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SIMULTANEOUS EXTENSIONS OF TURKEVICH'S INEQUALITY AND THE WEIGHTED AM-GM INEQUALITY

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Abstract. We establish a sharp homogeneous inequality which extends both the classical weighted AM-GM inequality and the Turkevich inequality.

1. Introduction and main results

Turkevich [1] discovered a neat 4-variable symmetric inequality of degree 4:

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$$

or

$$\left(a^2 - b^2\right)^2 + \left(c^2 - d^2\right)^2 \geq \left(a^2 + b^2\right)\left(c^2 + d^2\right) - (ab + cd)^2$$

for all non-negative real numbers a, b, c, d. Equality occurs if and only if either $a = b = c = d$ or if three of a, b, c, d are equal and the remaining one is zero.

Several generalizations of Turkevich's inequality are known; for example, Shleifer's inequality [1] says that, for $a_1, \ldots, a_n \geq 0$,

$$(n-1)\sum_{i=1}^{n} a_i^4 + n\left(a_1 \cdots a_n\right)^{\frac{4}{n}} \geq \left(\sum_{i=1}^{n} a_n^2\right)^2.$$

The main aim of this paper is to present a sharp weighted generalization of the AM-GM inequality, which also generalizes Turkevich's inequality.

In the following, let n be a positive integer with $n \geq 2$ and let $\omega_1, \ldots, \omega_n$ be positive real numbers with $\omega_1 + \cdots + \omega_n = 1$. Define $\omega = \min\{\omega_1, \ldots, \omega_n\} > 0$ and denote $\lambda = (1-\omega)^{-\frac{1-\omega}{\omega}} > 1$.

We now present our two main theorems, which will turn out to be equivalent.

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Theorem 1. *Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be non-negative real numbers ($n \geq 2$) and let $\omega_1, \ldots, \omega_n$ be positive weights with $\omega_1 + \cdots + \omega_n = 1$. We have*

$$\lambda \sum_{k=1}^{n} \omega_k (a_k^2 - b_k^2)^2 + \left(2 \sum_{k=1}^{n} \omega_k a_k b_k\right)^2 \geq \left(a_1^2 + b_1^2\right)^{2\omega_1} \cdots \left(a_n^2 + b_n^2\right)^{2\omega_n}. \tag{1.1}$$

Equality in (1.1) occurs if and only if we have either $a_1 = \cdots = a_n = b_1 = \cdots = b_n$, or if we have

$$|a_k^2 - b_k^2| = \begin{cases} a & \text{if } k = i_0 \\ 0 & \text{if } k \neq i_0 \end{cases} \text{ and } 2a_k b_k = \begin{cases} 0 & \text{if } k = i_0 \\ b & \text{if } k \neq i_0 \end{cases}$$

for some integer $i_0 \in \{1, \ldots, n\}$ with $\omega_{i_0} = \omega$ and for some $a, b \geq 0$ for which $\lambda a^2 = b^2(1 - \omega)$.

The existence of the equality condition guarantees the minimality of the optimal coefficient λ in inequality (1.1). Theorem 1 is an n-variable generalization of Turkevich's inequality [1]; the original inequality of Turkevich can be obtained by letting $n = 2$ and $\omega_1 = \omega_2 = \frac{1}{2}$, in which case $\lambda = 2$.

To establish Theorem 1, we will use the following theorem, which is a non-symmetric equivalent to Theorem 1.

Theorem 2. *Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be non-negative real numbers ($n \geq 2$) and let $\omega_1, \ldots, \omega_n$ be positive weights with $\omega_1 + \cdots + \omega_n = 1$. Then we have*

$$\lambda \sum_{k=1}^{n} \omega_k a_k^2 + \left(\sum_{k=1}^{n} \omega_k b_k\right)^2 \geq \left(a_1^2 + b_1^2\right)^{\omega_1} \cdots \left(a_n^2 + b_n^2\right)^{\omega_n}. \tag{1.2}$$

Equality in (1.2) occurs if and only if we either have $a_1 = \cdots = a_n = 0$ and $b_1 = \cdots = b_n$ or we have

$$a_k = \begin{cases} a & \text{if } k = i_0 \\ 0 & \text{if } k \neq i_0 \end{cases} \text{ and } b_k = \begin{cases} 0 & \text{if } k = i_0 \\ b & \text{if } k \neq i_0 \end{cases}$$

for some integer $i_0 \in \{1, \ldots, n\}$ with $\omega_{i_0} = \omega$ and for some $a, b \geq 0$ for which $\lambda a^2 = b^2(1 - \omega)$.

