WEIGHTED PALEY–WIENER SPACES
SHARING A MAJORANT-WEIGHT

PHILIPPE POULIN

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Abstract. We point out a need to slightly modify the statement of Theorem 2 in Yurii Lyubarskii and Kristian Seip’s work Weighted Paley–Wiener spaces. This last theorem lists all weighted Paley–Wiener spaces (in reduced form) sharing a prescribed majorant-weight. Attention is called to a part of its proof that requires an additional argument. Such an argument, based on a new characterization of Beurling’s lower uniform density, is then presented.

1. Introduction

In the explorative paper [1], Lyubarskii and Seip introduced a family of de Branges spaces subject to a natural axiom: in these spaces, the norm of a function is comparable to its $L^2$-norm against $M(x)^{-2} \, dx$, where $M(x)$ denotes the norm of the reproducing kernel at $x \in \mathbb{R}$ (see Section 2). In such circumstances, $M(x)$ is said to be a majorant-weight, while the de Branges space is said to be a weighted Paley–Wiener space.

From a deep study of the Hermite–Biehler function associated with weighted Paley–Wiener spaces (Theorem 1 in [1]), they showed that the majorant-weight of a weighted Paley–Wiener space is always comparable to a function of the form $e^{g(z)} e^{-\omega_m(z)}$, where $g(z)$ is real-entire (that is, real on the real line and entire), $m(x)$ is comparable to a constant, and $\omega_m(z)$ is the potential of $m(x) \, dx$ (see Section 2).

Furthermore, they proved that any weighted Paley–Wiener space is of the form $e^{g} PW(m)$ for such a $g$ and an $m$, where

$$PW(m) = \{ f \text{ entire} ; \| f e^{-\omega_m} \|_2 < \infty, \| f(z) e^{-\omega_m(z)} \|_2 < \infty, \| f(z) e^{-\omega_m(z)} \|_2 \leq C e^{|z|} \}.$$  

They then aimed to list all weighted Paley–Wiener spaces of the form $PW(\cdot)$ whose majorant-weight is comparable to a prescribed $e^{\omega_m(z)}$. They obtained such a list (Theorem 2 in [1]), which consists exactly of the following spaces,

$$PW_{-b}(m) = \{ f \text{ entire} ; \| f e^{-\omega_m} \|_2 < \infty, \| f(z) e^{-\omega_m(z)} + \pi b |z| \|_2 \leq C e^{|z|} \},$$

where $b$ is any real number inferior to the lower uniform density of $m$. This last is defined as

$$D_m = \lim_{R \to \infty} \inf_{x \in \mathbb{R}} \frac{1}{2R} \int_{-R}^{R} m(x+t) \, dt.$$
However, as we shall discuss later, a close examination of their proof reveals its incompleteness, and new ideas seem necessary for completing their work.

The present paper aims to remedy the situation. Section 2 provides the reader with the main definitions used in the sequel. In Section 3.1, we shall question the original argument that $D_m$ is a majorant of $b$ and then provide our own proof, based on a new characterization of the lower uniform density. Finally, in Section 3.2, we shall show that $D_m$ is the least majorant of $b$.

2. Definitions

In the sequel, given two nonnegative functions $f$ and $g$, $f \lesssim g$ indicates that $f \leq Cg$ for a positive constant $C$, and $f \simeq g$ indicates that $f$ is comparable to $g$ (that is, $f \lesssim g$ and $g \lesssim f$).

A Hilbert space $\mathcal{H}$ of entire functions is a de Branges space \cite{2} if it satisfies the following axioms:

1. The linear functional $\mathcal{H} \to \mathbb{C}$, $f \mapsto f(z_0)$ is bounded for all $z_0 \in \mathbb{C}$.
2. If $f(z) \in \mathcal{H}$, then $f^*(z) = \overline{f(z)}$ also belongs to $\mathcal{H}$ and has the same norm as $f(z)$.
3. If $f(z) \in \mathcal{H}$ and $f(z_0) = 0$, then $f(z)\frac{z - z_0}{z - z_0}$ also belongs to $\mathcal{H}$ and has the same norm as $f(z)$.

