2\epsilon) = \{x \in M : d(x, K) < 2\epsilon\} \). Let
\[ \hat{\chi}(x) = (1 \wedge (2 - 3\tilde{r}(x)/\epsilon))_+, \]
where \( \tilde{r} \) is the distance from \( K \). The function \( \hat{\chi} \) is 1 on \( N(K; \epsilon/3) \) and has support in \( N(K; 2\epsilon/3) \), and it satisfies \( \|\nabla\hat{\chi}\|_{L^\infty} \leq 3/\epsilon \). However, since \( \hat{\chi} \) does not need to be in the Sobolev space \( H^2 \) of order \( (2, 2) \), we apply the Friedrichs mollifier \( j \) with radius \( \delta > 0 \) for \( \hat{\chi} \) to find the smooth function \( \chi = j * \hat{\chi} \). If \( \delta < \epsilon/3 \), then \( \chi \) satisfies
\[ \begin{cases} \chi(x) = 1 & \text{for } x \in K, \\ \text{supp}(\chi) \subset N(K; 2\epsilon/3), \\ \|\nabla\chi\|_{L^\infty} \leq 3/\epsilon, \\ \Delta\chi \in L^\infty, \end{cases} \]
namely, condition (2). On the other hand, since \( K \) is compact, \( N_\epsilon \) and \( N \) are relatively compact in \( M \) with sufficiently small \( \epsilon > 0 \). Hence, \( N \) has finite volume and finite diameter, and the Poincaré inequality holds on \( N \). It follows that the embedding \( H_0^2(N; d\mu_g) \subset L^2(N; d\mu_g) \) is compact, that is, condition (3). We obtain the inclusion: \( \sigma_{\text{ess}}(\Delta) \subset \sigma_{\text{ess}}(\Delta') \).

This argumentation holds true if we replace \( g \) by \( g' \), and we arrive at the conclusion. \qed

3. **Examples**

In this section, we present examples of manifolds for which Theorem 1 can be applied.

**Example 1 (see [8]).** Let \((M, g)\) be an \( m \)-dimensional complete Riemannian manifold and let \( \Sigma \subset M \) be an \( n \)-dimensional compact manifold with \( m \geq n + 2 \). Assume that \( M \) has the product structure \( M^{m-n} \times M^n \) near \( \Sigma \) and that \( g \) can be diagonalized. Choose local coordinates in a neighborhood \( K \) of \( \Sigma \) so that
\[ g = g_1 \oplus g_2 \]
in $K$, where $g_1$ is a metric on $M^{m-n}$ and $g_2$ is a metric on $M^n$. Let $g'$ be another smooth metric on $M \setminus \Sigma$ so that

$$g' = \begin{cases} f^2 g_1 \oplus g_2, & \text{on } K, \\ g, & \text{on } M \setminus K. \end{cases}$$

If $m = 2$, assume that $f \in L^{2+\epsilon}(K; d\mu_g)$ for some $\epsilon \in (0, \infty)$. If $m = 3$, assume that $\inf(f) > 0$ and $f \in L^{(m(m-2)/2)+\epsilon}(K; d\mu_g)$ for some $\epsilon \in (0, \infty)$.

Then the manifold $M \setminus \Sigma$ with metrics $g$ and $g'$ satisfies the assumption of Theorem 1. In particular, if $M$ is compact, $\Delta'$ on $(M \setminus \Sigma, g')$ has discrete spectrum, which satisfies the Weyl asymptotic formula [8].

In the next example, the manifold has fractal singularity.

**Example 2.** Let $(M, g)$ be a complete Riemannian manifold with dimension greater than 2. Let $\Sigma \subset M$ be the Cantor set, $r$ the distance in $M$ from $\Sigma$, and $B(R)$ the $R$-neighborhood of $\Sigma$. Set

$$g' = f^2 g,$$

where

$$f(x) = \begin{cases} r^\epsilon, & x \in B = B(1), \\ 1, & x \in M \setminus B. \end{cases}$$

It is shown in [9] that $\Sigma$ is the almost polar Cauchy boundary of $(M \setminus \Sigma; g')$ if

$$\epsilon > \frac{\ln 2 - \ln 3}{2\ln 3 - \ln 2}.$$  

The compact inclusion

$$H^1_0(B \setminus \Sigma) \subset L^2(B \setminus \Sigma)$$

can be seen as follows. By definition, $H^1_0(B \setminus \Sigma) \subset H^1_0(B)$, and the inclusion $H^1_0(B) \subset L^2(B) = L^2(B \setminus \Sigma)$ is compact; thus, it suffices to show

$$H^1_0(B \setminus \Sigma) \subset H^1_0(B).$$

Let $u \in H^1_0(B)$. Since $L^\infty \cap H^1_0(B) \subset H^1_0(B)$ is dense, we may assume that $u \in L^\infty$ without loss of generality. Let $e_n$ be the equilibrium potential as in the proof of Theorem 1. Then $u_n = u(1 - e_n) \in H^1_0(B \setminus \Sigma)$ and

$$u_n \rightarrow u \text{ in } H^1_0(B; d\mu_g),$$

and hence $u \in H^1_0(B \setminus \Sigma)$. The function $\chi$ can be found as the relative equilibrium potential of $B(1)$ and $B(2)$ applied to the Friedrichs mollifier. Therefore, $M \setminus \Sigma$ together with $g$ and $g'$ satisfies the condition of Theorem 1. Let us point out the following:

- $(M \setminus \Sigma, g')$ is $C^{1,1}$ and is not smooth, but Theorem 1 can be applied to this setting.
- We can show the compact embedding $H^1_0(B \setminus \Sigma; d\mu_{g'}) \subset L^2(B \setminus \Sigma; d\mu_{g'})$ only for $\epsilon \geq 0$.

In the next example, $K$ has infinite volume with $g'$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
**Example 3.** Let $M$ be a 2-dimensional complete Riemannian manifold. Delete a point $p \in M$ and set 

$$g' = f^2 g,$$

where

$$f(x) = \begin{cases} r^{-\epsilon}, & x \in B = B(1), \\ 1, & x \in M \setminus B, \end{cases}$$

and $r$ is the distance from $p$. For any $\epsilon \geq 1$, $(M \setminus \{p\}, g')$ is complete and $\mu_{g'}(B \setminus \{p\}) = \infty$.

More generally, if $M$ is a complete manifold and $\Sigma \subset M$ is a compact set, then there is a smooth Riemannian metric $g'$ on $M \setminus \Sigma$ and a compact set $K \subset M$ such that $g = g'$ on $M \setminus K$, $(M \setminus \Sigma; g')$ is complete and there exist a function $\chi$ and a subset $N$ of $M \setminus \Sigma$ satisfying conditions (2) and (3), respectively.

**ACKNOWLEDGMENT**

The author would like to thank the referee for a careful reading and constructive discussions.

**REFERENCES**


**DEPARTMENT OF MATHEMATICAL SCIENCES, WORCESTER POLYTECHNIC INSTITUTE, 100 INSTITUTE ROAD, WORCESTER, MASSACHUSETTS 01609-2280**

Current address: Department of Mathematics and Statistics, Pennsylvania State University-Altoona, 3000 Iveyside Park, Altoona, Pennsylvania 16601

E-mail address: jum35@psu.edu