ORDINARY VARIETIES AND THE COMPARISON BETWEEN
MULTIPLIER IDEALS AND TEST IDEALS II

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Abstract. We consider the following conjecture: if $X$ is a smooth $n$-dimensional projective variety in characteristic zero, then there is a dense set of reductions $X_s$ to positive characteristic such that the action of the Frobenius morphism on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective. We also consider the conjecture relating the multiplier ideals of an ideal $a$ on a nonsingular variety in characteristic zero, and the test ideals of the reductions of $a$ to positive characteristic. We prove that the latter conjecture implies the former one.

1. Introduction

This paper is motivated by the joint paper with V. Srinivas [MS], aimed at understanding the following conjecture relating invariants of singularities in characteristic zero with corresponding invariants in positive characteristic. For a discussion of the notions involved, see below.

Conjecture 1.1. Let $Y$ be a smooth, irreducible variety over an algebraically closed field $k$ of characteristic zero and $a$ a nonzero ideal on $Y$. Given any model $Y_A$ and $a_A$ for $Y$ and $a$ over a subring $A$ of $k$, finitely generated over $\mathbb{Z}$, there is a dense set of closed points $S \subset \text{Spec } A$ such that

$$J(Y, a^\lambda)_s = \tau(Y_s, a_s^\lambda)$$

for every $\lambda \in \mathbb{R}_{\geq 0}$ and every $s \in S$.

In the conjecture, we denote by $Y_s$ the fiber of $Y_A$ over $s \in S$, and $a_s$ is the ideal on $Y_s$ induced by $a_A$. The ideals $J(Y, a^\lambda)$ are the multiplier ideals of $a$. These are fundamental invariants of the singularities of $a$, which have seen a lot of recent applications due to their appearance in vanishing theorems (see Laz, Chapter 9). The ideals $\tau(Y_s, a_s^\lambda)$ are the (generalized) test ideals of Hara and Yoshida [HY], defined in positive characteristic using the Frobenius morphism. The above conjecture asserts therefore that for a dense set of closed points, we have the equality between the test ideals of $a$ and the reductions of the multiplier ideals of $a$ for all exponents. We note that it is shown in [HY] that if $\lambda \in \mathbb{R}_{>0}$ is fixed, then the equality in (1) holds for every $s$ in an open subset of the closed points in $\text{Spec } A$. 

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The following conjecture was proposed in [MS].

**Conjecture 1.2.** Let $X$ be a smooth, irreducible $n$-dimensional projective variety defined over an algebraically closed field $k$ of characteristic zero. If $X_A$ is a model of $X$ defined over a subring $A$ of $k$, finitely generated over $Z$, then there is a dense set of closed points $S \subseteq \text{Spec } A$ such that the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective for every $s \in S$.

It is expected, in fact, that there is a set $S$ as in Conjecture 1.2 such that $X_s$ is ordinary in the sense of Bloch and Kato [BK] for every $s \in S$. In particular, this would imply that the action of the Frobenius on each cohomology group $H^i(X_s, \mathcal{O}_{X_s})$ is bijective (see [MS] Remark 5.1). The main result of [MS] is that Conjecture 1.2 implies Conjecture 1.1. In this paper we show that the converse is true:

