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SECONDARY CHERN-EULER FORMS AND THE LAW OF VECTOR FIELDS

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ABSTRACT. The Law of Vector Fields is a term coined by Gottlieb for a relative Poincaré-Hopf theorem. It was first proved by Morse and expresses the Euler characteristic of a manifold with boundary in terms of the indices of a generic vector field and the inner part of its tangential projection on the boundary. We give two elementary differential-geometric proofs of this topological theorem in which secondary Chern-Euler forms naturally play an essential role. In the first proof, the main point is to construct a chain away from some singularities. The second proof employs a study of the secondary Chern-Euler form on the boundary, which may be of independent interest. More precisely, we show by explicitly constructing a primitive that away from the outward and inward unit normal vectors, the secondary Chern-Euler form is exact up to a pullback form. In either case, Stokes' theorem is used to complete the proof.

1. Introduction

Let X be a smooth oriented compact Riemannian manifold with boundary M. Throughout the paper we fix $\dim X = n \geq 2$ and hence $\dim M = n-1$. On M, we have a canonical decomposition

$$TX|_M \cong \nu \oplus TM, \tag{1.1}$$

where ν is the rank 1 trivial normal bundle of M.

Let V be a smooth vector field on X. We assume that V has only isolated singularities, i.e., the set $\operatorname{Sing} V := \{x \in X | V(x) = 0\}$ is finite, and that the restriction $V|_M$ is nowhere zero. Define the index $\operatorname{Ind}_x V$ of V at an isolated singularity x as usual (see, e.g., [Hir76, p. 136]), and let $\operatorname{Ind} V = \sum_{x \in \operatorname{Sing} V} \operatorname{Ind}_x V$ denote the sum of the local indices.

1.2. As an important special case, let $\vec{n}$ be the outward unit normal vector field of M, and $\vec{N}$ a generic extension of $\vec{n}$ to X. Then by definition,

$$\operatorname{Ind} \vec{N} = \chi(X), \tag{1.3}$$

where $\chi(X)$ is the Euler characteristic of X (see, e.g., [Hir76, p. 135]).

For a general V, let ∂V be the projection of $V|_M$ to TM according to (1.1), and let $\partial_- V$ (resp. $\partial_+ V$) be the restriction of ∂V to the subspace of M, where V points inward (resp. outward) to X. Generically $\partial_\pm V$ have isolated singularities.

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(A non-generic V can always be modified by adding an extension to X of a normal vector field or a tangent vector field to M.)

Using the flow along $-V$ and counting fixed points with multiplicities, we have the following *Law of Vector Fields*:

$$\operatorname{Ind} V + \operatorname{Ind} \partial_- V = \chi(X). \tag{1.4}$$

Naturally this is a relative Poincaré-Hopf theorem. It was first proved by Morse [Mor29] and later on publicized by Gottlieb, who also coined the name "Law of Vector Fields".

One main purpose of this paper is to give two elementary differential-geometric proofs of this theorem (1.4).

In his famous proof [Che44] of the Gauss-Bonnet theorem, Chern constructed a differential form Φ (see (2.7)) of degree $n-1$ on the tangent sphere bundle STX, consisting of unit vectors in TX, satisfying the following two conditions:

$$d\Phi = -\Omega,$$

where Ω is the Euler curvature form of X (pulled back to STX), which is defined to be 0 when $\dim X$ is odd (see (2.11)), and

$$\widetilde{\Phi_0} = \widetilde{d\sigma}_{n-1},$$

i.e., the 0th term $\widetilde{\Phi_0}$ of Φ is the relative unit volume form for the fibration $S^{n-1} \to STX \to X$ (see (2.8)). We call Φ the *secondary Chern-Euler form*.

Define $\alpha_V : M \to STX|_M$ by rescaling V, i.e., $\alpha_V(x) = \frac{V(x)}{|V(x)|}$ for $x \in M$. Then Chern's basic method [Che45, §2], [BC65, §6] using the above two conditions and Stokes' theorem gives

$$\int_X \Omega = -\int_{\alpha_V(M)} \Phi + \operatorname{Ind} V \tag{1.5}$$

(see (2.12)). Applying (1.5) to the $\vec{n}$ and $\vec{N}$ in subsection 1.2 and using (1.3), one gets the following relative Gauss-Bonnet theorem in [Che45]:

$$\int_X \Omega = -\int_{\vec{n}(M)} \Phi + \operatorname{Ind}(\vec{N}) = -\int_{\vec{n}(M)} \Phi + \chi(X). \tag{1.6}$$

Comparison of (1.5) and (1.6) gives

$$\chi(X) = \operatorname{Ind} V + \int_{\vec{n}(M)} \Phi - \int_{\alpha_V(M)} \Phi. \tag{1.7}$$

The following is our main result that identifies (1.7) with the Law of Vector Fields (1.4).

