HANDLE NUMBER ONE LINKS
AND GENERALIZED PROPERTY R

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ABSTRACT. It is shown that if the exterior of a link L in the three sphere admits a genus 2 Heegaard splitting, then L has Generalized Property R.

1. INTRODUCTION

A knot K in the three sphere $\mathbb{S}^3$ is said to have **Property R** if K is the unknot, or 0-framed Dehn surgery on K does not yield $\mathbb{S}^1 \times \mathbb{S}^2$. Gabai [2] proved that every knot in $\mathbb{S}^3$ has Property R. A generalization to links is the following:

Generalized Property R Conjecture. *If Dehn surgery on an n-component framed link $L \subset \mathbb{S}^3$ yields $\#_n(\mathbb{S}^1 \times \mathbb{S}^2)$, then there is a sequence of handleslides which converts L to the n-component 0-framed unlink.*

Recall that the *handleslide* is one of the basic operations of Kirby calculus in which a framed link component gets replaced by its band connected sum with the framing of another framed component; see [4, Chapter 4] for more details. All the framings on a link L satisfying the hypothesis of Generalized Property R must be the 0-framing, and the components of L must be algebraically unlinked; this is due to classical homological arguments (see [3, Proposition 2.2]). We will say that a link L has **Generalized Property R** if L satisfies the the Generalized Property R Conjecture.

Only a few (easily describable) infinite families of links are known to have Generalized Property R. These include the tunnel number one links [3, Proposition 3.1] and the 2-component links which have at least one unknotted component [3, Proposition 3.2].

Our main result below is inspired by [3, Proof of Proposition 3.1], and resolves the Generalized Property R Conjecture for all links whose exteriors admit genus 2 Heegaard splittings.

Main Theorem. *Suppose that $L \subset \mathbb{S}^3$ is an n-component link $(n > 1)$ whose exterior admits a genus 2 Heegaard splitting. Then L has Generalized Property R. Furthermore, if 0-framed surgery on L yields $\#_n(\mathbb{S}^1 \times \mathbb{S}^2)$, then $n = 2$ and the components of L are unknots.*

Indeed, our main theorem generalizes Generalized Property R for *tunnel number one* links to *handle number one* links; see Section 2 for definitions. For example, any link consisting of a "couple of 2-bridge knots" (as defined in [7, Section 3.1]) has handle number one but not tunnel number one; such links also have linking number zero. So our main theorem implies that 0-framed surgery on a "couple of 2-bridge knots" can never yield $\#_2(\mathbb{S}^1 \times \mathbb{S}^2)$.

Part of the significance of the Generalized Property R Conjecture lies in the realm of smooth 4-dimensional handle structures. The closed 3-manifold M obtained by integral Dehn surgery on an n-component link L in $\mathbb{S}^3$ is the boundary of a compact, orientable, simply connected 4-manifold X obtained by attaching 2-handles to the standard smooth 4-ball B^4 along L with framings given by the Dehn surgery slopes; see [4, Chapters 4 and 5] for details. If $M = \#_n(\mathbb{S}^1 \times \mathbb{S}^2)$, then the $\mathbb{S}^2$ factors of the $\mathbb{S}^1 \times \mathbb{S}^2$ summands serve as attaching regions for attaching 3-handles to X; this yields a smooth homotopy 4-ball X' with $\partial X' = \mathbb{S}^3$. If L has Generalized Property R, then the 2-handles and 3-handles can be arranged to cancel in pairs, giving a standard handle structure on B^4. Hence X' is diffeomorphic to B^4, preventing X' from generating a counterexample to the Smooth Poincaré Conjecture in dimension 4. For more information, see [8, Section 2] and [1, Section 3].

Our notation for Dehn surgery will be the following. Given a link $L \subset \mathbb{S}^3$ with components $L_1, \ldots, L_n$ and corresponding framings $r_1, \ldots, r_n \in \mathbb{Z}$, let $L(r_1, \ldots, r_n)$ denote the 3-*manifold obtained by Dehn surgery on the framed link* L. If all the r_i are equal to a particular $r \in \mathbb{Z}$, we will refer to L as an r-framed link.