**Theorem 1.3.** If Conjecture 1.1 holds, then so does Conjecture 1.2.

The following is an outline of the proof. Given a variety $X$ as in Conjecture 1.2, we embed it in a projective space $\mathbb{P}^N_k$ such that $r := N - n \geq n + 1$, and the ideal $\mathfrak{a} \subseteq k[x_0, \ldots, x_N]$ defining $X$ is generated by quadrics. In this case it is easy to compute the multiplier ideals $\mathcal{J}(A^{N+1}_k, \mathfrak{a}^\lambda)$ for $\lambda < r$, and in particular we see that $(x_0, \ldots, x_N)^{2r-N-1} \subseteq \mathcal{J}(A^{N+1}_k, \mathfrak{a}^\lambda)$ for every $\lambda < r$. It follows from a general property of multiplier ideals that if $g_1, \ldots, g_r$ are general linear combinations of a system of generators of $\mathfrak{a}$, and if $h = g_1 \cdots g_r$, then $\mathcal{J}(A^{N+1}_k, \mathfrak{a}^\lambda) = \mathcal{J}(A^{N+1}_k, h^{\lambda/r})$ for every $\lambda < r$. In this case, Conjecture 1.1 implies that for a dense set of closed points $s \in \text{Spec } A$, the ideal $(x_0, \ldots, x_N)^{2r-N-1}$ is contained in $\tau(A^{N+1}_k(s), h^\mu)$ for every $\mu < 1$. Using some basic properties of test ideals, we deduce that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective, where $D_s \subseteq \mathbb{P}^N_{k(s)}$ is the hypersurface defined by $h_s$. We show that this in turn implies the bijectivity of the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$, hence proves the theorem.

## 2. Proof of the Main Result

We start by recalling the definition of multiplier ideals and test ideals. Suppose first that $Y$ is a smooth, irreducible variety over an algebraically closed field $k$ of characteristic zero, and $\mathfrak{a}$ is a nonzero ideal on $Y$. A log resolution of $\mathfrak{a}$ is a projective, birational morphism $\pi: W \to Y$, with $W$ smooth, such that $\mathfrak{a} \cdot \mathcal{O}_W$ is the ideal of a divisor $D$ on $W$, with $D + K_{W/Y}$ having simple normal crossings (as usual, $K_{W/Y}$ denotes the relative canonical divisor of $W$ over $Y$). With this notation, for every $\lambda \in \mathbb{R}_{\geq 0}$ we have

$$\mathcal{J}(Y, \mathfrak{a}^\lambda) = \pi_* \mathcal{O}_W(K_{W/Y} - \lfloor \lambda D \rfloor).$$

Recall that if $E = \sum a_i E_i$ is a divisor with $\mathbb{R}$-coefficients, then $[E] = \sum [a_i] E_i$, where $[t]$ is the largest integer \leq t. It is a well-known fact that the above definition is independent of the choice of log resolution. For this and other basic facts about multiplier ideals, see [Laz] Chapter 9.

Suppose now that $Y = \text{Spec } R$ is an affine smooth, irreducible scheme of finite type over a perfect field $L$ of positive characteristic $p$ (in the case of interest for us, $L$ will be a finite field). Under these assumptions, the test ideals admit the following simple description that we will use; see [BMS2]. Recall that for an ideal $J$ and for $e \geq 1$, one denotes by $J^{[p^e]}$ the ideal $(h^{pf} \mid h \in J)$. One can show that
given an ideal \( b \) in \( R \), there is a unique smallest ideal \( J \) such that \( b \subseteq J^{[1/p^n]} \); this ideal is denoted by \( b^{[1/p^n]} \). Suppose now that \( a \) is an ideal in \( R \) and \( \lambda \in \mathbb{R}_{\geq 0} \). One can show that for every \( e \geq 1 \) we have the inclusion
\[
(a^{[\lambda p^n]})^{[1/p^e]} \subseteq (a^{[\lambda p^{n+1}]})^{[1/p^e]},
\]
where \([t]\) denotes the smallest integer \( \geq t \). Since \( R \) is Noetherian, it follows that \((a^{[\lambda p^n]})^{[1/p^e]}\) is constant for \( e \gg 0 \). This is the test ideal \( \tau(Y, a^\lambda) \). For details and a discussion of basic properties of test ideals in this setting, we refer to [BMS2]. For a comparison of general properties of multiplier ideals and test ideals, see [HY] and [MY].