Theorem 1.8. *The following formula holds:*

$$\int_{\vec{n}(M)} \Phi - \int_{\alpha_V(M)} \Phi = \operatorname{Ind} \partial_- V.$$

A first proof of the above theorem is given in Section 3. The main point of this first proof is to construct, away from some singularities, a chain connecting $\alpha_V(M)$ to $\vec{n}(M)$ and then to apply Stokes' theorem.

A second proof of Theorem 1.8 to be given in Section 4 employs a study of the secondary Chern-Euler form on the boundary, i.e., when the structure group is reduced from SO(n) to $1 \times$ SO($n-1$). This study may be of some independent interest.

In more detail, the images $\vec{n}(M)$ and $(-\vec{n})(M)$ in $STX|_M$ are the spaces of outward and inward unit normal vectors of M. Define

$$CSTM := STX|_M \backslash (\vec{n}(M) \cup (-\vec{n})(M)) \tag{1.9}$$

(C for cylinder) to be the complement. Also let $\pi : STX|_M \to M$ be the natural projection.

Theorem 1.10. *There exists a differential form Γ of degree $n-2$ on CSTM, such that after restricting to $CSTM$,*

$$\Phi - \pi^* \vec{n}^* \Phi = d\Gamma. \tag{1.11}$$

The form Γ is defined in (4.21), and the above theorem is proved right after that by utilizing Propositions 4.1 and 4.12.

At the end of Section 4, we employ Stokes' theorem to give a second proof of Theorem 1.8, and hence of the Law of Vector Fields (1.4), using Theorem 1.10.

Remark 1.12. Unlike in [Sha99] or [Nie11b], we do not assume that the metric on X is locally a product near its boundary M. Therefore our results in this paper deal with the general case and generalize those in [Nie11b].

Remark 1.13. We would like to emphasize the elementary nature of our approaches, in the classical spirit of Chern in [Che44, Che45]. Transgression of Euler classes has gone through some modern development utilizing Berezin integrals. The Thom class in a vector bundle and its transgression are studied in [MQ86]. This Mathai-Quillen form is further studied in [BZ92] and [BM06]. For the modern developments, we refer the reader to the above references and two books, [BGV92] and [Zha01], on this subject.

2. Secondary Chern-Euler forms

In this section, we review the construction, properties and usage of the secondary Chern-Euler form Φ in [Che44], which plays an essential role in our approaches.

Throughout the paper, c_{r-1} denotes the volume of the unit $(r-1)$-sphere S^{r-1}. We also agree on the following ranges of indices:

$$1 \leq A, B \leq n, \ 2 \leq \alpha, \beta \leq n-1, \ 2 \leq s, t \leq n. \tag{2.1}$$

The secondary Chern-Euler form Φ is defined as follows. Choose oriented local orthonormal frames $\{e_1, e_2, \cdots, e_n\}$ for the tangent bundle TX. Let (ω_{AB}) and (Ω_{AB}) be the $\mathfrak{so}(n)$-valued connection forms and curvature forms for the Levi-Civita connection ∇ of the Riemannian metric on X defined by

$$\nabla e_A = \sum_{B=1}^{n} \omega_{AB} e_B, \tag{2.2}$$

$$\Omega_{AB} = d\omega_{AB} - \sum_{C=1}^{n} \omega_{AC}\omega_{CB}. \tag{2.3}$$

(In this paper, products of differential forms always mean "exterior products", although we omit the notation $\wedge$ for simplicity. Also, we closely follow Chern's notation and convention in [Che44] and [Che45]. In particular we follow his convention in choosing the row and column indices in (2.2), which may not be the most standard.) Let the u_A be the coordinate functions on STX in terms of the frames defined by

$$v = \sum_{A=1}^{n} u_A(v) e_A, \quad \forall v \in STX. \tag{2.4}$$

Let the θ_A be the 1-forms on STX defined by

$$\theta_A = du_A + \sum_{B=1}^{n} u_B \omega_{BA}. \tag{2.5}$$

For $k = 0, 1, \cdots, [\frac{n-1}{2}]$ (with $[-]$ standing for the integral part), define degree $n-1$ forms on STX by