By the first axiom, $\mathcal{H}$ admits a reproducing kernel, that is, a function $k_w(z)$ of the variables $w, z \in \mathbb{C}$ such that $k_w(z) \in \mathcal{H}$ for all $w \in \mathbb{C}$ and

$$\langle f, k_w \rangle_{\mathcal{H}} = f(w) \quad \text{for all } f \in \mathcal{H}.$$ 

The majorant of $\mathcal{H}$ at $z \in \mathbb{C}$ is then defined as

$$M(z) = \|k_z\|_{\mathcal{H}} = \sup_{\|f\|_{\mathcal{H}} = 1} |f(z)|.$$ 

Let $M(x)$ be the restriction of $M$ to the real axis. Following Lyubarskii and Seip, we shall say that $M(x)$ is a majorant-weight if

1. $M(x) > 0$ for all $x \in \mathbb{R}$;
2. $\|f\|_{\mathcal{H}} \simeq \|f/M\|_2$ for all $f \in \mathcal{H}$.

Then, the corresponding $\mathcal{H}$ is called a weighted Paley–Wiener space.

A Hermite–Biehler function $E$ is an entire function satisfying $|E(z)| > |E(\mathbb{R})|$ for all $z \in \mathbb{C}^+$. Such a function may be factorized as

$$(2.1) \quad E(z) = Cz^me^{h(z)e^{-i\alpha z}} \prod_{\gamma \in \Gamma} (1 - z/\gamma)e^{\zeta_R(1/\gamma)},$$

where $C \in \mathbb{C}$, $h(z)$ is real-entire, $\alpha \geq 0$, and $\Gamma$ is a family of nonzero elements lying in the closed lower half-plane (with possible repetitions). Conversely, given such a $C$, $h(z)$, $\alpha$, and $\Gamma$, if the right-hand side in (2.1) defines an entire function, then it is in the Hermite–Biehler class (provided that $\Gamma \not\subseteq \mathbb{R}$ or $\alpha \neq 0$).

In the case where $E$ does not have real zeroes, its restriction to the real axis may be written

$$E(x) = |E(x)|e^{-i\varphi(x)},$$
where the phase, \( \varphi(x) \), is real-analytic and well-defined (up to the addition of \( 2k\pi \)). The factorization (2.1) then implies
\[
\varphi'(x) = \alpha + \sum_{\xi-i\eta \in \Gamma} \frac{\eta}{(x-\xi)^2 + \eta^2}.
\]

From an arbitrary Hermite–Biehler function \( E \), one may build a prototypical example of a de Branges space, namely
\[
\mathcal{H}(E) = \left\{ f \text{ entire} : \|f/E\|_2 < \infty, \ |f^2(z)/E(z)| \leq C e^{\epsilon|z|} \text{ for } \exists \ z \geq 0 \right\},
\]
for \( f^2 \) running over \( \{f, f^*\} \). It is equipped with the norm \( \|f\|_{\mathcal{H}(E)} = \|f/E\|_2 \). In fact, a theorem of de Branges (Theorem 23 in [2]) shows that every de Branges space is isometrically equal to a space of the form \( \mathcal{H}(E) \), where \( E \) is not unique in general.

The reproducing kernel in \( \mathcal{H}(E) \) is given by
\[
k_w(z) = \frac{E^*(z)E^+(w) - E(z)E^+(w)}{2\pi i (z - w)}.
\]
In particular, if \( E \) does not have real zeroes,
\[
M(x) = \sqrt{k_x(x)} = \frac{1}{\sqrt{\pi}} \sqrt{\varphi'(x)|E(x)|}
\]
for all \( x \in \mathbb{R} \).

