SIMULTANEOUS EXTENSIONS
OF TURKEVICH’S INEQUALITY
AND THE WEIGHTED AM-GM INEQUALITY

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ABSTRACT. We establish a sharp homogeneous inequality which extends both the classical weighted AM-GM inequality and the Turkevich inequality.

1. Introduction and main results

Turkevich [1] discovered a neat 4-variable symmetric inequality of degree 4:

\[ a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 \]

or

\[ (a^2 - b^2)^2 + (c^2 - d^2)^2 \geq (a^2 + b^2)(c^2 + d^2) - (ab + cd)^2 \]

for all non-negative real numbers \(a, b, c, d\). Equality occurs if and only if either \(a = b = c = d\) or if three of \(a, b, c, d\) are equal and the remaining one is zero.

Several generalizations of Turkevich’s inequality are known; for example, Shleifer’s inequality [1] says that, for \(a_1, \ldots, a_n \geq 0\),

\[ (n-1) \sum_{i=1}^{n} a_i^4 + n (a_1 \cdots a_n)^{\frac{4}{n}} \geq \left( \sum_{i=1}^{n} a_i^2 \right)^2. \]

The main aim of this paper is to present a sharp weighted generalization of the AM-GM inequality, which also generalizes Turkevich’s inequality.

In the following, let \(n\) be a positive integer with \(n \geq 2\) and let \(\omega_1, \ldots, \omega_n\) be positive real numbers with \(\omega_1 + \cdots + \omega_n = 1\). Define \(\omega = \min\{\omega_1, \ldots, \omega_n\} > 0\) and denote \(\lambda = (1 - \omega)^{-\frac{1}{n-1}} > 1\).

We now present our two main theorems, which will turn out to be equivalent.
**Theorem 1.** Let \(a_1, \ldots, a_n, b_1, \ldots, b_n\) be non-negative real numbers \((n \geq 2)\) and let \(\omega_1, \ldots, \omega_n\) be positive weights with \(\omega_1 + \cdots + \omega_n = 1\). We have

\[
\lambda \sum_{k=1}^{n} \omega_k (a_k^2 - b_k^2)^2 + \left(2 \sum_{k=1}^{n} \omega_k a_k b_k \right)^2 \geq (a_1^2 + b_1^2)^{2\omega_1} \cdots (a_n^2 + b_n^2)^{2\omega_n}.
\]

Equality in \((1.1)\) occurs if and only if we have either \(a_1 = \cdots = a_n = b_1 = \cdots = b_n\), or if we have

\[
|a_k^2 - b_k^2| = \begin{cases} a & \text{if } k = i_0 \\ 0 & \text{if } k \neq i_0 \end{cases} \quad \text{and} \quad 2a_k b_k = \begin{cases} 0 & \text{if } k = i_0 \\ b & \text{if } k \neq i_0 \end{cases}
\]

for some integer \(i_0 \in \{1, \ldots, n\}\) with \(\omega_{i_0} = \omega\) and for some \(a, b \geq 0\) for which \(\lambda a^2 = b^2(1 - \omega)\).

The existence of the equality condition guarantees the minimality of the optimal coefficient \(\lambda\) in inequality \((1.1)\). Theorem 1 is an \(n\)-variable generalization of Turkevich’s inequality \([1]\); the original inequality of Turkevich can be obtained by letting \(n = 2\) and \(\omega_1 = \omega_2 = \frac{1}{2}\), in which case \(\lambda = 2\).

To establish Theorem 1, we will use the following theorem, which is a non-symmetric equivalent to Theorem 1.

**Theorem 2.** Let \(a_1, \ldots, a_n, b_1, \ldots, b_n\) be non-negative real numbers \((n \geq 2)\) and let \(\omega_1, \ldots, \omega_n\) be positive weights with \(\omega_1 + \cdots + \omega_n = 1\). Then we have

\[
\sum_{k=1}^{n} \omega_k a_k^2 + \left(\sum_{k=1}^{n} \omega_k b_k \right)^2 \geq (a_1^2 + b_1^2)^{\omega_1} \cdots (a_n^2 + b_n^2)^{\omega_n}.
\]

Equality in \((1.2)\) occurs if and only if we either have \(a_1 = \cdots = a_n = 0\) and \(b_1 = \cdots = b_n\) or we have

\[
a_k = \begin{cases} a & \text{if } k = i_0 \\ 0 & \text{if } k \neq i_0 \end{cases} \quad \text{and} \quad b_k = \begin{cases} 0 & \text{if } k = i_0 \\ b & \text{if } k \neq i_0 \end{cases}
\]

for some integer \(i_0 \in \{1, \ldots, n\}\) with \(\omega_{i_0} = \omega\) and for some \(a, b \geq 0\) for which \(\lambda a^2 = b^2(1 - \omega)\).