If \( a \) is an ideal in the polynomial ring \( k[x_0, \ldots, x_N] \), where \( k \) is a field of characteristic zero, a model of \( a \) over a subring \( A \) of \( k \), finitely generated over \( \mathbb{Z} \), is an ideal \( a_A \) in \( A[x_0, \ldots, x_N] \) such that \( a_A \cdot k[x_0, \ldots, x_N] = a \). We can obtain such a model by simply taking \( A \) to contain all the coefficients of a finite system of generators of \( a \). Of course, we may always replace \( A \) by a larger ring with the same properties; in particular, we may replace \( A \) by a localization \( A_a \) at a nonzero element \( a \in A \). If \( s \in \text{Spec} A \) and if \( a_A \) is a model of \( a \), then we obtain a corresponding ideal \( a_s \) in \( k(s)[x_0, \ldots, x_N] \). Note that if \( s \) is a closed point, then the residue field \( k(s) \) is a finite field.

Suppose now that \( X \subseteq \mathbb{P}^N_k \) is a projective subscheme defined by the homogeneous ideal \( a \subseteq k[x_0, \ldots, x_N] \). If \( a_A \subseteq A[x_0, \ldots, x_N] \) is a model of \( a \) over \( A \), which we may assume to be homogeneous, then the subscheme \( X_A \) of \( \mathbb{P}^N_A \) defined by \( a_A \) is a model of \( X \) over \( A \). If \( s \in \text{Spec} A \), then the subscheme \( X_s \subseteq \mathbb{P}^N_{k(s)} \) is defined by the ideal \( a_s \). We refer to [MS §2.2] for some of the standard facts about reduction to positive characteristic. We note that given \( a \) as above, we may consider simultaneously all the reductions \( J(A_k^{N+1}, a^\lambda)_s \) for all \( \lambda \in \mathbb{R}_{\geq 0} \). This is due to the fact that for bounded \( \lambda \) we only have to deal with finitely many ideals, while for \( \lambda \gg 0 \), the multiplier ideals are determined by the lower ones via a Skoda-type theorem (see [MS §3.2] for details).

We can now give the proof of our main result stated in the introduction.

**Proof of Theorem 1.3.** Let \( X \) be a smooth, irreducible \( n \)-dimensional projective variety over an algebraically closed field \( k \) of characteristic zero, with \( n \geq 1 \). It is clear that the assertion we need is independent of the model \( X_A \) that we choose. Consider a closed embedding \( X \hookrightarrow \mathbb{P}^N_k \). After replacing this by a composition with a \( d \)-tuple Veronese embedding, for \( d \gg 0 \), we may assume that the saturated ideal \( a \subset R = k[x_0, \ldots, x_N] \) defining \( X \) is generated by homogeneous polynomials of degree two (see [ERT] Proposition 5). Furthermore, we may clearly assume that \( r := N - n \geq n + 1 \). Under these assumptions, it is easy to determine the multiplier ideals of \( a \) of exponent \( < r \).

**Lemma 2.1.** With the above notation, if \( m = (x_0, \ldots, x_N) \), then
\[
J(A_k^{N+1}, a^\lambda) = \begin{cases} R, & \text{if } 0 \leq \lambda < \frac{N+1}{2}; \\ m^{[2\lambda]} - N, & \text{if } \frac{N+1}{2} \leq \lambda < r. \end{cases}
\]

**Proof.** Let us fix \( \lambda \in \mathbb{R}_{\geq 0} \), with \( \lambda < r \). We denote by \( Z \) the subscheme of \( \mathbb{A}^{N+1}_k \) defined by \( a \). Let \( \varphi \colon W \to \mathbb{A}^{N+1}_k \) be the blowup of the origin, with exceptional divisor \( E \). Since \( a \) is generated by homogeneous polynomials of degree two, it follows
that \(a \cdot \mathcal{O}_W = \mathcal{O}_W(-2E) \cdot \bar{a}\), where \(\bar{a}\) is the ideal defining the strict transform \(\bar{Z}\) of \(Z\) on \(W\). We have \(K_{N}^{\mathbf{A}^{N+1}_{k}} = NE\); hence the change of variable formula for multiplier ideals (see [Laz, Theorem 9.2.33]) implies that

\[
\mathcal{J}(\mathbf{A}^{N+1}_{k}, a^\lambda) = \varphi_*(\mathcal{J}(W, (a \cdot \mathcal{O}_W)^\lambda) \otimes \mathcal{O}_W(NE)).
\]