**Example 2.1.** Let \( \varphi \) be the phase of a Hermite–Biehler function \( E \) without real zero. If \( \varphi'(x) \approx 1 \), then \( \mathcal{H}(E) \) is obviously a weighted PW-space. The converse statement however does not hold in general (Remark 3 on the last page of [1]).

Lyubarskii and Seip made a bridge between Hermite–Biehler functions and a certain kind of potentials, namely, potentials of measures of the form \( m(x) \) dx for \( m(x) \) measurable, positive, and \( \approx 1 \). Such a potential cannot be defined as \( \int_{-\infty}^{\infty} \log |1 - z/t|m(t) \ dt \). In fact, this last integral does not exist due to the dominating term in the expansion
\[
\log |1 - z/t| = -x/t - \sum_{n=2}^{\infty} (1/n) \Re(z^n)/t^n
\]
for \( |t| \) large, where \( z = x + iy \). It suggests defining
\[
\omega_m(z) = \int_{-\infty}^{\infty} \log^* |1 - z/t|m(t) \ dt
\]
where \( \log^* |1 - z/t| = \log |1 - z/t| + \chi(t)x/t, \ \chi(t) = 1 - \chi_{[-1,1]}(t) \).

We first show that \( \omega_m(z) \) is well-defined, indeed, that the above integral is absolutely convergent. The previous expansion gives, for \( |t| \) large,
\[
|\log^* |1 - z/t| | \leq \sum_{n=2}^{\infty} (1/n)|z/t|^n
\]
\[
\leq |z/t|^2(1/2 - \log(1 - |z/t|) \).
\]
Since \( m(x) \approx 1 \), it suffices to show that \( \int_R^{\infty} -1/(t^2) \log(1 - |z/t|) \ dt < \infty \) for \( R \) large. This last relation follows from the substitution \( u = 1 - |z/t| \). Therefore,
\[
\int_{-\infty}^{\infty} \log^* |1 - z/t| \ dt < \infty.
\]
The inequality (2.3) and the dominated convergence theorem also yield that \( \omega_m \) is continuous. The dominated convergence theorem then implies that for \( z \notin \mathbb{R} \)

\[
\partial_y \omega_m(z) = \pi P_m(z),
\]

(2.4)

where

\[
P_m(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} m(t) \, dt
\]

is the Poisson transform of \( m(x) \) dx. Similarly,

\[
\partial_x \omega_m(z) = \int_{-\infty}^{\infty} \left( \frac{x-t}{(x-t)^2 + y^2} + \frac{\chi(t)}{t} \right) m(t) \, dt
\]

for \( z \notin \mathbb{R} \). Finally, a straight adaptation of the classical argument (Theorem 3.7.4 in [3]) gives \( \Delta \omega_m = 2\pi m(x) \, dx \, d\delta_0(y) \) in the sense of distribution, where \( \delta_0 \) denotes the 1-dimensional Dirac measure at 0.

**Example 2.2.** For \( m(x) = 1 \) and \( z \notin \mathbb{R} \), \( \partial_y \omega_1(z) = \pi \text{sgn}(y) \), while \( \partial_x \omega_1(z) = 0 \). Hence, \( \omega_1(z) = \pi |y| + C \). By continuity, this last relation applies for all \( z \in \mathbb{C} \). Since \( \omega_1(0) = 0 \), we deduce \( \omega_1(z) = \pi |y| \).

The aforementioned link between Hermite–Biehler functions and potentials of the form \( \omega_m \) is given by the following **multiplier lemma**:

**Proposition 2.3.** Let \( m(x) \simeq 1 \) be a measurable function. There exists a Hermite–Biehler function \( E_m \) which satisfies

\[
|E_m(z)| \simeq e^{\omega_m(z)} \quad \text{when} \quad \exists z \geq 0
\]

and whose zeroes are simple and of the form \( \xi_k - i \), where \( \xi_{k+1} - \xi_k \simeq 1 \), \( \xi_k \in \mathbb{R} \).