$$\Phi_k = \sum_A \epsilon(A) u_{A_1} \theta_{A_2} \cdots \theta_{A_{n-2k}} \Omega_{A_{n-2k+1} A_{n-2k+2}} \cdots \Omega_{A_{n-1} A_n}, \tag{2.6}$$

where the summation runs over all permutations A of $\{1, 2, \cdots, n\}$, and $\epsilon(A)$ is the sign of A. (The index k stands for the number of curvature forms involved. Hence we have the restriction $0 \le k \le [\frac{n-1}{2}]$. This convention applies throughout the paper.) Define the secondary Chern-Euler form as

$$\Phi = \frac{1}{(n-2)!!c_{n-1}} \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \frac{1}{2^k k!(n-2k-1)!!} \Phi_k =: \sum_{k=0}^{[\frac{n-1}{2}]} \widetilde{\Phi_k}. \tag{2.7}$$

The Φ_k and hence Φ are invariant under SO(n)-transformations of the local frames and hence are intrinsically defined. Note that the 0th term

$$\widetilde{\Phi_0} = \frac{1}{(n-2)!!c_{n-1}} \frac{1}{(n-1)!!} \Phi_0 = \frac{1}{c_{n-1}} d\sigma_{n-1} = \widetilde{d\sigma}_{n-1} \tag{2.8}$$

is the relative unit volume form of the fibration $S^{n-1} \to STX \to X$, since by (2.6),

$$\Phi_0 = \sum_A \epsilon(A) u_{A_1} \theta_{A_2} \cdots \theta_{A_n} = (n-1)! d\sigma_{n-1} \tag{2.9}$$

(see [Che44, (26)]).

Then [Che44, (23)] and [Che45, (11)] prove that

$$d\Phi = -\Omega, \tag{2.10}$$

where

$$\Omega = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ (-1)^m \frac{1}{(2\pi)^m 2^m m!} \sum_A \epsilon(A) \Omega_{A_1 A_2} \cdots \Omega_{A_{n-1} A_n}, & \text{if } n = 2m \text{ is even} \end{cases} \tag{2.11}$$

is the Euler curvature form of X.

Now we review Chern's basic method [Che45, §2], [BC65, §6] of relating indices, Φ and Ω, using Stokes' theorem. Similar procedures will be employed twice later. Let V be a generic vector field on X with isolated singularities $\operatorname{Sing} V$. Let $B_r^X(\operatorname{Sing} V)$ (resp. $S_r^X(\operatorname{Sing} V)$) denote the union of small open balls (resp. spheres) of radii r in X around the finite set of points $\operatorname{Sing} V$. Define $\alpha_V : X \backslash B_r^X(\operatorname{Sing} V) \to STX$

by rescaling V. Then using (2.10) and Stokes' theorem, one proves (1.5) as

$$\begin{aligned}\int_X \Omega &= \lim_{r\to 0}\int_{\alpha_V(X-B_r^X(\operatorname{Sing} V))}\Omega = \lim_{r\to 0}\int_{\alpha_V(X-B_r^X(\operatorname{Sing} V))} -d\Phi \\ &= -\int_{\alpha_V(M)}\Phi + \lim_{r\to 0}\int_{\alpha_V(S_r^X(\operatorname{Sing} V))}\Phi = -\int_{\alpha_V(M)}\Phi + \operatorname{Ind} V,\end{aligned} \tag{2.12}$$

where the last equality follows from the definition of the index and (2.8).

3. First proof by constructing a chain

In this section, we give a first proof of Theorem 1.8 by constructing a chain, away from $\operatorname{Sing}\partial_- V$, connecting $\alpha_V(M)$ to $\vec{n}(M)$.

First proof of Theorem 1.8. By definition, $\operatorname{Sing}\partial_- V$ consists of a finite number of points $x \in M$ such that $\alpha_V(x) = -\vec{n}(x)$. For $x \notin \operatorname{Sing}\partial_- V$, let C_x be the unique directed shortest great circle segment pointing from $\alpha_V(x)$ to $\vec{n}(x)$ in ST_xX. With the obvious notation from before, let $U_r = M\backslash B_r^M(\operatorname{Sing}\partial_- V)$ denote the complement in M of the union of open balls of radii r in M around $\operatorname{Sing}\partial_- V$. Obviously its boundary $\partial U_r = -S_r^M(\operatorname{Sing}\partial_- V)$. Then

$$\partial\left(\bigcup_{x\in U_r} C_x\right) = \bigcup_{x\in U_r}\partial C_x - \bigcup_{x\in \partial U_r} C_x = \vec{n}(U_r) - \alpha_V(U_r) + W_r, \tag{3.1}$$