Inequality \((1.2)\) is clearly a generalization of the weighted AM-GM inequality, as can be seen by substituting \(a_1 = \cdots = a_n = 0\). That it is a strict generalization, can be seen from the additional equality conditions, where \(a_1 = \cdots = a_n = 0\) does not necessarily hold.

Several specific estimations on the optimal coefficient \(\lambda\) in Theorems 1 and 2 can be made. First, as the following proposition shows, both inequalities \((1.1)\) and \((1.2)\) still hold when replacing \(\lambda\) with Euler’s constant \(\epsilon\).

**Proposition 3.** Let \(n \geq 2\). We have \(\epsilon > \lambda\) for any positive weights \(\omega_1, \ldots, \omega_n\) with \(\omega_1 + \cdots + \omega_n = 1\).

Second, the following proposition indicates that the resulting inequalities are still sharp, in the sense that \(\epsilon\) cannot be replaced by a smaller constant.

**Proposition 4.** Let \(n \geq 2\). Suppose that \(C\) is a positive real constant for which

\[
C \sum_{k=1}^{n} \omega_k a_k^2 + \left(\sum_{k=1}^{n} \omega_k b_k \right)^2 \geq (a_1^2 + b_1^2)^{\omega_1} \cdots (a_n^2 + b_n^2)^{\omega_n}
\]
holds for all positive weights \(\omega_1, \ldots, \omega_n\) with \(\omega_1 + \cdots + \omega_n = 1\) and for all non-negative real numbers \(a_1, \ldots, a_n, b_1, \ldots, b_n\). Then \(C \geq e\).

If \(\omega_1 = \ldots = \omega_n = \frac{1}{n}\), we have \(\lambda = \left(1 + \frac{1}{n-1}\right)^{n-1}\). This gives our inequalities simple forms for the uniform weight distribution \(\omega_1 = \ldots = \omega_n = \frac{1}{n}\), and it is sharper than replacing \(\lambda = \left(1 + \frac{1}{n-1}\right)^{n-1}\) by Euler’s constant \(e\).

Theorems 1 and 2 are the main theorems of this paper. In Section 2, we present a proof of our main theorems, as well as a proof for the propositions above.

2. PROOF OF THE MAIN THEOREMS AND THE PROPOSITIONS

In this section we give the proof of our main theorems. First we introduce a useful notation and we present an observation on the minimal optimal coefficient \(\lambda\). Given a proper subset \(I\) of \(\{1, \ldots, n\}\), we denote

\[
\lambda_I = \left(\sum_{i \in I} \omega_i\right)^{-\frac{\sum_{i \in I} \omega_i}{\sum_{i \in I} \omega_i}} = f\left(\sum_{i \in I} \omega_i\right),
\]

where we define \(f(x) = (1 - x)^{-1/x}\). We then recall the definitions in Section 1:

\[
\omega = \min\{\omega_1, \ldots, \omega_n\} > 0 \quad \text{and} \quad \lambda = f(\omega) = (1 - \omega)^{-\frac{1}{\omega}} > 1.
\]

Since the function \(f\) is decreasing on \([0, 1]\), we have that \(\lambda_I \leq \lambda\) for each non-empty proper subset \(I \subset \{1, \ldots, n\}\). In particular, because the function \(f\) is decreasing,

\[
\lambda = \max\{\lambda_I \mid I \text{ is a non-empty proper subset of } \{1, \ldots, n\}\}
\]

and this maximum is attained when \(\sum_{i \in I} \omega_i\) is minimal, i.e. when \(I = \{i_0\}\), where \(i_0\) is any index for which \(\omega_{i_0} = \omega\). This maximality of the minimal optimal coefficient \(\lambda = f(\omega)\) is crucial to the proof of Theorem 2. We start by proving Theorem 2.