Inequality (1.2) is clearly a generalization of the weighted AM-GM inequality, as can be seen by substituting $a_1 = \ldots = a_n = 0$. That it is a strict generalization, can be seen from the additional equality conditions, where $a_1 = \ldots = a_n = 0$ does not necessarily hold.

Several specific estimations on the optimal coefficient λ in Theorems 1 and 2 can be made. First, as the following proposition shows, both inequalities (1.1) and (1.2) still hold when replacing λ with Euler's constant e.

Proposition 3. Let $n \geq 2$. We have $e > \lambda$ for any positive weights $\omega_1, \ldots, \omega_n$ with $\omega_1 + \cdots + \omega_n = 1$.

Second, the following proposition indicates that the resulting inequalities are still sharp, in the sense that e cannot be replaced by a smaller constant.

Proposition 4. Let $n \geq 2$. Suppose that $\mathcal{C}$ is a positive real constant for which

$$\mathcal{C} \sum_{k=1}^{n} \omega_k a_k^2 + \left(\sum_{k=1}^{n} \omega_k b_k\right)^2 \geq \left(a_1^2 + b_1^2\right)^{\omega_1} \cdots \left(a_n^2 + b_n^2\right)^{\omega_n} \tag{1.3}$$

holds for all positive weights $\omega_1, \ldots, \omega_n$ with $\omega_1 + \cdots + \omega_n = 1$ and for all non-negative real numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$. Then $\mathcal{C} \geq e$.

If $\omega_1 = \ldots = \omega_n = \frac{1}{n}$, we have $\lambda = \left(1 + \frac{1}{n-1}\right)^{n-1}$. This gives our inequalities simple forms for the uniform weight distribution $\omega_1 = \ldots = \omega_n = \frac{1}{n}$, and it is sharper than replacing $\lambda = \left(1 + \frac{1}{n-1}\right)^{n-1}$ by Euler's constant e.

Theorems 1 and 2 are the main theorems of this paper. In Section 2, we present a proof of our main theorems, as well as a proof for the propositions above.

2. Proof of the main theorems and the propositions

In this section we give the proof of our main theorems. First we introduce a useful notation and we present an observation on the minimal optimal coefficient λ. Given a proper subset I of $\{1, \ldots, n\}$, we denote

$$\lambda_I = \left(\sum_{i \notin I} \omega_i\right)^{-\frac{\sum_{i \notin I} \omega_i}{\sum_{i \in I} \omega_i}} = f\left(\sum_{i \in I} \omega_i\right),$$

where we define $f(x) = (1-x)^{-\frac{1-x}{x}}$. We then recall the definitions in Section 1:

$$\omega = \min\{\omega_1, \ldots, \omega_n\} > 0 \ \text{ and } \ \lambda = f(\omega) = (1-\omega)^{-\frac{1-\omega}{\omega}} > 1.$$

Since the function f is decreasing on $]0, 1[$, we have that $\lambda_I \leq \lambda$ for each non-empty proper subset $I \subset \{1, \ldots, n\}$. In particular, because the function f is decreasing,

$$\lambda = \max\{\lambda_I \mid I \text{ is a non-empty proper subset of } \{1, \ldots, n\}\}$$

and this maximum is attained when $\sum_{i \in I} \omega_i$ is minimal, i.e. when $I = \{i_0\}$, where i_0 is any index for which $\omega_{i_0} = \omega$. This *maximality* of the *minimal* optimal coefficient $\lambda = f(\omega)$ is crucial to the proof of Theorem 2. We start by proving Theorem 2.

Proof of Theorem 2. Let $p_i = \sqrt{a_i^2 + b_i^2}$ for all integers i, with $1 \leq i \leq n$. If there is any integer i, with $1 \leq i \leq n$, for which $p_i = 0$, then the right hand side equals 0 and the inequality holds trivially. In this case equality occurs if and only if $a_1 = \ldots = a_n = b_1 = \ldots = b_n = 0$.