It is clear that \(\bar{Z}\) is nonisomorphic over \(\mathbf{A}^{N+1}_{k} \setminus \{0\}\). Since \(\bar{Z} \cap E \subseteq E \simeq \mathbb{P}^N\) is isomorphic to the scheme \(X\), hence it is nonisomorphic, it follows that \(\bar{Z}\) is nonisomorphic, and \(\bar{Z}\) and \(E\) have simple normal crossings. Let \(\psi: \bar{W} \to W\) be the blowup of \(W\) along \(\bar{Z}\), with exceptional divisor \(T\), and let \(\bar{E}\) be the strict transform of \(E\). Note that \(\bar{W}\) is nonisomorphic, and \(\bar{E} + T\) has simple normal crossings. We have \(K_{\bar{W}/W} = (r - 1)T\) and \(a \cdot \mathcal{O}_{\bar{W}} = \mathcal{O}_{\bar{W}}(-2\bar{E} - T)\). Therefore \(\psi\) is a log resolution of \(a \cdot \mathcal{O}_W\), and by definition we have

\[
\mathcal{J}(W, (a \cdot \mathcal{O}_W)^\lambda) = \psi_* (\mathcal{J}(\bar{W}, (a \cdot \mathcal{O}_{\bar{W}})^\lambda) \otimes \mathcal{O}_W(\lambda E)).
\]

(recall that \(\lambda < r\)). The formula in the lemma follows from (3), (4), and the fact that \(\varphi_*(\mathcal{O}_W(-iE)) = m^i\) for every \(i \in \mathbb{Z}_{\geq 0}\). \(\square\)

Let \(f_1, \ldots, f_m\) be a system of generators of \(a\), with each \(f_i\) homogeneous of degree two. We fix \(g_1, \ldots, g_r\), general linear combinations of the \(f_i\), with coefficients in \(k\), and put \(h = g_1 \cdots g_r\). In this case, we have

\[
\mathcal{J}(\mathbf{A}^{N+1}_{k}, a^\lambda) = \mathcal{J}(\mathbf{A}^{N+1}_{k}, h^{\lambda/r})
\]

for every \(\lambda < r\) (see [Laz, Proposition 9.2.28]).

Suppose now that \(a_A\) and \(h_A\) are homogeneous models of \(a\), and respectively \(h\), over \(A\). Let \(X_A, D_A \subset P^N_A\) be the projective schemes defined by \(a_A\) and \(h_A\), respectively. Note that \(g_1, \ldots, g_r\) being general linear combinations of the \(f_i\), the subscheme \(V(g_1, \ldots, g_r) \subset P^N_k\) has pure codimension \(r\). Therefore we may assume that for every \(s \in \text{Spec} A\), the scheme \(V((g_1)s, \ldots, (g_r)s)\) has pure codimension \(r\) in \(P^N_{k(s)}\). We need to show that given models as above, there is a dense set of closed points \(S \subset \text{Spec} A\) such that the Frobenius action on \(H^n(X_s, \mathcal{O}_{X_s})\) is bijective for every \(s \in S\). The next lemma shows that, in fact, it is enough to find \(S\) as above such that the Frobenius action on \(H^{N-1}(D_s, \mathcal{O}_{D_s})\) is bijective for all \(s \in S\).

**Lemma 2.2.** Let \(L\) be a finite field, and \(D_1, \ldots, D_r\) hypersurfaces in \(P^N_L\), with \(r \leq N\), such that the intersection scheme \(Y = D_1 \cap \ldots \cap D_r\) has pure codimension \(r\) in \(P^N\). If the Frobenius action on \(H^{N-1}(D, \mathcal{O}_D)\) is bijective, where \(D = \sum_{i=1}^r D_i\), then for every closed subscheme \(X \subset Y\), the Frobenius action on \(H^{N-r}(X, \mathcal{O}_X)\) is bijective.