**Proof of Theorem 2.** Let \(p_i = \sqrt{a_i^2 + b_i^2}\) for all integers \(i\), with \(1 \leq i \leq n\). If there is any integer \(i\), with \(1 \leq i \leq n\), for which \(p_i = 0\), then the right hand side equals 0 and the inequality holds trivially. In this case equality occurs if and only if \(a_1 = \ldots = a_n = b_1 = \ldots = b_n = 0\).

Hence we may assume that \(p_i > 0\) for all integers \(i\), \(1 \leq i \leq n\). We can rewrite the claimed estimation as

\[
\lambda \left(\sum_{k=1}^{n} \omega_k (p_k^2 - b_k^2) + \left(\sum_{k=1}^{n} \omega_k b_k\right)^2\right) \geq p_1^{2\omega_1} \cdots p_n^{2\omega_n}.
\]

If we now fix the variables \(p_1, \ldots, p_n, b_1, \ldots, b_{i-1}\) and \(b_{i+1}, \ldots, b_n\), for some integer \(i\), with \(1 \leq i \leq n\), then we find that the right hand side is a constant, while the left hand side is a quadratic function of \(b_i\) with leading coefficient \(\omega_i(\omega_i - \lambda)\). Since \(\lambda > 1 > \omega_i > 0\), this leading coefficient is negative; thus the left hand side is a concave function in the variable \(b_i\). Therefore, the smallest value of the left hand side is attained either when \(b_i = 0\) or \(b_i = p_i\). Since this holds for any integer \(i\), with \(1 \leq i \leq n\), we may assume that \(b_i \in \{0, p_i\}\) for each integer \(i\), with \(1 \leq i \leq n\).

Let \(m\) be the number of integers \(i\), with \(1 \leq i \leq n\), for which \(b_i = 0\). We may permute the indices such that \(b_1 = b_2 = \ldots = b_m = 0\) and \(b_{m+1} = p_{m+1} > 0, \ldots, b_n = p_n > 0\); we denote this permutation by \(\sigma\). With these observations, it is
sufficient to prove the following inequality for arbitrary positive weights \(\omega_1, \ldots, \omega_n\) with \(\omega_1 + \cdots + \omega_n = 1\) and arbitrary positive reals \(p_1, \ldots, p_n\):

\[
\lambda \sum_{k=1}^{n} \omega_k p_k^2 + \left( \sum_{k=m+1}^{n} \omega_k p_k \right)^2 \geq p_1^{2\omega_1} \cdots p_n^{2\omega_n}.
\]

(2.1)

Now there are three cases: either \(m = 0, m = n,\) or \(1 \leq m \leq n - 1\). If \(m = 0\), then (2.1) is simply the AM-GM inequality for \(p_1, \ldots, p_n\). Equality hence occurs if and only if \(p_1 = \cdots = p_n\), which in the original problem can be written as \(a_1 = \cdots = a_n = 0\) and \(b_1 = \cdots = b_n\).

If \(m = n\), then

\[
\lambda \sum_{k=1}^{n} \omega_k p_k^2 > \sum_{k=1}^{n} \omega_k p_k^2 \geq p_1^{2\omega_1} \cdots p_n^{2\omega_n},
\]

by the AM-GM inequality for \(p_1^2, \ldots, p_n^2\). Equality cannot be attained in this case.

Hence, we are left with the case \(1 \leq m \leq n - 1\). Define

\[
U = \omega_1 + \cdots + \omega_m, \quad V = \omega_{m+1} + \cdots + \omega_n,
\]

\[
A = (p_1^{\omega_1} \cdots p_m^{\omega_m})^{1/U} \quad \text{and} \quad B = (p_{m+1}^{\omega_{m+1}} \cdots p_n^{\omega_n})^{1/V}.
\]

Applying the weighted AM-GM inequality twice to the left hand side then yields

\[
\lambda \sum_{k=1}^{n} \omega_k p_k^2 + \left( \sum_{k=m+1}^{n} \omega_k p_k \right)^2 \geq \lambda \cdot U A^2 + (V B)^2.
\]

On the other hand, using the same notation, the right hand side of (2.1) can be written as \(p_1^{2\omega_1} \cdots p_n^{2\omega_n} = A^2 U B^2 V\), and hence we are left to prove that

\[
\lambda \cdot U A^2 + (V B)^2 \geq A^2 U B^2 V.
\]