Let \( \varphi \) be the phase of \( E_m \). By (2.2),

\[
\varphi'(x) = \sum_k \frac{1}{(x-\xi_k)^2 + 1}.
\]

The condition \( \xi_{k+1} - \xi_k \simeq 1 \) then implies \( \varphi'(x) \simeq 1 \). Therefore \( \mathcal{H}(E_m) \) is a weighted PW-space. By the multiplier lemma, \( |E_m| \) may be replaced with \( e^{\omega_m} \) in the definition of \( \mathcal{H}(E_m) \). Since \( \omega_m(\infty) = \omega_m(z) \), it follows that \( \mathcal{H}(E_m) \) is equal with equivalent norms to the space

\[
PW(m) = \{ f \text{ entire} : \| fe^{-\omega_m} \|_2 < \infty, | f(z) | e^{-\omega_m(z)} < C e^{c|z|} \text{ for } z \in \mathbb{C} \},
\]

equipped with the norm \( \| f \|_{PW(m)} = \| fe^{-\omega_m} \|_2 \).

In particular, for any measurable \( m(x) \simeq 1 \), \( PW(m) \) is a weighted PW-space whose majorant-weight is comparable to \( |E_m| \simeq e^{\omega_m(z)} \). Consequently, given a real-entire \( g \), \( e^g PW(m) \) is also a weighted PW-space; its majorant-weight is comparable to \( e^g e^{\omega_m} \). The converse statement constitutes the remarkable achievement in [1]: Lyubarskii and Seip proved that all weighted PW-spaces have a representation \( e^g PW(m) \), where \( g \) is real-entire and \( m \simeq 1 \) is measurable.

**Example 2.4.** We have seen that \( \omega_1(z) = \pi |y| \). Consequently, \( PW(1) \) is the classical Paley–Wiener space, \( L_x^2 \).
3. Spaces sharing a given majorant-weight

In [1], Lyubarskii and Seip investigated the following question: which weighted PW-spaces share a prescribed majorant-weight? They gave special attention to spaces of the form $e^{az}PW(m)$, $a \in \mathbb{R}$, which we will call linearly reduced PW-spaces. They obtained an answer for these last spaces, involving the following object: for $m$ measurable and $1 \leq \tau \in \mathbb{R}$,

$$PW_{\tau}(m) = \{f \text{ entire}; \|fe^{-\omega_m}\|_{2} < \infty, |f(z)|e^{-\omega_m(z)} \leq C_{e}e^{\tau|z|}e^{\pi|\Im z|}\}. $$

Notice that $PW_{\tau}(m) = PW(m + \tau)$ if $m + \tau \simeq 1$, but this last relation is not assumed.

**Proposition 3.1.** Suppose $e^{ax}e^{\omega_{m_{0}}(x)} \simeq e^{ax}e^{\omega_{m}(x)}$ on the real axis, where $a, a_{0}$ are in $\mathbb{R}$ and $m, m_{0}$ are measurable and $\simeq 1$. Then, there exists a real number $b$ such that

$$e^{ax}e^{\omega_{m_{0}}(z)} \simeq e^{ax}e^{\omega_{m}(z)}e^{-\pi b|y|}$$

on the whole complex plane, where $z = x + iy$.

**Proof.** By hypothesis $|\omega_{m_{0}}(x) + (a_{0} - a)x|$ is bounded, and hence there exists a $C > 0$ such that

$$|\omega_{m_{0}}(x) + (a_{0} - a)x| \leq |\omega_{m_{0}}(x) - \omega_{m_{0}}(x)(x) + |\omega_{m_{0}}(x)(x) + (a_{0} - a)x| \leq C(|y| + 1).$$

In particular, $C_{y} + \omega_{m_{0}}(x) + (a_{0} - a)x$ is bounded below on the upper half-plane and hence admits a Poisson representation

$$Dy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\omega_{m_{0}}(t) + (a_{0} - a)t}{(x - t)^{2} + y^{2}} dt \quad (y > 0).$$

Observe that the last term in the previous expression is bounded. Consequently, letting $b = (C - D)/\pi$,

$$|\pi by + \omega_{m_{0}}(z) + (a_{0} - a)x| \lessgtr 1 \quad (\Im z \geq 0),$$

that is, $e^{ax}e^{\omega_{m_{0}}(z)} \simeq e^{ax}e^{\omega_{m}(z)}e^{-\pi b|\Im z|}$ when $\Im z \geq 0$. The result follows.

