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ON THE RELATIVE WEAK ASYMPTOTIC HOMOMORPHISM PROPERTY FOR TRIPLES OF GROUP VON NEUMANN ALGEBRAS

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Abstract. A triple of finite von Neumann algebras $B \subset N \subset M$ is said to have the relative weak asymptotic homomorphism property if there exists a net of unitaries $(u_i)_{i\in I} \subset U(B)$ such that

$$\lim_{i\in I} \|\mathbb{E}_B(xu_iy) - \mathbb{E}_B(\mathbb{E}_N(x)u_i\mathbb{E}_N(y))\|_2 = 0$$

for all $x, y \in M$. Recently, J. Fang, M. Gao and R. Smith proved that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if N contains the set of all $x \in M$ such that $Bx \subset \sum_{i=1}^n x_iB$ for finitely many elements $x_1, \dots, x_n \in M$. Furthermore, if $H < G$ is a pair of groups, they get a purely algebraic characterization of the weak asymptotic homomorphism property for the pair of von Neumann algebras $L(H) \subset L(G)$, but their proof requires a result which is very general and whose proof is rather long. We extend the result to the case of a triple of groups $H < K < G$, we present a direct and elementary proof of the above-mentioned characterization, and we introduce three more equivalent conditions on the triple $H < K < G$, one of them stating that the subspace of H-compact vectors of the quasi-regular representation of H on $\ell^2(G/H)$ is contained in $\ell^2(K/H)$.

1. Introduction

Let $1 \in B \subset N \subset M$ be a triple of finite von Neumann algebras endowed with a fixed, normal, finite, faithful and normalized trace τ. Then $\mathbb{E}_N$ (resp. $\mathbb{E}_B$) denotes the τ-preserving conditional expectation from M onto N (resp. B); we also set $M \ominus N = \{x \in M : \mathbb{E}_N(x) = 0\}$.

Following [3], we say that the triple $B \subset N \subset M$ has the *relative weak asymptotic homomorphism property* if there exists a net of unitaries $(u_i)_{i\in I} \subset U(B)$ such that, for all $x, y \in M$,

$$\lim_{i\in I} \|\mathbb{E}_B(xu_iy) - \mathbb{E}_B(\mathbb{E}_N(x)u_i\mathbb{E}_N(y))\|_2 = 0.$$

Using the identity

$$\mathbb{E}_B(xuy) - \mathbb{E}_B(\mathbb{E}_N(x)u\mathbb{E}_N(y)) = \mathbb{E}_B([x - \mathbb{E}_N(x)]u[y - \mathbb{E}_N(y)])$$

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which holds for every $u \in U(B)$ and all $x, y \in M$, it is readily seen that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if one can find a net $(u_i)_{i\in I} \subset U(B)$ such that

$$\lim_{i\in I} \|\mathbb{E}_B(xu_iy)\|_2 = 0$$

for all $x, y \in M \ominus N$.

The *one-sided quasi-normalizer of B in M* is the set of elements $x \in M$ for which there exist finitely many elements $x_1, \dots, x_n \in M$ such that $Bx \subset \sum_{i=1}^n x_iB$. It is denoted by $q\mathcal{N}_M^{(1)}(B)$.

Inspired by [2], the authors of [3] prove in Theorem 3.1 that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if $q\mathcal{N}_M^{(1)}(B) \subset N$. Furthermore, they also study the case of group algebras that we recall now.

Let G be a discrete group and let H be a subgroup of G. Then there is a natural analogue of the one-sided quasi-normalizer for such a pair of groups: we denote by $q\mathcal{N}_G^{(1)}(H)$ the set of elements $g \in G$ for which there exist finitely many elements $g_1, \dots, g_n \in G$ such that $Hg \subset \bigcup_{i=1}^n g_iH$.

Thus, if $H < K < G$ is a triple of groups and if $B = L(H) \subset N = L(K) \subset M = L(G)$ denotes the triple of von Neumann algebras associated to $H < K < G$, then it is reasonable to ask whether $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if $q\mathcal{N}_G^{(1)}(H) \subset K$. Corollary 5.4 in [3] states that this is indeed true when $K = H$, but the proof presented there relies heavily on the main theorem of the article. It is thus natural to look for a more direct and elementary proof of the above-mentioned result, and the aim of the present paper is to provide such a proof and to add three more equivalent conditions.