2. Handle number one links

A **system of unknotting tunnels** for a link L in $\mathbb{S}^3$ is a collection of disjoint properly embedded arcs $\gamma_1, \ldots, \gamma_k$ in the link exterior $E(L)$ with the property that drilling $\bigcup \gamma_i$ out of $E(L)$ yields a handlebody. The **tunnel number $t(L)$** for a link L is the least number required in a system of unknotting tunnels for L. A tunnel system for $E(L)$ induces a Heegaard splitting for $E(L)$ in which all of the boundary components of $E(L)$ lie on one side of the splitting surface. Only the unknot satisfies $t(L) = 0$.

Now suppose that L is a link in $\mathbb{S}^3$ with tunnel number one (i.e. $t(L) = 1$). Then $n \leq 2$, and we see that $E(L)$ is obtained by attaching a 2-handle to a genus 2 handlebody H along a simple closed curve $\alpha \subset \partial H$. Note that $n = 2$ if and only if α separates ∂H.

A **system of unknotting handles** for a link L in $\mathbb{S}^3$ is a collection of disjoint properly embedded arcs $\gamma_1, \ldots, \gamma_k$ in the exterior $E(L)$ with the property that drilling $\bigcup \gamma_i$ out of $E(L)$ yields a compression body; this clearly generalizes a system of unknotting tunnels. The **handle number $h(L)$**, following [9], for a link L is the least number required in a system of unknotting handles for L. We note that $h(L) \leq t(L)$. A handle system for $E(L)$ induces a Heegaard splitting for $E(L)$ in which the boundary components of $E(L)$ do not necessarily lie on one side of the splitting surface. Only the unknot and the Hopf link satisfy $h(L) = 0$.

Now suppose that L is a link in $\mathbb{S}^3$ with handle number one (i.e. $h(L) = 1$). Then $n \leq 4$, and we see that $E(L)$ is obtained by attaching a 2-handle to a compression body W along a simple closed curve $\alpha \subset \partial_+ W$ where $\partial_+ W$ has genus 2.

3. Proof of Main Theorem

The proof will involve fundamental group calculations. In order to keep our notation as simple as possible, we will adopt the following convention: if M is a connected 3-manifold with basepoint $p \in M$ and if $\gamma \subset M$ is a loop, then we will also let γ denote the homotopy class of γ in $\pi_1(M, p)$, possibly under an appropriate change-of-basepoint isomorphism when $p \notin \gamma$. If G is a group and $S \subset G$, let $\langle\langle S \rangle\rangle$ denote the **normal closure** of S in G.

Proof of Main Theorem. Let $L \subset \mathbb{S}^3$ be an n-component link $n > 1$ whose exterior $E(L)$ admits a genus 2 Heegaard splitting. Therefore, any Dehn surgery on L admits a genus 2 Heegaard splitting. We also observe that $n \leq 4$. By additivity of Heegaard genus under connected sums, we see that $\#_m(\mathbb{S}^1 \times \mathbb{S}^2)$ has Heegaard genus m for any $m \in \mathbb{N}$. By assumption, 0-framed Dehn surgery on L yields $\#_n(\mathbb{S}^1 \times \mathbb{S}^2)$, so $n = 2$.

Let L_1 and L_2 be the components of L with respective regular neighborhoods $N(L_1)$ and $N(L_2)$. So L has exterior $E(L) = \overline{\mathbb{S}^3 - (N(L_1) \cup N(L_2))}$. Let ∂_1 and ∂_2 be the boundary components of $E(L)$, so that ∂_i corresponds to L_i. Let m_i, l_i be a standard oriented meridian-longitude pair for ∂_i for $i = 1, 2$.