**Corollary 3.2.** Each linearly reduced weighted PW-space whose majorant-weight is comparable to $e^{ax}e^{\omega_{m}(x)}$ is of the form $e^{az}PW_{-b}(m)$ for a certain $b \in \mathbb{R}$.

**Remark 3.3.** In their original paper Lyubarskii and Seip stated their result not in terms of linearly reduced PW-spaces, but in terms of linearly reduced majorant-weights (that is, weights of the form $e^{ax}e^{\omega_{m}(x)}$). Their statement is too general: clearly, $\exp(\exp(-z^{2}))PW(1)$ is a weighted PW-space whose majorant-weight is $\simeq 1$, but it is not equal to $PW_{-b}(1) = L^{2}_{\pi(1-b)}$ for any $b$.

Linearly reduced PW-spaces of majorant $e^{ax}e^{\omega_{m}}$ ($a \in \mathbb{R}$) thus take the form $e^{az}PW_{-b}(m)$, but for which possible $b$? Clearly each $b < \inf_{x \in \mathbb{R}} m(x)$ is possible, since then $PW_{-b}(m) = PW(m - b)$ with $m - b \simeq 1$. However, this majoration cannot be optimal in general, since it is easy to change the infimum of $m$ without altering $e^{\omega_{m}}$ (using for instance Proposition 3.3). Lyubarskii and Seip stated that the optimal majoration is $b < D_{m}$, where

$$D_{m} = \lim_{R \to \infty} \inf_{z \in \mathbb{R}} \frac{1}{2R} \int_{-R}^{R} m(x + t) dt$$
is the uniform lower density of \( m \). Notice that this last limit exists by Fekete’s lemma, since \( \inf_x \int_{-R}^{R} m(x+t) \, dt = \inf_x \int_0^{2R} m(x+t) \, dt \) is superadditive.

### 3.1. The lower density is a majorant.

For showing that \( b < D_m \) is necessary, Lyubarskii and Seip argued by contradiction: they assumed that given an \( \varepsilon > 0 \), for all sufficiently large \( R \) there exists an \( x_R \) satisfying

\[
(3.1) \quad \int_{-R}^{R} (m(x_R + s) - b) \, ds \leq \varepsilon R.
\]

For convenience they set \( x_R = 0 \). Let \( e^{az} PW(m_0) \) be a representation of \( e^{az} PW_{-z}(m) \) as a weighted Paley–Wiener space. By an elegant argument based on Green’s formula they proved

\[
\left| \int_{-R}^{R} (m_0(t) - m(t) + b)(R^2 - t) \, dt \right| \lesssim R^2.
\]

They claimed however that (3.1) would imply

\[
2 \int_0^R t \int_{-t}^t (m_0(s) - m(s) + b) \, ds \, dt \geq (\inf m_0) \frac{4}{3} R^3 - \varepsilon R^3,
\]

a contradiction. Unfortunately the use of the estimate

\[
2 \int_0^R t \int_{-t}^t (m(s) - b) \, ds \, dt \leq \varepsilon R^3
\]

is not explicitly justified. A conscientious reader may get puzzled: one cannot for instance restrict the domain of integration to large \( t \) and invoke (3.1) with \( t \) instead of \( R \), since \( x_t \neq x_R \).