Hence we may assume that $p_i > 0$ for all integers i, $1 \leq i \leq n$. We can rewrite the claimed estimation as

$$\lambda \sum_{k=1}^{n} \omega_k (p_k^2 - b_k^2) + \left(\sum_{k=1}^{n} \omega_k b_k\right)^2 \geq p_1^{2\omega_1} \cdots p_n^{2\omega_n}.$$

If we now fix the variables $p_1, \ldots, p_n$, $b_1, \ldots, b_{i-1}$ and $b_{i+1}, \ldots, b_n$, for some integer i, with $1 \leq i \leq n$, then we find that the right hand side is a constant, while the left hand side is a quadratic function of b_i with leading coefficient $\omega_i(\omega_i - \lambda)$. Since $\lambda > 1 > \omega_i > 0$, this leading coefficient is negative; thus the left hand side is a concave function in the variable b_i. Therefore, the smallest value of the left hand side is attained either when $b_i = 0$ or $b_i = p_i$. Since this holds for any integer i, with $1 \leq i \leq n$, we may assume that $b_i \in \{0, p_i\}$ for each integer i, with $1 \leq i \leq n$.

Let m be the number of integers i, with $1 \leq i \leq n$, for which $b_i = 0$. We may permute the indices such that $b_1 = b_2 = \ldots = b_m = 0$ and $b_{m+1} = p_{m+1} > 0, \ldots, b_n = p_n > 0$; we denote this permutation by σ. With these observations, it is

sufficient to prove the following inequality for arbitrary positive weights $\omega_1, \ldots, \omega_n$ with $\omega_1 + \cdots + \omega_n = 1$ and arbitrary positive reals $p_1, \ldots, p_n$:

$$\lambda \sum_{k=1}^{m} \omega_k p_k^2 + \left(\sum_{k=m+1}^{n} \omega_k p_k \right)^2 \geq p_1^{2\omega_1} \cdots p_n^{2\omega_n}. \tag{2.1}$$

Now there are three cases: either $m = 0$, $m = n$, or $1 \leq m \leq n-1$. If $m = 0$, then (2.1) is simply the AM-GM inequality for $p_1, \ldots, p_n$. Equality hence occurs if and only if $p_1 = \ldots = p_n$, which in the original problem can be written as $a_1 = \ldots = a_n = 0$ and $b_1 = \ldots = b_n$.

If $m = n$, then

$$\lambda \sum_{k=1}^{n} \omega_k p_k^2 > \sum_{k=1}^{n} \omega_k p_k^2 \geq p_1^{2\omega_1} \cdots p_n^{2\omega_n},$$

by the AM-GM inequality for $p_1^2, \ldots, p_n^2$. Equality cannot be attained in this case.

Hence, we are left with the case $1 \leq m \leq n-1$. Define

$$U = \omega_1 + \cdots + \omega_m, \quad V = \omega_{m+1} + \cdots + \omega_n,$$
$$A = (p_1^{\omega_1} \cdots p_m^{\omega_m})^{1/U} \text{ and } B = \left(p_{m+1}^{\omega_{m+1}} \cdots p_n^{\omega_n}\right)^{1/V}.$$

Applying the weighted AM-GM inequality twice to the left hand side then yields

$$\lambda \sum_{k=1}^{m} \omega_k p_k^2 + \left(\sum_{k=m+1}^{n} \omega_k p_k \right)^2 \geq \lambda \cdot UA^2 + (VB)^2.$$

On the other hand, using the same notation, the right hand side of (2.1) can be written as $p_1^{2\omega_1} \cdots p_n^{2\omega_n} = A^{2U} B^{2V}$, and hence we are left to prove that

$$\lambda \cdot UA^2 + \left(VB\right)^2 \geq A^{2U} B^{2V}.$$

Now, let $I = \{\sigma^{-1}(1), \ldots, \sigma^{-1}(m)\}$ in the original definition of λ_I. Then at this point in the proof (after rearranging our indices) we have $\sigma(I) = \{1, 2, \ldots, m\}$. Hence, $\lambda_{\sigma(I)} = (1-U)^{-\frac{1-U}{U}} = f(U)$. Then, the maximality of $\lambda = f(\omega)$ implies

$$\lambda \geq \lambda_{\sigma(I)} = (1-U)^{-\frac{1-U}{U}} = \left(\frac{1}{V}\right)^{\frac{V}{U}}.$$