2. The main result

Before stating our result, let us fix some additional notation. For each element $g \in G$ we denote by λ_g the unitary operator acting by left translation on $\ell^2(G)$, i.e. $(\lambda_g\xi)(g') = \xi(g^{-1}g')$ for every $\xi \in \ell^2(G)$ and every $g' \in G$. We denote also by $L_f(G)$ the subalgebra of all elements of $L(G)$ with finite support; i.e., $L_f(G)$ is the linear span of $\lambda(G)$ in $B(\ell^2(G))$.

We fix a triple of groups $H < K < G$ for the rest of the article.

Let π denote the quasi-regular representation of G on $\ell^2(G/H)$; we denote by $[g]$ the equivalence class $[g] = gH$, so that $\pi(g)\xi([g']) = \xi([g^{-1}g'])$ for all $g, g' \in G$ and $\xi \in \ell^2(G/H)$. Following [1], we say that a vector $\xi \in \ell^2(G/H)$ is *H-compact* if the norm closure of its H-orbit $\{\pi(h)\xi : h \in H\}$ is a compact subset of $\ell^2(G/H)$. The set of all H-compact vectors is a closed subspace of $\ell^2(G/H)$ that we denote by $\ell^2(G/H)_{c,H}$. We also set $\ell^2(G/H)^H = \{\xi \in \ell^2(G/H) : \pi(h)\xi = \xi \ \forall h \in H\}$, which is the subspace of all H-invariant vectors of $\ell^2(G/H)$. It is contained in $\ell^2(G/H)_{c,H}$.

Theorem 2.1. *Let $H < K < G$ and $B = L(H) \subset N = L(K) \subset M = L(G)$ be as above. Then the following conditions are equivalent:*

(1) *There exists a net $(h_i)_{i\in I} \subset H$ such that, for all $x, y \in M \ominus N$, one has*

$$\lim_{i\in I} \|\mathbb{E}_B(x\lambda_{h_i}y)\|_2 = 0;$$

i.e., the net of unitaries in the relative weak asymptotic homomorphism property may be chosen in the subgroup $\lambda(H)$ of $U(B)$.

(2) *The triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property.*

(3) *If $g \in G$ and $F \subset G$ finite are such that $Hg \subset FH$, then $g \in K$, i.e. $q\mathcal{N}_G^{(1)}(H) \subset K$.*

(4) *The subspace of H-compact vectors $\ell^2(G/H)_{c,H}$ is contained in $\ell^2(K/H)$.*

(5) *The subspace of H-fixed vectors $\ell^2(G/H)^H$ is contained in $\ell^2(K/H)$.*

(6) *For every nonempty finite set $F \subset G \setminus K$, there exists $h \in H$ such that*

$$FhF \cap H = \emptyset.$$

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Observe that condition (3) is equivalent to the following statement (since, if $g \notin K$, then $HgH \cap K = \emptyset$):

For every $g \in G \setminus K$, and for every nonempty finite set $F \subset G \setminus K$, there exists $h \in H$ such that $Fhg \cap H = \emptyset$.

Thus, let us assume that condition (3) does not hold. There exists $g \in G \setminus K$ and a nonempty finite set $F \subset G \setminus K$ such that $Fhg \cap H \neq \emptyset$ for every $h \in H$. Then let $u \in U(B)$. One has:

$$\begin{aligned}
\sum_{g' \in F} \|\mathbb{E}_B(\lambda_{g'} u \lambda_g)\|_2^2 &= \sum_{g' \in F} \left(\sum_{h \in H, g'hg \in H} |u(h)|^2 \right) \\
&= \sum_{h \in H} \left(\sum_{g' \in F, g'hg \in H} |u(h)|^2 \right) \\
&\geq \sum_{h \in H} |u(h)|^2 = \|u\|_2^2 = 1
\end{aligned}$$

since, for every $h \in H$, one can find $g'(h) \in F$ such that $g'(h)hg \in H$. Hence there cannot exist a net $(u_i)_{i \in I} \subset U(B)$ as above, and the triple $B \subset N \subset M$ does not have the relative weak asymptotic homomorphism property.