**Proof.** If \(r = N\), then \(X\) is zero-dimensional and the Frobenius action on \(\Gamma(X, \mathcal{O}_X)\) is bijective since \(L\) is perfect. Therefore from now on we may assume that \(r \leq N - 1\).

For every subset \(J \subseteq \{1, \ldots, r\}\), let \(D_J = \bigcap_{j \in J} D_J\). By assumption, \(Y\) is a complete intersection; hence there is an exact complex

\[
C^\bullet : 0 \to C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{r-1}} C^r \to 0,
\]

where \(C^0 = \mathcal{O}_D\), and \(C^m = \bigoplus_{|J|=m} \mathcal{O}_{D_J}\) for \(m \geq 1\). Note that we have a morphism of complexes \(C^\bullet \to F_\lambda(C^\bullet)\), where \(F\) is the absolute Frobenius morphism on \(X\). It follows that if we break up \(C^\bullet\) into short exact sequences, the maps in the corresponding long exact sequences for cohomology are compatible with the Frobenius action.
Let \( M^i = \text{Im}(d') \); hence \( M^0 \simeq C^0 = O_D \) and \( M^{r-1} = C^r = O_Y \). Since each \( D_J \) is a complete intersection in \( \mathbb{P}^N \), it follows that \( H^i(D_J, O_{D_J}) = 0 \) for every \( i \) with \( 1 \leq i < \dim(D_J) = N - |J| \). We deduce that for every \( i \) with \( 0 \leq i \leq r - 2 \), the short exact sequence
\[
0 \to M^i \to C^{i+1} \to M^{i+1} \to 0
\]
gives an exact sequence
\[
0 = H^{N-i-2}(\mathbb{P}^N, C^{i+1}) \to H^{N-i-2}(\mathbb{P}^N, M^{i+1}) \to H^{N-i-1}(\mathbb{P}^N, M^i).
\]

Therefore we have a sequence of injective maps
\[
H^{N-r}(Y, O_Y) \to H^{N-r+1}(\mathbb{P}^N, M^{r-2}) \to \ldots \to H^{N-2}(\mathbb{P}^N, M^1) \to H^{N-1}(D, O_D),
\]
compatible with the Frobenius action. Since this action is bijective on \( H^{N-1}(D, O_D) \) by hypothesis, it follows that it is bijective also on \( H^{N-r}(Y, O_Y) \) (see, for example, [MS, Lemma 2.4]).

On the other hand, since \( \dim(Y) = N - r \), the surjection \( O_Y \to O_X \) induces a surjection \( H^{N-r}(Y, O_Y) \to H^{N-r}(X, O_X) \), compatible with the Frobenius action. As we have seen, the Frobenius action is bijective on \( H^{N-r}(Y, O_Y) \), hence on every quotient (see [MS, Lemma 2.4]). This completes the proof of the lemma.

Returning to the proof of Theorem 1.3, we see that it is enough to show that there is a dense set of closed points \( S \subset \text{Spec } A \) such that Frobenius acts bijectively on \( H^{N-1}(D_s, O_{D_s}) \) for \( s \in S \). We assume that Conjecture 1.1 holds; hence there is a dense set of closed points \( S \subset \text{Spec } A \) such that \( \tau(A^{N+1}_k, h^\lambda) = J(A^{N+1}_k, h^\lambda)s \) for every \( \lambda \in \mathbb{R}_{\geq 0} \) and every \( s \in S \). In particular, it follows from Lemma 2.1 and [5] that \( (x_0, \ldots, x_N)^{2r-N-1} \subseteq \tau(A^{N+1}_k, h^\lambda) \) for every \( \lambda < 1 \). Since \( \deg(h_s) = 2r \geq (N + 1) \), Proposition 2.3 below implies that the Frobenius action on \( H^{N-1}(D_s, O_{D_s}) \) is bijective for all \( s \in S \). As we have seen, this completes the proof of Theorem 1.3.