with

$$W_r := \bigcup_{x\in S_r^M(\operatorname{Sing}\partial_- V)} C_x. \tag{3.2}$$

Note the negative sign from the graded differentiation of chains in the second expression of (3.1). From (2.10) and (2.11), we have

$$d\Phi = 0 \text{ on } STX|_M, \tag{3.3}$$

since even if $\dim X$ is even, $\Omega|_M = 0$ by a dimensional reason. (3.1), Stokes' theorem and (3.3) imply that

$$\begin{aligned}\int_{\vec{n}(M)}\Phi - \int_{\alpha_V(M)}\Phi &= \lim_{r\to 0}\left(\int_{\vec{n}(U_r)}\Phi - \int_{\alpha_V(U_r)}\Phi\right) \\ &= -\lim_{r\to 0}\int_{W_r}\Phi = -\lim_{r\to 0}\int_{W_r}\widetilde{\Phi_0},\end{aligned} \tag{3.4}$$

where the last equality follows from (2.7) and $\lim_{r\to 0}\int_{W_r}\widetilde{\Phi_k} = 0$ for $k \geq 1$, since such $\widetilde{\Phi_k}$'s in (2.6) involve curvature forms and do not contribute in the limit (see [Che45, §2]). By (2.8), $\widetilde{\Phi_0} = \frac{1}{c_{n-1}}d\sigma_{n-1}$ is the relative unit volume form. We then compute the RHS of (3.4) using spherical coordinates.

3.5. At $TX|_M$, we choose oriented local orthonormal frames $\{e_1, e_2, \cdots, e_n\}$ such that $e_1 = \vec{n}$ is the outward unit normal vector of M. Therefore $(e_2, \cdots, e_n)$ are oriented local orthonormal frames for TM. Let ϕ be the angle coordinate on $STX|_M$ defined by

$$\phi(v) = \angle(v, e_1) = \angle(v, \vec{n}),\ \forall v \in STX|_M. \tag{3.6}$$

We have from (2.4)

$$u_1 = \cos\phi. \tag{3.7}$$

Let

$$p : CSTM = STX|_M \backslash (\vec{n}(M) \cup (-\vec{n})(M)) \to STM;\ v \mapsto \frac{\partial v}{|\partial v|} \tag{3.8}$$
$$(\text{in coordinates})(\cos\phi, u_2, \cdots, u_n) \mapsto \frac{1}{\sin\phi}(u_2, \cdots, u_n)$$

be the projection to the equator STM. By definition,

$$p \circ \alpha_V = \alpha_{\partial V} \text{ when } \partial V \neq 0. \tag{3.9}$$

Therefore the image of W_r in (3.2) under the above projection is

$$p(W_r) = \bigcup_{x \in S_r^M(\operatorname{Sing}\partial_- V)} p(C_x) = \bigcup_{x \in S_r^M(\operatorname{Sing}\partial_- V)} \alpha_{\partial V}(x) = \alpha_{\partial V}\big(S_r^M(\operatorname{Sing}\partial_- V)\big).$$

On C_x for $x \in M$, the ϕ (3.6) ranges from $\phi(\alpha_V(x))$ to 0.

The relative volume forms $d\sigma_{n-1}$ of $S^{n-1} \to STX|_M \to M$ and $d\sigma_{n-2}$ of $S^{n-2} \to STM \to M$ are related by

$$d\sigma_{n-1} = \sin^{n-2}\phi\, d\phi\, p^* d\sigma_{n-2} + \text{terms involving } \omega_{1s} \text{ or } \Omega^M_{\alpha\beta}. \tag{3.10}$$

(See (4.7) for the definition of the curvature forms $\Omega^M_{\alpha\beta}$. Also compare (4.18) when $k = 0$ in view of (2.9). In the case of one fixed sphere and its equator, (3.10) without the extra terms is easy and follows from using spherical coordinates.) In the limit when $r \to 0$, the integrals of the terms involving ω_{1s} or $\Omega^M_{\alpha\beta}$ are zero by the same reason as in the last step of (3.4).