Now, if L has tunnel number one, then we are done by [3, Proposition 3.1]. So we may assume that there is a genus 2 Heegaard splitting of $E(L)$ which separates at least two boundary components. Hence the exterior of L can be realized as $E(L) = W \cup_\alpha$ (2-handle) where W is a compression body with $\partial_- W = \partial_1$, genus$(\partial_+ W){=}2$, and α is a nonseparating curve on $\partial_+ W$. Note that W is obtained by attaching a 1-handle to (torus) $\times [0, 1]$ on (torus) $\times \{1\}$. If we slide the endpoints of the core of the 1-handle together, we obtain a simple closed curve τ which meets (torus) $\times \{1\}$ in one point; we push τ toward $\partial_- W = $ (torus) $\times \{0\} = \partial_1$ to meet it in a point p. Give τ an arbitrary orientation and situate the curves m_1 and l_1 on ∂_1 so that $m_1 \cap l_1 = p$; see Figure 1. It is now clear that

$$\pi_1(W, p) \cong \langle m_1, l_1, \tau : m_1 l_1 = l_1 m_1 \rangle \cong (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} \ ,$$
$$\pi_1(E(L), p) \cong \pi_1(W, p)/\langle\langle \alpha \rangle\rangle \ , \text{ and}$$
$$\pi_1(L(0, 0), p) \cong \pi_1(E(L), p)/\langle\langle l_1, l_2 \rangle\rangle \cong \#_2(\mathbb{S}^1 \times \mathbb{S}^2) \cong \mathbb{Z} * \mathbb{Z} \ .$$

To simplify notation, the basepoint $p \in W \subset E(L) \subset L(0, 0)$ will be suppressed in fundamental group expressions for the remainder of the proof. We have epimorphisms

$$\pi_1(W) \xrightarrow{\ \phi\ } \pi_1(E(L)) \xrightarrow{\ \rho\ } \pi_1(L(0, 0)) \ ,$$

where ϕ is reduction mod $\langle\langle \alpha \rangle\rangle$ and ρ is reduction mod $\langle\langle l_1, l_2 \rangle\rangle$. Consider another sequence of epimorphisms,

$$\pi_1(W) \xrightarrow{\ \phi\ } \pi_1(E(L)) \xrightarrow{\ \rho_0\ } \pi_1(E(L))/\langle\langle l_1 \rangle\rangle \ ,$$

where ρ_0 is reduction mod $\langle\langle l_1 \rangle\rangle$. Note that the epimorphism ρ *factors through* ρ_0; that is, there is an epimorphism

$$\pi_1(E(L))/\langle\langle l_1 \rangle\rangle \xrightarrow{\ \rho_1\ } \pi_1(L(0, 0)) \ ,$$
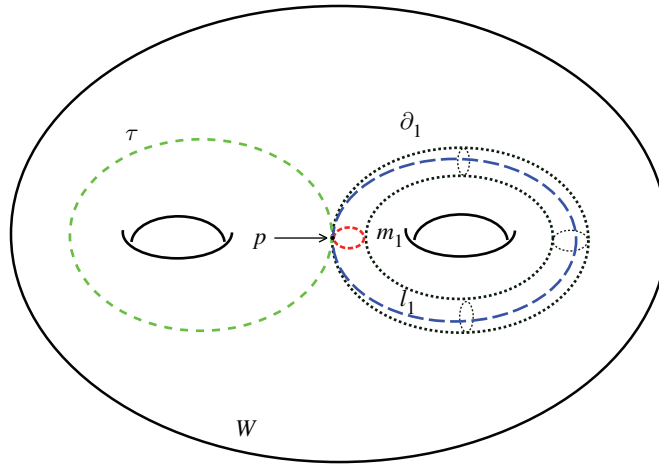
FIGURE 1. The compression body W with curves m_1 and l_1 on $\partial_- W = \partial_1$ and the "tunnel curve" τ. The curves m_1, l_1, and τ all meet the basepoint p. Note that $\tau \cup \partial_1$ forms a spine for W.

namely reduction mod $\langle\langle l_2 \rangle\rangle$, so that $\rho = \rho_1 \circ \rho_0$. Since $(\rho_0 \circ \phi)(l_1) = 1$, the epimorphism ϕ naturally descends to an epimorphism

$$\pi_1(W)/\langle\langle l_1 \rangle\rangle \xrightarrow{\ \phi'\ } \pi_1(E(L))/\langle\langle l_1 \rangle\rangle .$$