**Proposition 2.3.** Let \( L \) be a perfect field of characteristic \( p > 0 \), and let \( h \in R = L[x_0, \ldots, x_N] \) be a homogeneous polynomial of degree \( d \geq N + 1 \), with \( N \geq 2 \). If \( (x_0, \ldots, x_N)^{d-N-1} \subseteq \tau(A^{N+1}_L, h^{1-\frac{1}{p}}) \), then the Frobenius action on \( H^{N-1}(D, O_D) \) is bijective, where \( D \subset P^N_L \) is the hypersurface defined by \( h \).

**Proof.** In the case \( d = N + 1 \), this is a reformulation of a well-known fact due to Fedder [F2]. We follow the argument from [MTW] Proposition 2.16, which extends to our more general setting. It is enough to show that the Frobenius action on \( H^{N-1}(D, O_D) \) is injective (see [MS, §2.1]).

Note first that \( \tau(A^{N+1}_L, h^{1-\frac{1}{p}}) = (h^{p-1})^{1/p} \) (see [BMS1] Lemma 2.1); hence by assumption \( m^{d-N-1} \subseteq (h^{p-1})^{1/p} \), where \( m = (x_0, \ldots, x_N) \). It is convenient to use the interpretation of the ideal \( (h^{p-1})^{1/p} \) in terms of local cohomology. Let \( E = H^{N+1}_m(R) \). Recall that this is a graded \( R \)-module, carrying a natural action of the Frobenius, which we denote by \( F_E \). There is an isomorphism
\[
E \simeq R_{x_0 \cdots x_N} / \sum_{i=0}^N R_{x_0 \cdots \hat{x}_i \cdots x_N}.
\]
Via this isomorphism, \( F_E \) is induced by the Frobenius morphism on \( R_{x_0 \cdots x_N} \).
The annihilator of \((h^{p-1})^{1/p}\) in \(E\) is equal to \(\text{Ker}(h^{p-1}F_E)\) (see, for example, [BMS2 §2.3]). Therefore we have

\[
\text{Ker}(h^{p-1}F_E) \subseteq \text{Ann}_E(m^{d-N-1}) = \bigoplus_{i \geq -d+1} E_i.
\]

On the other hand, the exact sequence

\[
0 \to R(-d) \xrightarrow{h} R \to R/(h) \to 0
\]

induces an isomorphism

\[
H^N_m(R/(h)) \cong \{u \in E \mid hu = 0\}(-d)
\]

such that the Frobenius action on \(H^N_m(R/(h))\) is given by \(h^{p-1}F_E\). Since \(H^{N-1}(D, O_D) \cong H^N_m(R/(h))_0 \to E_{-d}\), (6) implies that the Frobenius action is injective on \(H^{N-1}(D, O_D)\). This completes the proof of the proposition. \(\Box\)

Remark 2.4. In the proof of Theorem 1.3 we only used the inclusion "\(\subseteq\)" in Conjecture 1.1. However, this is the interesting inclusion: the reverse one is known; see [HY] or [MS, Proposition 4.2]. It is more interesting that we only used Conjecture 1.1 when \(Y = A^{n+1}_k\), \(a\) is principal and homogeneous, and \(\lambda = 1 - \frac{1}{p}\). By combining Theorem 1.3 with the main result in [MS], we see that in order to prove Conjecture 1.1 in general, it is enough to consider the case when \(Y = A^n_k\) and \(a = (f)\) is principal and homogeneous, and to show the following: if \(b = \mathcal{J}(Y, a^{1-\varepsilon})\) for \(0 < \varepsilon \ll 1\) and if \(f_A \in A[x_1, \ldots, x_n] \) is a model for \(f\), then there is a dense set of closed points \(S \subset \text{Spec} \; A\) such that

\[
b_s \subseteq (f_s^{p-1})^{1/p}
\]

for every \(s \in S\), where \(p = \text{char}(k(s))\).

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