We prefer to present another proof, based on a new characterization of the uniform lower density:

**Proposition 3.4.** If \( m \simeq 1 \) is measurable, then

\[
D_m = \lim_{R \to \infty} \inf_{x \in \mathbb{R}} \frac{1}{R} \int_{-R}^R \frac{1}{2} \int_{-\rho}^\rho m(x+t) \, dt \, d\rho.
\]

**Proof.** For \( R > 0 \) and \( x \in \mathbb{R} \), let us write

\[
m_R(x) = \frac{1}{2R} \int_{-R}^R m(x+t) \, dt \quad \text{and} \quad A_R(x) = \frac{1}{R} \int_0^R m_\rho(x) \, d\rho.
\]

In this notation we want to prove \( D_m = \lim_{R \to \infty} \inf_x A_R(x) \), given that \( D_m = \lim_{R \to \infty} \inf_x m_R(x) \).

Given \( \varepsilon > 0 \), let \( R_0 \) be such that \( |\inf_x m_\rho(x) - D_m| < \varepsilon \) when \( \rho \geq R_0 \). Then, for \( R \) large

\[
\inf_x A_R(x) \geq \frac{1}{R} \left( \int_0^{R_0} + \int_{R_0}^R \right) \inf_x m_\rho(x) \, d\rho \geq \frac{R_0}{R} \inf_m m + \frac{R - R_0}{R} (D_m - \varepsilon).
\]

Letting \( R \to \infty \) along an appropriate sequence, we conclude

\[
\lim \inf_{R \to \infty} \inf_x A_R(x) \geq D_m.
\]
Let us derive the converse inequality. Let $0 < \varepsilon < 1/4$ be given, and define $\eta = \varepsilon / \log(1/2\varepsilon)$. There exists an $R_0$ depending on $\varepsilon$ such that
\[
\inf_x m_\rho(x) > D_m - \eta \quad \text{whenever} \quad \rho \geq R_0.
\]
Let $R \geq R_0$ be arbitrarily fixed, and let $S = R/\varepsilon$. By the previous relation there exists an $x^*$ such that $m_S(x^*) < D_m + 2\eta$. Consequently, for $\rho < S - 2R$
\[
\frac{S - \rho}{2S} m_{\frac{S - \rho}{2}}(x^*) - \frac{S + \rho}{2} m_\rho(x^*) + \frac{S - \rho}{2S} m_{\frac{S - \rho}{2}}(x^*) + \frac{S + \rho}{2} < D_m + 2\eta
\]
(since the left-hand side is equal to $m_S(x^*)$), while
\[
m_{\frac{S - \rho}{2}}(x^*) \geq m_{\frac{S + \rho}{2}}(x) > D_m - \eta.
\]
It follows that $m_\rho(x^*) < D_m + \frac{3\eta S}{\rho}$ for such $\rho$. Therefore, for $\rho \in [R, S - 2R]$
\[
A_\rho(x^*) = \frac{1}{\rho} \left( \int_R^\rho + \int_0^\rho \right) m_\rho(x^*) \, dr \leq \frac{R}{\rho} \sup m + \frac{3\eta S}{\rho} \log \left( \frac{\rho}{R} \right) + D_m.
\]
Letting $\rho = S/2 = R/2\varepsilon$, the definition of $\eta$ yields
\[
\inf_{x \in \mathbb{R}} A_{R/2\varepsilon}(x) \leq A_{R/2\varepsilon}(x^*) \leq D_m + C\varepsilon,
\]
where $C = 2 \sup m + 6$. This last relation holds for all $R \geq R_0$. In other words
\[
\sup_{R \geq R_0/2 \varepsilon} \inf_{x \in \mathbb{R}} A_R(x) \leq D_m + C\varepsilon.
\]
Therefore, $\limsup_{R \to \infty} \inf_{x \in \mathbb{R}} A_R(x) \leq D_m + C\varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, we conclude
\[
\limsup_{R \to \infty} \inf_{x \in \mathbb{R}} A_R(x) \leq D_m,
\]
as desired. \qed