Therefore, continuing (3.4) and using iterated integrals, we have

$$\begin{aligned}
&\int_{\vec{n}(M)} \Phi - \int_{\alpha_V(M)} \Phi = -\lim_{r\to 0}\int_{W_r} \widetilde{\Phi_0} = -\frac{1}{c_{n-1}}\lim_{r\to 0}\int_{W_r} d\sigma_{n-1} \\
&= -\frac{1}{c_{n-1}}\lim_{r\to 0}\int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_- V))}\left(\int_{\phi(\alpha_V(x))}^{0}\sin^{n-2}\phi\, d\phi\right) d\sigma_{n-2} \\
&\overset{(1)}{=} \frac{1}{c_{n-1}}\left(\int_0^\pi \sin^{n-2}\phi\, d\phi\right)\lim_{r\to 0}\int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_- V))} d\sigma_{n-2} \\
&\overset{(2)}{=} \frac{1}{c_{n-2}}\lim_{r\to 0}\int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_- V))} d\sigma_{n-2} \overset{(3)}{=} \operatorname{Ind}\partial_- V.
\end{aligned}$$

Here equality (1) uses

$$\phi(\alpha_V(x)) \to \pi \text{ for } x \in S_r^M(\operatorname{Sing}\partial_- V), \text{ as } r \to 0, \tag{3.11}$$

equality (2) uses the basic knowledge

$$c_{n-1} = c_{n-2}\int_0^\pi \sin^{n-2}\phi\, d\phi, \tag{3.12}$$

and equality (3) is by the definition of index. □

Remark 3.13. The construction of the chain $\bigcup_{x\in U_r} C_x$ is reminiscent of the topological method [Mor29] of attaching $M \times I$ to X and extending $V|_M$ to a vector field on $M \times I$ whose value at $(x,t) \in M \times I$ is $(1-t)V(x) + t\vec{n}(x)$.

Remark 3.14. The homology group $H_{n-1}(STX|_M, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ has two generators as the image $\vec{n}(M)$ and a fiber sphere ST_xM for $x \in M$ (see [Nie11a]). Our proof shows that as a homology class,

$$\alpha_V(M) = \vec{n}(M) + (\operatorname{Ind} \partial_- V) ST_xM.$$

4. Second proof by transgressing Φ

In this section, we present a second transgression of the secondary Chern-Euler form Φ on $CSTM \subset STX|_M$ (1.9), leading to a proof of Theorem 1.10 and a second proof of Theorem 1.8 using that.

Recall the definition of the angle coordinate ϕ (3.6). Then $d\phi$ and $\frac{\partial}{\partial \phi}$ are a well-defined 1-form and a vector field on $CSTM$. We write d for exterior differentiation on $CSTM$, and $\iota_{\frac{\partial}{\partial \phi}}$ for interior product with $\frac{\partial}{\partial \phi}$.

Proposition 4.1. *On $CSTM$, let*

$$\Upsilon = \iota_{\frac{\partial}{\partial \phi}} \Phi. \tag{4.2}$$

Then the Lie derivative

$$\mathcal{L}_{\frac{\partial}{\partial \phi}} \Phi = d\Upsilon. \tag{4.3}$$

Therefore

$$\Phi - \pi^* \vec{n}^* \Phi = d \int_0^{\phi} \Upsilon \, dt. \tag{4.4}$$

Proof. (4.3) follows from the Cartan homotopy formula (see, e.g., [KN63, Proposition I.3.10])

$$\mathcal{L}_{\frac{\partial}{\partial \phi}} \Phi = (d \, \iota_{\frac{\partial}{\partial \phi}} + \iota_{\frac{\partial}{\partial \phi}} d) \Phi = d\Upsilon,$$

by (4.2) and $d\Phi = 0$ (3.3).

(4.4) then follows by integration since $\pi^* \vec{n}^* \Phi$ corresponds to the evaluation of Φ at $\phi = 0$ by the definition of ϕ (3.6), and we have for any fixed ϕ:

$$\Phi - \pi^* \vec{n}^* \Phi = \int_0^{\phi} \mathcal{L}_{\frac{\partial}{\partial \phi}} \Phi \, dt = \int_0^{\phi} d\Upsilon \, dt = d \int_0^{\phi} \Upsilon \, dt. \qquad \square$$