(3) $\Rightarrow$ (4). We choose a set of representatives $T \ni e$ of left classes so that $G = \bigsqcup_{t \in T} tH$, and let $\xi \neq 0$ be an H-compact vector.

Let $s \in T$ be such that $\epsilon := |\xi([s])| > 0$. There exist then finitely many vectors $\xi_1, \ldots, \xi_n \in \ell^2(G/H)$ such that, for every $h \in H$, there exists $1 \leq j \leq n$ such that $\|\pi(h)\xi - \xi_j\| \leq \epsilon/2$. Set

$$F = \bigcup_{j=1}^{n} \{t \in T : |\xi_j([t])| \geq \epsilon/2\},$$

which is a finite set. Then we claim that $Hs \subset FH$. Indeed, if $h \in H$, let $t \in T$ be such that $[hs] = [t]$, and let j be such that $\|\pi(h)\xi - \xi_j\| \leq \epsilon/2$. Then

$$\epsilon - |\xi_j([t])| = |\xi([s])| - |\xi_j([t])| \leq |\xi([s]) - \xi_j([hs])| \leq \|\pi(h)\xi - \xi_j\| \leq \epsilon/2;$$

hence $\epsilon/2 \leq |\xi_j([t])|$ and $t \in F$. Thus $Hs \subset FH$, and condition (3) implies that $s \in K$. This proves that $\xi \in \ell^2(K/H)$.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (6). Let us assume that the triple $H < K < G$ satisfies condition (5) but not (6). Then there exists a finite set $F = F^{-1} \subset G \setminus K$ such that $FhF \cap H \neq \emptyset$

for every $h \in H$. Set

$$\xi = \sum_{g \in F} \delta_{[g]}.$$

Then $\xi \perp \ell^2(K/H)$, and one has for every $h \in H$:

$$\langle \pi(h)\xi, \xi \rangle = \sum_{g,g' \in F} \langle \delta_{[hg]}, \delta_{[g']} \rangle \geq 1$$

since the condition on F implies that for every $h \in H$, there exist $g, g' \in F$ such that $hgH = g'H$. Let C be the closed convex hull of $\{\pi(h)\xi : h \in H\}$. Then it is easy to see that $\langle \zeta, \xi \rangle \geq 1$ for every $\zeta \in C$. Let $\eta \in C$ be the vector with minimal norm. By its uniqueness, it is H-invariant and nonzero by the above observation. Thus, η is supported in K/H and orthogonal to $\ell^2(K/H)$ since ξ is. This is the expected contradiction.

(6) $\Rightarrow$ (1). Let $I = \{F \subset G \setminus K : F \neq \emptyset, \text{ finite}\}$ be the directed set of all nonempty finite subsets of $G \setminus K$. Condition (6) states that, for every $F \in I$, there exists $h_F \in H$ such that $Fh_FF \cap H = \emptyset$. Let x and y in $L_f(G)$ satisfy $\mathbb{E}_N(x) = \mathbb{E}_N(y) = 0$. Then let $F_0 \in I$ be chosen so that the supports of x and y are contained in F_0. Then $x\lambda_{h_F}y = \sum_{g,g' \in F_0} x(g)y(g')\lambda_{gh_Fg'}$ for every $F \supset F_0$; thus $\mathbb{E}_B(x\lambda_{h_F}y) = 0$ for every $F \supset F_0$. This proves that the triple $B \subset N \subset M$ satisfies condition (1) by the density of $L_f(G)$ in $L(G)$. $\square$

Remark 2.2. In the case of a pair of groups $H < G$, which corresponds to $H = K$, condition (4) means that all H-invariant vectors in $\ell^2(G/H)$ are multiples of $\delta_{[e]}$, and this means that the unitary representation ρ of H on the subspace $\ell^2(G/H) \ominus \mathbb{C}\delta_{[e]}$ is ergodic in the sense of [1].

References

1. V. Bergelson and J. Rosenblatt, *Mixing actions of groups*, Illinois J. Math. **32** (1988), 65–80. MR921351 (89g:28029)
2. I. Chifan, *On the normalizing algebra of a MASA in a II_1 factor*, arXiv:math.OA/0606225, 2006.
3. J. Fang, M. Gao and R. R. Smith, *The relative weak asymptotic homomorphism property for inclusions of finite von Neumann algebras*, arXiv:math.OA/1005.3049 v1, 2010.

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