The following identity is also useful:

**Lemma 3.5.** For $m$ measurable and bounded and $R > 0$,
\[
\frac{1}{\pi} \int_0^\pi \omega_m(Re^{i\theta}) \, d\theta = \int_0^R \frac{1}{\rho} \int_{-\rho}^\rho m(t) \, dt \, d\rho.
\]

**Proof.** Fubini’s theorem and the relation $\int_0^\pi \cos \theta \, d\theta = 0$ give
\[
\frac{1}{\pi} \int_0^\pi \omega_m(Re^{i\theta}) \, d\theta = \frac{1}{\pi} \int_{-\infty}^\infty m(t) \int_0^\pi \log \left( 1 - \frac{Re^{i\theta}}{t} \right) \, d\theta \, dt.
\]
Moreover, Jensen’s formula gives
\[
\frac{1}{\pi} \int_0^\pi \log \left( 1 - \frac{Re^{i\theta}}{t} \right) \, d\theta = \frac{1}{2\pi} \int_{-\pi}^\pi \log \left| 1 - \frac{Re^{i\theta}}{t} \right| \, d\theta = \chi_{[-R,R]}(t) \log(R/|t|).
\]
Therefore,
\[
\frac{1}{\pi} \int_0^\pi \omega_m(Re^{i\theta}) \, d\theta = \int_{-R}^R \log(R/|t|) m(t) \, dt = \int_0^R \frac{1}{\rho} \int_{-\rho}^\rho m(t) \, dt \, d\rho,
\]
by Fubini’s theorem again. \qed

**Proposition 3.6.** In Corollary 3.2 $b < D_m$. 


Proof. Let $e^{a_0 z} \text{PW}(m_0) = e^{a_0 z} \text{PW}_-X(m)$ be the linearly reduced space in question, and suppose by contradiction that $b \geq D_m$. By our characterization of $D_m$, for any large $R$ there exists an $x_R$ such that

$$
\frac{1}{R} \int_0^R \frac{1}{2\rho} \int_{-\rho}^\rho (m(x_R + t) - b) \, dt \, d\rho \leq \frac{\inf m_0}{2}.
$$

In particular, for $R$ large

$$
\frac{1}{R} \int_0^R \frac{1}{2\rho} \int_{-\rho}^\rho (m_0(x_R + t) - m(x_R + t) + b) \, dt \, d\rho \geq \frac{\inf m_0}{2}.
$$

The last lemma then implies

$$
\frac{1}{2\pi R} \int_0^\pi (\omega_{m_0(x_R+)}(Re^{i\theta}) - \omega_{m_0(x_R+)}(Re^{i\theta}) + \pi b R \sin \theta) \, d\theta \geq \frac{\inf m_0}{2}.
$$

Notice that in general

$$
\omega_{M(X+)}(z) = \omega_{M}(z + X) - \omega_{M}(X) + \Re z \int_{-\infty}^\infty \left( \frac{\chi(t - X)}{t - X} - \frac{\chi(t)}{t} \right) M(t) \, dt,
$$

and hence

$$
\int_0^\pi \omega_{M(X+)}(Re^{i\theta}) \, d\theta = \int_0^\pi (\omega_{M}(Re^{i\theta} + X) - \omega_{M}(X)) \, d\theta.
$$

Therefore,

$$
\frac{1}{2\pi R} \int_0^\pi (\omega_{m_0-m}(Re^{i\theta} + x_R) + \pi b R \sin \theta - \omega_{m_0-m}(x_R)) \, d\theta \geq \frac{\inf m_0}{2}.
$$

However, Proposition 3.1 implies that

$$
|\omega_{m_0-m}(Re^{i\theta} + x_R) + \pi b R \sin \theta + (a_0 - a)(R \cos \theta + x_R)| \leq 1,
$$

while $|\omega_{m_0-m}(x_R) + (a_0 - a)x_R| \leq 1$. Since $\int_0^\pi R \cos \theta \, d\theta = 0$, the integral in the relation (3.2) is bounded, a contradiction.