Now we calculate Υ explicitly. Since Φ (2.7) is invariant under $\mathrm{SO}(n)$-changes of local frames, we adapt an idea from [Che45] to use a nice frame for $TX|_M$ to facilitate the calculations about Φ on $CSTM$. Choose e_1 as in subsection 3.5. For $v \in CSTM$, let

$$e_n = p(v) \tag{4.5}$$

as defined in (3.8). Choose $e_2, \cdots, e_{n-1}$ so that $\{e_1, e_2, \cdots, e_{n-1}, e_n\}$ is a positively oriented frame for $TX|_M$. (Therefore we need $n \geq 3$ from now on, with the $n = 2$ case being simple.) Then in view of (3.6),

$$v = \cos \phi \, e_1 + \sin \phi \, e_n. \tag{4.6}$$

Let (Ω^M_{st}) denote the curvature forms on M of the induced metric from X. In view of (2.3),

$$\Omega^M_{st} = d\omega_{st} - \sum_{r=2}^{n} \omega_{sr} \omega_{rt}, \tag{4.7}$$

$$\Omega_{st} = \Omega^M_{st} + \omega_{1s} \omega_{1t}. \tag{4.8}$$

Define the following differential forms on STM, regarded to be pulled back to $CSTM$ by p (3.8), of degree $n-2$:

$$\Phi^M(i,j) = \sum_\alpha \epsilon(\alpha)\omega_{1\alpha_2}\cdots\omega_{1\alpha_{n-2i-j-1}}\Omega^M_{\alpha_{n-2i-j}\alpha_{n-2i-j+1}}\cdots\Omega^M_{\alpha_{n-j-2}\alpha_{n-j-1}} \omega_{\alpha_{n-j}n}\cdots\omega_{\alpha_{n-1}n}, \tag{4.9}$$

where the summations run over all permutations α of $\{2,\cdots,n-1\}$. It is easy to check that these $\Phi^M(i,j)$ are invariant under $\mathrm{SO}(n-2)$-changes of the partial frames $\{e_2,\cdots,e_{n-1}\}$. Here the two parameters i and j stand for the numbers of curvature forms and $\omega_{\alpha n}$'s involved. Define the following region of the indices i,j:

$$D_1 = \{(i,j)\in\mathbb{Z}\times\mathbb{Z}\,|\, i\geq 0, j\geq 0, 2i+j\leq n-2\}.$$

Then

$$\Phi^M(i,j)\neq 0 \Rightarrow (i,j)\in D_1.$$

Remark 4.10. Our choice of the letter Φ^M is due to the following special case when there are no $\omega_{1\alpha}$'s:

$$\Phi^M(i,n-2i-2)=\Phi^M_i,$$

where Φ^M_i are forms on STM defined by Chern [Che45]. Since we are considering the case of a boundary, we have the extra $\omega_{1\alpha}$'s in our more general forms.

Also note that the $\omega_{1s}=0$ if the metric on X is locally a product near the boundary M. Therefore a lot of our forms vanish in that simpler case as considered in [Nie11b].

We also introduce the following functions of ϕ (3.6), for non-negative integers p and q:

$$T(p,q)(\phi)=\cos^p\phi\sin^q\phi,$$

$$I(p,q)(\phi)=\int_0^\phi T(p,q)(t)\,dt. \tag{4.11}$$

Proposition 4.12. *We have the following concrete formulas:*

$$\Upsilon = \iota_{\frac{\partial}{\partial\phi}}\Phi = \frac{1}{(n-2)!!c_{n-1}}\sum_{(i,j)\in D_1} a(i,j)(\phi)\,\Phi^M(i,j), \tag{4.13}$$

$$a(i,j)(\phi)=\sum_{k=i}^{[\frac{n-j}{2}]-1}(-1)^{n+j+k}\frac{(n-2k-2)!!}{2^k j!(n-2k-j-2)!i!(k-i)!}T(n-2k-j-2,j)(\phi). \tag{4.14}$$

Proof. From (2.4), (2.5) and (4.6), we have

$$u_1=\cos\phi,\ u_n=\sin\phi,\ u_\alpha=0; \tag{4.15}$$

$$\theta_1=-\sin\phi\,(d\phi+\omega_{1n}),\ \theta_n=\cos\phi\,(d\phi+\omega_{1n}), \tag{4.16}$$

$$\theta_\alpha=\cos\phi\,\omega_{1\alpha}-\sin\phi\,\omega_{\alpha n}. \tag{4.17}$$

From (4.15), there are only two non-zero coordinates u_1 and u_n. Hence there are four cases for the positions of the indices 1 and n in Φ_k (2.6):

(i) $n-2k-1$ possibilities of $u_1\theta_n$;

(ii) $2k$ possibilities of $u_1\Omega_{\alpha n}$;
(iii) $n-2k-1$ possibilities of $u_n\theta_1$;
(iv) $2k$ possibilities of $u_n\Omega_{1\alpha}$.