Finally, we can combine this with the weighted AM-GM inequality to deduce

$$\begin{aligned} \lambda \cdot UA^2 + (VB)^2 &\geq \left(\tfrac{1}{V}\right)^{V/U} \cdot UA^2 + (VB)^2 \\ &= U \cdot \left(\tfrac{A^2}{V^{V/U}}\right) + V \cdot (VB^2) \\ &\geq \left(\tfrac{A^2}{V^{V/U}}\right)^U \cdot (VB^2)^V \\ &= A^{2U} B^{2V} \end{aligned}$$

as claimed. This proves inequality (1.2).

Equality in the above occurs only if $\lambda = \lambda_{\sigma(I)} = \left(\frac{1}{V}\right)^{\frac{V}{U}}$ and $\lambda A^2 = VB^2$. Filling in the definitions of U and V, we see that $\lambda = \lambda_{\sigma(I)}$ implies that $\sigma(I) = \{i_0\}$ with $\omega_{i_0} = \omega$. Hence, this is exactly the claimed equality condition; this proves the 'only if' part. For the 'if' part, let $I = \{i_0\}$ and let a, b be non-negative real numbers satisfying the given conditions. Denoting $u = \sum_{k \in I} \omega_k = \omega$ and $v = 1 - u = \sum_{k \notin I} \omega_k = 1 - \omega$, we have $\lambda = \lambda_I = v^{-v/u}$ and we have to show that

$v^{-v/u}ua^2 + v^2b^2 = a^{2u}b^{2v}$, which is equivalent to $u\left(\frac{a^2}{v^{v/u}}\right) + v(vb^2) = a^{2u}b^{2v}$. Since we are given that $\lambda_I a^2 = b^2 \sum_{k\notin I} \omega_k$, we know that $\frac{a^2}{v^{v/u}} = b^2 v$, yielding

$$u\left(\frac{a^2}{v^{v/u}}\right) + v(vb^2) = vb^2 = (vb^2)^u \cdot (vb^2)^v = \left(\frac{a^2}{v^{v/u}}\right)^u \cdot (vb^2)^v = a^{2u}b^{2v}.$$

Hence the statement about the equality condition follows. □

We have proven Theorem 2. Theorem 1 is now a straightforward corollary.

Proof of Theorem 1. For each integer i, with $1 \le i \le n$, we substitute (a_i, b_i) by $(|a_i^2 - b_i^2|, 2a_ib_i)$ in inequality (1.2). Then inequality (1.2) in Theorem 2 reduces to inequality (1.1) in Theorem 1. □

Now we prove the propositions from Section 1.

Proof of Proposition 3. We use the inequality $e^t > 1 + t$ for $t > 0$ to deduce

$$\lambda = (1-\omega)^{-\frac{1-\omega}{\omega}} = \left(\frac{1}{1-\omega}\right)^{\frac{1-\omega}{\omega}} = \left(1 + \frac{\omega}{1-\omega}\right)^{\frac{1-\omega}{\omega}} < \left(e^{\frac{\omega}{1-\omega}}\right)^{\frac{1-\omega}{\omega}} = e,$$

as claimed. □

Proof of Proposition 4. Substituting $\omega_1 = \ldots = \omega_n = \frac{1}{n}$, $b_1 = a_2 = \ldots = a_n = 0$, $a_1 = \left(1 - \frac{1}{n}\right)^{\frac{n}{2}}$ and $b_2 = \ldots = b_n = 1$ in inequality (1.3) yields

$$\mathcal{C}\left(1 - \frac{1}{n}\right)^n + \left(\frac{n-1}{n}\right)^2 \ge 1 - \frac{1}{n},$$

or equivalently,

$$\mathcal{C} \ge \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

Taking the limit for $n \to +\infty$, we meet the desired estimation $\mathcal{C} \ge e$. □

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