3.2. The lower density is the least majorant. For showing that any $b < D_m$ is suitable, Lyubarskii and Seip replaced $m$ with a smoothing of $m$ of the form

$$
m_R(x) = \frac{1}{2R} \int_{-R}^R m(x + t) \, dt.
$$

They justified this replacement by the relation $|\omega_m(z) - \omega_{m_R}(z)| \leq 1$, which is essentially right (after addition of a linear term $\alpha x$). In fact, $\omega_{m_R}(z) - \omega_{m}(z)$ is equal to

$$
\frac{1}{2R} \int_{-R}^R \left( \int_{-\infty}^\infty \log^+ |1 - z/t|m(t + s) \, dt - \int_{-\infty}^\infty \log^+ |1 - z/t|m(t) \, dt \right) \, ds
$$

$$
= \frac{1}{2R} \int_{-R}^R \int_{-\infty}^\infty \left( \log \left| 1 - \frac{z}{t - s} \right| + \frac{\chi(t - s)}{t - s} x - \log \left| 1 - \frac{z}{t} \right| - \frac{\chi(t)}{t} x \right) m(t) \, dt \, ds
$$

$$
= F(z) + \alpha x,
$$
where

\[ F(z) = \frac{1}{2R} \int_{-R}^{R} \int_{-\infty}^{\infty} \log \left| \frac{1 - z/(t-s)}{|1 - z/t|} \right| m(t) \, dt \, ds \quad \text{and} \quad \alpha = \frac{1}{2R} \int_{-R}^{R} \int_{-\infty}^{\infty} \left( \frac{\chi(t-s)}{t-s} - \frac{\chi(t)}{t} \right) m(t) \, dt \, ds. \]

Notice that both \( \alpha \) and \( \omega_{m_R-m}(z) \) are absolutely convergent, forcing \( F(z) \) to be such.

Let us prove that \( F(z) \) is bounded. In fact, \( \int_{-\infty}^{\infty} \log \left| \frac{1 - x/(t-s)}{|1 - x/t|} \right| m(t) \, dt \) is equal to

\[ \omega_{m(x+)}(s) - \omega_{m}(s) + s \int_{-\infty}^{\infty} \left( \frac{\chi(t)}{t} - \frac{\chi(t-x)}{t-x} \right) m(t) \, dt. \]

Clearly \( |\omega_{m(x+)}(s)| \lesssim \int_{-\infty}^{\infty} |\log^* \left| 1 - (s/t) \right| | \, dt < \infty \) uniformly in \( s \in [-R,R] \), and similarly for \( |\omega_{m}(s)| \). Moreover, the last term in (3.3) disappears when averaging over \( s \in [-R,R] \). Hence, \( |F(z)| \lesssim 1 \).

In total, \( e^{\omega_{m}(z)} \) is comparable to \( e^{\omega_{m_R}(z)} e^{-\alpha x} \), and hence \( PW_{-b}(m) \) equals \( e^{-\alpha z} PW(m_R - b) \), which is a weighted PW-space for all \( b < \inf_x m_R(x) \), eventually for all \( b < D_m \).

Joining this result with Corollary 3.2 and Proposition 3.6 we have completed the proof of the following theorem:

**Theorem 3.7.** Let \( m \simeq 1 \) be measurable and \( a \in \mathbb{R} \). The family of linearly reduced weighted Paley–Wiener spaces whose majorant-weight is comparable to \( e^{\alpha z} e^{\omega_{m}(x)} \) consists of all \( e^{\alpha z} PW_{-b}(m) \) with \( b < D_m \).

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**References**


Department of Mathematical Sciences, United Arab Emirates University, P.O. Box 17551, Al Ain, Abu Dhabi, United Arab Emirates

E-mail address: PhilippePoulin@uaeu.ac.ae