Only cases (i) and (iii) contribute $d\phi$ in view of (4.16), and hence we are only concerned with these two cases for the computation of $\Upsilon = \iota_{\frac{\partial}{\partial\phi}}\Phi$. Starting with (2.6), taking signs into consideration, by (4.15) and (4.16), by $\cos^2\phi + \sin^2\phi = 1$, (4.17), (4.8) and the multinomial theorem, we have

(4.18)

$$\begin{aligned}
\Phi_k &= (n-2k-1)(-1)^n \cos^2\phi(d\phi+\omega_{1n}) \\
&\qquad \sum_\alpha \epsilon(\alpha)\theta_{\alpha_2}\cdots\theta_{\alpha_{n-2k-1}}\Omega_{\alpha_{n-2k}\alpha_{n-2k+1}}\cdots\Omega_{\alpha_{n-2}\alpha_{n-1}} \\
&\quad + (n-2k-1)(-1)^n \sin^2\phi(d\phi+\omega_{1n}) \\
&\qquad \sum_\alpha \epsilon(\alpha)\theta_{\alpha_2}\cdots\theta_{\alpha_{n-2k-1}}\Omega_{\alpha_{n-2k}\alpha_{n-2k+1}}\cdots\Omega_{\alpha_{n-2}\alpha_{n-1}} \\
&\quad + \cdots \\
&= (-1)^n(n-2k-1)(d\phi+\omega_{1n})\sum_\alpha \epsilon(\alpha)(\cos\phi\,\omega_{1\alpha_2} - \sin\phi\,\omega_{a_2 n})\cdots \\
&\quad (\cos\phi\,\omega_{1\alpha_{n-2k-1}} - \sin\phi\,\omega_{a_{n-2k-1}n})(\Omega^M_{\alpha_{n-2k}\alpha_{n-2k+1}} + \omega_{1\alpha_{n-2k}}\omega_{1\alpha_{n-2k+1}})\cdots \\
&\quad (\Omega^M_{\alpha_{n-2}\alpha_{n-1}} + \omega_{1\alpha_{n-2}}\omega_{1\alpha_{n-1}}) \\
&\quad + \cdots \\
&= (-1)^n(n-2k-1)(d\phi+\omega_{1n}) \\
&\quad \sum_{\substack{0\le i\le k \\ 0\le j\le n-2k-2}} \frac{(n-2k-2)!}{j!(n-2k-j-2)!}\cos^{n-2k-j-2}\phi\,(-\sin\phi)^j\frac{k!}{i!(k-i)!}\Phi^M(i,j) \\
&\quad + \cdots \\
&= \sum_{\substack{0\le i\le k \\ 0\le j\le n-2k-2}} (-1)^{n+j}\frac{(n-2k-1)!k!}{j!(n-2k-j-2)!i!(k-i)!}T(n-2k-j-2,j)(\phi) \\
&\qquad\qquad (d\phi+\omega_{1n})\Phi^M(i,j) + \cdots .
\end{aligned}$$

From (2.7) and the above, we get (4.13) and the coefficients $a(i,j)(\phi)$ in (4.14), after some immediate cancellations. □

Definition 4.19. For $(i,j)\in D_1$, define the following functions on $CSTM$:

$$\begin{aligned}
A(i,j)(\phi) &= \int_0^\phi a(i,j)(t)\,dt \\
&= \sum_{k=i}^{[\frac{n-j}{2}]-1} (-1)^{n+j+k}\frac{(n-2k-2)!!}{2^k j!(n-2k-j-2)!i!(k-i)!}I(n-2k-j-2,j)(\phi),
\end{aligned} \tag{4.20}$$

in view of (4.14) and (4.11). Also define the differential form of degree $n-2$ on $CSTM$:

$$\Gamma = \frac{1}{(n-2)!!c_{n-1}} \sum_{(i,j)\in D_1} A(i,j)(\phi)\Phi^M(i,j). \tag{4.21}$$

Proof of Theorem 1.10. We just need to notice that $\Gamma = \int_0^\phi \Upsilon\, dt$ by Proposition 4.12 and use (4.4) in Proposition 4.1. □

Remark 4.22. Our first proof of Theorem 1.10 was through very explicit differentiations. Write $\Phi = d\phi\,\Upsilon + \Xi$ in view of (4.2). We can compute Ξ explicitly. After correctly guessing the Γ in (4.21), we prove Theorem 1.10 by some differentiation formulas of differential forms in the spirit of [Che45], and some induction formulas for the functions $I(p,q)(\phi)$ in (4.11) through integration by parts.