Now, let \(I = \{\sigma^{-1}(1), \ldots, \sigma^{-1}(m)\}\) in the original definition of \(\lambda_I\). Then at this point in the proof (after rearranging our indices) we have \(\sigma(I) = \{1, 2, \ldots, m\}\). Hence, \(\lambda_{\sigma(I)} = (1 - U)^{-\frac{1}{2U}} = f(U)\). Then, the maximality of \(\lambda = f(\omega)\) implies

\[
\lambda \geq \lambda_{\sigma(I)} = (1 - U)^{-\frac{1}{2U}} = \left( \frac{1}{V} \right)^{\frac{U}{V}}.
\]

Finally, we can combine this with the weighted AM-GM inequality to deduce

\[
\lambda \cdot U A^2 + (V B)^2 \geq \left( \frac{U}{V} \right)^{\frac{U}{V}} \cdot U A^2 + (V B)^2 = U \cdot \left( \frac{A^2}{V^{\frac{U}{V}}} \right) + V \cdot (V B^2) \geq \left( \frac{A^2}{V^{\frac{U}{V}}} \right) \cdot (V B^2)^V = A^2 V B^2 V
\]

as claimed. This proves inequality (1.2).

Equality in the above occurs only if \(\lambda = \lambda_{\sigma(I)} = \left( \frac{1}{V} \right)^{\frac{U}{V}}\) and \(\lambda A^2 = V B^2\). Filling in the definitions of \(U\) and \(V\), we see that \(\lambda = \lambda_{\sigma(I)}\) implies that \(\sigma(I) = \{i_0\}\) with \(\omega_{i_0} = \omega\). Hence, this is exactly the claimed equality condition; this proves the ‘only if’ part. For the ‘if’ part, let \(I = \{i_0\}\) and let \(a, b\) be non-negative real numbers satisfying the given conditions. Denoting \(u = \sum_{k \notin I} \omega_k = 1 - \omega\), we have \(\lambda = \lambda_I = v^{-v/u}\) and we have to show that
\[ v^{\frac{v}{u}}u a^2 + v^2 b^2 = a^2 v b^2 v, \] which is equivalent to \( u \left( \frac{a^2}{v^{v/u}} \right) + v (v b^2) = a^2 v b^2 v. \) Since we are given that \( \lambda_I a^2 = b^2 \sum_{k \in I} \omega_k, \) we know that \( \frac{a^2}{v^{v/u}} = b^2 v, \) yielding

\[ u \left( \frac{a^2}{v^{v/u}} \right) + v (v b^2) = v b^2 = (v b^2)^u. \]

Hence the statement about the equality condition follows. \( \square \)

We have proven Theorem 2. Theorem 1 is now a straightforward corollary.

Proof of Theorem 1. For each integer \( i, \) with \( 1 \leq i \leq n, \) we substitute \((a_i, b_i)\) by \((|a_i^2 - b_i^2|, 2a_i b_i)\) in inequality (1.2). Then inequality (1.2) in Theorem 2 reduces to inequality (1.1) in Theorem 1. \( \square \)

Now we prove the propositions from Section 1.

Proof of Proposition 3. We use the inequality \( e^t > 1 + t \) for \( t > 0 \) to deduce

\[ \lambda = (1 - \omega)^{-\frac{1-\omega}{1-\omega}} = \left( \frac{1}{1 - \omega} \right)^{\frac{1-\omega}{1-\omega}} = (1 + \frac{\omega}{1-\omega})^{\frac{1-\omega}{1-\omega}} < (e^{\frac{1}{1-\omega}})^{\frac{1-\omega}{1-\omega}} = e, \]

as claimed. \( \square \)

Proof of Proposition 4. Substituting \( \omega_1 = \ldots = \omega_n = \frac{1}{n}, \) \( b_1 = a_2 = \ldots = a_n = 0, \) \( a_1 = (1 - \frac{1}{n})^\frac{1}{2} \) and \( b_2 = \ldots = b_n = 1 \) in inequality (1.3) yields

\[ C \left( 1 - \frac{1}{n} \right)^n + \left( n - 1 \right) n \geq 1 - \frac{1}{n}, \]

or equivalently,

\[ C \geq \left( 1 + \frac{1}{n-1} \right)^{n-1}. \]

Taking the limit for \( n \to +\infty, \) we meet the desired estimation \( C \geq e. \) \( \square \)

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References