AN ALMOST SCHUR THEOREM ON 4-DIMENSIONAL MANIFOLDS

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Abstract. In this short paper we prove that the almost Schur theorem, introduced by De Lellis and Topping, is true on 4-dimensional Riemannian manifolds of nonnegative scalar curvature and discuss some related problems on other dimensional manifolds.

1. Introduction

Very recently, De Lellis and Topping proved an interesting result about a generalization of Schur’s theorem.

Theorem 1 (Almost Schur Theorem [1]). For \( n \geq 3 \), if \((M^n, g)\) is a closed Riemannian manifold with nonnegative Ricci tensor, then

\[
\int_M |\text{Ric} - \frac{R}{n}g|^2 dv(g) \leq \frac{n^2}{(n-2)^2} \int_M |\text{Ric} - \frac{R}{n}g|^2 dv(g),
\]

where \( \overline{R} = \text{vol}(g)^{-1} \int_M R dv(g) \) is the average of the scalar curvature \( R \) of \( g \).

It is clear that the Schur theorem follows directly from Theorem 1. The latter can be seen as a quantitative version or a stability result of the Schur Theorem. In [1] they also showed that the constant in inequality (1) is optimal and the nonnegativity of the Ricci tensor cannot be removed in general: When \( n \geq 5 \) they gave examples of metrics on \( S^n \) which make the ratio of the left-hand side of (1) to the right-hand side of (1) arbitrarily large. When \( n = 3 \), they found manifolds which make the ratio arbitrary. At the end they left an open question: Inequalities of this form may hold for \( n = 3 \) and \( n = 4 \) with constants depending on the topology of \( M \).

In this short paper we will show that Theorem 1 holds under the condition of nonnegativity of the scalar curvature for dimension \( n = 4 \).

Theorem 2. If \( n = 4 \) and if \((M^4, g)\) is a closed Riemannian manifold with nonnegative scalar curvature, then (1) holds. Moreover, equality holds if and only if \((M^4, g)\) is an Einstein manifold.

We first observe that inequality (1) is equivalent to

\[
\left( \int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} \text{vol}(g) \int_M \sigma_2(g) dv(g),
\]

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where \( \sigma_k(g) \) is the \( k \)-scalar curvature of metric \( g \). Its definition will be recalled in Section 2. Then we prove this inequality for \( n = 4 \) by using an argument given by Gursky \[3\].

2. Proof of Theorem \[2\]

Let us first recall the definition of the \( k \)-scalar curvature, which was first introduced by Viaclovsky \[4\] and has been intensively studied by many mathematicians. Let

\[
S_g = \frac{1}{n-2} \left( Ric_g - \frac{R_g}{2(n-1)} \cdot g \right)
\]

be the Schouten tensor of \( g \). For an integer \( k \) with \( 1 \leq k \leq n \) let \( \sigma_k \) be the \( k \)-th elementary symmetric function in \( \mathbb{R}^n \). The \( k \)-scalar curvature is

\[
\sigma_k(g) := \sigma_k(\Lambda_g),
\]

where \( \Lambda_g \) is the set of eigenvalues of the matrix \( g^{-1} \cdot S_g \). In particular, \( \sigma_1(g) = \text{tr} S \) and \( \sigma_2 = \frac{1}{2}((\text{tr} S)^2 - |S|^2) \). It is trivial to see that

\[
\sigma_1(g) = \frac{R}{2(n-1)},
\]

\[
\sigma_2(g) = \frac{1}{2(n-2)^2} \left\{ -|Ric|^2 + \frac{n}{4(n-1)} R^2 \right\},
\]

\[
\left| Ric - \frac{R}{n} g \right|^2 = |Ric|^2 - \frac{R^2}{n}.
\]

From the above it is easy to have the following observation.

**Observation 1.** Inequality (1) is equivalent to (2).

Hence, instead of proving Theorem \[2\] we actually prove

**Theorem 3.** If \( n = 4 \) and if \((M^n, g)\) is a closed Riemannian manifold with non-negative scalar curvature, then (2) holds. Moreover, equality holds if and only if \((M, g)\) is an Einstein metric.

The proof of Theorem \[3\] follows closely a nice argument of Gursky \[3\].

**Lemma 1.** For any \( n \geq 3 \) and any closed Riemannian manifold \((M^n, g)\), there exists a conformal metric \( g_1 \in [g] \) satisfying

\[
\frac{2n}{n-1} \int_M \sigma_2(g_1) dv(g_1) \leq Y_1([g])^2,
\]

where \( Y_1([g]) \) is the first Yamabe invariant defined by

\[
Y_1([g]) := \inf_{g \in [g]} \int_M \sigma_1(g) dv(g)
\]

and \([g]\) is the conformal class of the metric to \( g \).

Here our definition of the Yamabe constant is different from the standard one by a multiple factor \( \frac{1}{2(n-1)} \).
Proof of Lemma 1. The proof follows closely an argument given by Gursky in [3]. Let $g_1$ be a solution of the Yamabe problem. Thus the scalar curvature, and hence $\sigma_1(g_1)$, is constant. We have a simple fact: for any $n \times n$ symmetric matrix $A$ such that

$$(\sigma_1(A))^2 \geq \frac{2n}{n-1} \sigma_2(A),$$

equality holds if and only if the matrix is a multiple of the identity matrix. Now the following calculations lead to

$$(5) \quad \frac{2n}{n-1} \text{vol}(g_1) \int_M \sigma_2(g_1) dv(g_1) \leq \text{vol}(g_1) \int_M (\sigma_1(g_1))^2 dv(g_1) = \left( \int_M \sigma_1(g_1) dv(g_1) \right)^2.$$ 

Here we have used the fact that $\sigma_1(g_1)$ is a constant. Therefore,

$$\frac{2n}{n-1} \int_M \frac{\sigma_2(g_1) dv(g_1)}{\text{vol}(g_1)^{\frac{n-4}{n-2}}} \leq \left( \frac{\int_M \sigma_1(g_1) dv(g_1)}{\text{vol}(g_1)^{\frac{n-2}{n}}} \right)^2 = Y_1(\left[ g \right])^2,$$

since $g_1$ is a Yamabe solution.

Proof of Theorem 3. In the case of dimension $n = 4$, it is well known that $\int_M \sigma_2(g) dv(g)$ is constant in any given conformal class. Hence by Lemma 1 we have

$$\frac{2n}{n-1} \int_M \sigma_2(g) dv(g) = \frac{2n}{n-1} \int_M \sigma_2(g_1) dv(g_1) \leq Y_1(\left[ g \right])^2$$

$$\leq \left( \frac{\int_M \sigma_1(g) dv(g)}{\text{vol}(g)^{\frac{n-2}{n}}} \right)^2.$$ 

In the last inequality we have used the condition $\sigma_1(g) \geq 0$, which implies that $Y_1(\left[ g \right]) \geq 0$. The equality holds if and only if the Schouten tensor $S_g$ is proportional to the metric $g$; i.e., $g$ is an Einstein metric.

We conjecture that Theorem 2 is true for $n = 3$. To attack this conjecture one needs to study a corresponding Yamabe-type problem. The methods developed, especially in [2], for a $\sigma_k$-Yamabe problem would be helpful to study this problem.

Note added in proof.

1. The conjecture proposed at the end of the paper was proved in [6].

2. The rigidity of Theorem 1, i.e., equality in (1) implies that the metric is Einstein, was proved in [4], among other generalizations of Theorems 1 and 2. It was also proved independently in the last version of [1].

References


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