We finally arrive at

Second proof of Theorem 1.8. Let $B_r^M(\operatorname{Sing}\partial V)$ (resp. $S_r^M(\operatorname{Sing}\partial V)$) denote the union of small open balls (resp. spheres) of radii r in M around the finite set of points $\operatorname{Sing}\partial V$. Then by $\partial V(x) = 0 \Leftrightarrow \alpha_V(x) = \pm\vec{n}(x)$,

$$\alpha_V(M\backslash B_r^M(\operatorname{Sing}\partial V)) \subset CSTM.$$

By Theorem 1.10 and Stokes' theorem,

$$\begin{aligned}
&\int_{\alpha_V(M)} \Phi - \int_{\vec{n}(M)} \Phi = \int_{\alpha_V(M)} \Phi - \pi^*\vec{n}^*\Phi = \lim_{r\to 0} \int_{\alpha_V(M\backslash B_r^M(\operatorname{Sing}\partial V))} \Phi - \pi^*\vec{n}^*\Phi \\
&= \lim_{r\to 0} \int_{\alpha_V(M\backslash B_r^M(\operatorname{Sing}\partial V))} d\Gamma = -\lim_{r\to 0} \int_{\alpha_V(S_r^M(\operatorname{Sing}\partial V))} \Gamma \\
&= -\lim_{r\to 0} \int_{\alpha_V(S_r^M(\operatorname{Sing}\partial V))} \frac{1}{(n-2)!!c_{n-1}} A(0,n-2)(\phi)\Phi^M(0,n-2),
\end{aligned} \tag{4.23}$$

since all the other $A(i,j)(\phi)\Phi^M(i,j)$ in (4.21), for $(i,j) \in D_1$ and not equal to $(0,n-2)$, involve either curvature forms $\Omega^M_{\alpha\beta}$ or connection forms $\omega_{1\alpha}$ and hence do not contribute in the limit when integrated over small spheres.

We have by (4.20) and (4.9),

$$\begin{aligned}
&\frac{1}{(n-2)!!c_{n-1}} A(0,n-2)(\phi)\Phi^M(0,n-2) \\
&\quad = \frac{1}{(n-2)!!c_{n-1}} \frac{(n-2)!!}{(n-2)!} I(0,n-2)(\phi) \sum_\alpha \epsilon(\alpha)\omega_{\alpha_2 n}\cdots\omega_{\alpha_{n-1}n} \\
&\quad = \frac{1}{c_{n-1}} I(0,n-2)(\phi)\, p^* d\sigma_{n-2}
\end{aligned} \tag{4.24}$$

with $d\sigma_{n-2}$ being the relative volume form of $S^{n-2} \to STM \to M$, since

$$\sum_\alpha \epsilon(\alpha)\omega_{\alpha_2 n}\cdots\omega_{\alpha_{n-1}n} = (n-2)!\, p^* d\sigma_{n-2}$$

in view of (4.5) and by comparison with (2.9).

Continuing (4.23) and using (4.24), we have

$$\begin{aligned}
&\int_{\alpha_V(M)} \Phi - \int_{\vec{n}(M)} \Phi \\
&\quad = -\frac{1}{c_{n-1}} \lim_{r\to 0} \int_{\alpha_V(S_r^M(\operatorname{Sing}\partial_+ V)\cup S_r^M(\operatorname{Sing}\partial_- V))} I(0,n-2)(\phi)\, p^* d\sigma_{n-2} \\
&\quad \overset{(1)}{=} -\frac{1}{c_{n-1}} \big[I(0,n-2)(0) \lim_{r\to 0} \int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_+ V))} d\sigma_{n-2} \\
&\qquad\qquad + I(0,n-2)(\pi) \lim_{r\to 0} \int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_- V))} d\sigma_{n-2} \big] \\
&\quad \overset{(2)}{=} \frac{1}{c_{n-2}} \lim_{r\to 0} \int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_- V))} d\sigma_{n-2} \\
&\quad \overset{(3)}{=} -\operatorname{Ind}\partial_- V.
\end{aligned}$$

Here equality (1) uses (3.9), (3.11) and the similar

$$\phi(\alpha_V(x)) \to 0 \text{ for } x \in S_r^M(\operatorname{Sing}\partial_+ V), \text{ as } r \to 0.$$

In view of (4.11), we have

$$I(0,n-2)(0) = 0,\ I(0,n-2)(\pi) = \int_0^\pi \sin^{n-2}\phi\, d\phi.$$

Then equality (2) follows from (3.12). Equality (3) is by the definition of the index. □

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