Note that $W \cup_{l_1} (2\text{-handle})$ is a punctured genus 2 handlebody, so $\pi_1(W)/\langle\langle l_1 \rangle\rangle \cong \mathbb{Z} * \mathbb{Z}$. In summary, we have a commutative diagram of epimorphisms:

$$
\begin{array}{ccccccc}
(\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} & \!\!=\!\!=\!\!=\!\! & \pi_1(W) & \xrightarrow{\ \phi\ } & \pi_1(E(L)) & \xrightarrow{\ \rho\ } & \pi_1(L(0,0)) \ . \\
& & \downarrow & & {\scriptstyle\rho_0}\downarrow & {\scriptstyle\rho_1}\nearrow & \Vert \\
\mathbb{Z} * \mathbb{Z} & \!\!=\!\!=\!\!=\!\! & \pi_1(W)/\langle\langle l_1 \rangle\rangle & \xrightarrow{\ \phi'\ } & \pi_1(E(L))/\langle\langle l_1 \rangle\rangle & & \mathbb{Z} * \mathbb{Z}
\end{array}
$$

Since the group $\mathbb{Z} * \mathbb{Z}$ is Hopfian (*every self-epimorphism is an isomorphism*), we must have all isomorphisms in the sequence

$$\mathbb{Z} * \mathbb{Z} =\!\!=\!\!= \pi_1(W)/\langle\langle l_1 \rangle\rangle \xrightarrow{\ \phi'\ } \pi_1(E(L))/\langle\langle l_1 \rangle\rangle \xrightarrow{\ \rho_1\ } \pi_1(L(0,0)) =\!\!=\!\!= \mathbb{Z} * \mathbb{Z} \ .$$

We now see that $\pi_1(E_1) \cong \pi_1(E(L))/\langle\langle l_1 \rangle\rangle \cong \mathbb{Z} * \mathbb{Z}$, where E_1 is the 3-manifold obtained from $E(L)$ by Dehn filling along $l_1 \subset \partial \mathrm{N}(L_1)$. By [5, Theorem 5.2], the Prime Decomposition Theorem, and the fact that genus 2 3-manifolds satisfy the Poincaré Conjecture (see [6]), we conclude that

$$E_1 \cong (\mathbb{S}^1 \times D^2) \# (\mathbb{S}^1 \times \mathbb{S}^2) \ .$$

Furthermore, Dehn filling on E_1 along $l_2 \subset \partial \mathrm{N}(L_2)$ gives $L(0,0) \cong (\mathbb{S}^1 \times \mathbb{S}^2) \# (\mathbb{S}^1 \times \mathbb{S}^2)$, so l_2 corresponds to a meridian curve in the $\mathbb{S}^1 \times D^2$ connected-summand of E_1. Note that this is just the topological realization of the epimorphisms ρ_0 and ρ_1. It follows that m_2 generates a free factor of $\pi_1(E_1)$. Hence $\pi_1(E_1)/\langle\langle m_2 \rangle\rangle \cong \mathbb{Z}$.

Let $L_1(0)$ denote the 3-manifold obtained by 0-framed Dehn surgery on the knot L_1. We see that

$$
\begin{aligned}
\pi_1(L_1(0)) &\cong \pi_1(E(L))/\langle\langle l_1, m_2 \rangle\rangle \\
&\cong \big(\pi_1(E(L))/\langle\langle l_1 \rangle\rangle\big)/\langle\langle m_2 \rangle\rangle \\
&\cong \pi_1(E_1)/\langle\langle m_2 \rangle\rangle \\
&\cong \mathbb{Z} \ .
\end{aligned}
$$

By [5, Theorem 5.2], we have that $L_1(0) \cong \mathbb{S}^1 \times \mathbb{S}^2$. Applying [2, Corollary 8.3] establishes that L_1 must be the unknot. We can similarly establish that L_2 is also the unknot; this is accomplished by simply interchanging the roles of L_1 and L_2 throughout the proof. Finally, the result [3, Proposition 3.2] asserts that any 2-component link containing an unknot has Generalized Property R. This completes the proof. $\qquad\square$

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