

GENERALIZATION OF A THEOREM OF CLUNIE AND HAYMAN

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ABSTRACT. Clunie and Hayman proved that if the spherical derivative $\|f'\|$ of an entire function satisfies $\|f'\|(z) = O(|z|^\sigma)$, then $T(r, f) = O(r^{\sigma+1})$. We generalize this to holomorphic curves in projective space of dimension n omitting n hyperplanes in general position.

INTRODUCTION

We consider holomorphic curves $f : \mathbf{C} \rightarrow \mathbf{P}^n$; for the general background on the subject we refer to [7]. The Fubini–Study derivative $\|f'\|$ measures the length distortion from the Euclidean metric in \mathbf{C} to the Fubini–Study metric in \mathbf{P}^n . The explicit expression is

$$\|f'\|^2 = \|f\|^{-4} \sum_{i < j} |f'_i f_j - f_i f'_j|^2,$$

where (f_0, \dots, f_n) is a homogeneous representation of f (that is, the f_j are entire functions which never simultaneously vanish), and

$$\|f\|^2 = \sum_{j=0}^n |f_j|^2.$$

See [3] for a general discussion of the Fubini–Study derivative.

We recall that the Nevanlinna–Cartan characteristic is defined by

$$T(r, f) = \int_0^r \frac{dt}{t} \left(\frac{1}{\pi} \int_{|z| \leq t} \|f'\|^2(z) dm(z) \right),$$

where dm is the area element in \mathbf{C} . So the condition

$$(1) \quad \limsup_{z \rightarrow \infty} |z|^{-\sigma} \|f'(z)\| \leq K < \infty$$

implies

$$(2) \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{2\sigma+2}} < \infty.$$

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Clunie and Hayman [4] found that for curves $\mathbf{C} \rightarrow \mathbf{P}^1$ omitting one point in \mathbf{P}^1 , a stronger conclusion follows from (1), namely

$$(3) \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\sigma+1}} \leq KC(\sigma).$$

In the most important case of $\sigma = 0$, a different proof of this fact for $n = 1$ is due to Pommerenke [8]. Pommerenke’s method gives the exact constant $C(0)$. In this paper we prove that this phenomenon persists in all dimensions.

Theorem. *For holomorphic curves $f : \mathbf{C} \rightarrow \mathbf{P}^n$ omitting n hyperplanes in general position, condition (1) implies (3) with an explicit constant $C(n, \sigma)$.*

In [6], the case $\sigma = 0$ was considered. There it was proved that holomorphic curves in \mathbf{P}^n with bounded spherical derivative and omitting n hyperplanes in general position must satisfy $T(r, f) = O(r)$. With a stronger assumption that f omits $n + 1$ hyperplanes this was earlier established by Berteloot and Duval [2] and by Tsukamoto [9]. The proof in [6] has two drawbacks: it does not extend to arbitrary $\sigma \geq 0$, and it is non-constructive; unlike Clunie–Hayman and Pommerenke’s proofs mentioned above, it does not give an explicit constant in (3).

It is shown in [6] that the condition that n hyperplanes are omitted is exact: there are curves in any dimension n satisfying (1), $T(r, f) \sim cr^{2\sigma+2}$ and omitting $n - 1$ hyperplanes.

PRELIMINARIES

Without loss of generality we assume that the omitted hyperplanes are given in the homogeneous coordinates by the equations $\{w_j = 0\}$, $1 \leq j \leq n$. We fix a homogeneous representation (f_0, \dots, f_n) of our curve, where f_j are entire functions and $f_n = 1$. Then

$$(4) \quad u = \log \sqrt{|f_0|^2 + \dots + |f_n|^2}$$

is a positive subharmonic function, and Jensen’s formula gives

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta})d\theta - u(0) = \int_0^r \frac{n(t)}{t} dt,$$

where $n(t) = \mu(\{z : |z| \leq t\})$ and $\mu = \mu_u$ is the Riesz measure of u , that is, the measure with the density

$$(5) \quad \frac{1}{2\pi} \Delta u = \frac{1}{\pi} \|f'\|^2.$$

This measure μ is also called Cartan’s measure of f . Positivity of u and (2) imply that all f_j are of order at most $2\sigma + 2$, normal type. As $f_j(z) \neq 0$, $1 \leq j \leq n$, we conclude that

$$f_j = e^{P_j}, \quad 1 \leq j \leq n,$$

where

$$(6) \quad P_j \text{ are polynomials of degree at most } 2\sigma + 2.$$

We need two lemmas from potential theory.

Lemma 1 ([6]). *Let v be a non-negative harmonic function in the closure of the disc $B(a, R)$, and assume that $v(z_1) = 0$ for some point $z_1 \in \partial B(a, R)$. Then*

$$v(a) \leq 2R|\nabla v(z_1)|.$$

We include a proof, suggested by the referee, which is simpler than that given in [6]. Without loss of generality, assume that $a = 0, R = 1, z_1 = 1$. Then Harnack's inequality gives

$$\frac{v(0)}{1+r} \leq \frac{v(r)}{1-r} = \frac{v(r) - v(1)}{1-r}.$$

Passing to the limit as $r \rightarrow 1$, we obtain the result.

Lemma 2. *Let v be a non-negative superharmonic function in the closure of the disc $B(a, R)$ and suppose that $v(z_1) = 0$ for some $z_1 \in \partial B(a, R)$. Then*

$$|\mu_v(B(a, R/2))| \leq 3R \left| \frac{\partial v}{\partial n}(z_1) \right|.$$

By $|\partial v / \partial n|$ we mean here $\liminf |v(rz_1)| / (R(1-r))$ as $r \rightarrow 1-$.

Proof. The function $v(a + Rz)$ satisfies the conditions of the lemma with $R = 1$. So it is enough to prove the lemma with $a = 0$ and $R = 1$. Let

$$w(z) = \int_{|\zeta| \leq 1/2} G(z, \zeta) d\mu_v(\zeta)$$

be the Green potential of the restriction of μ_v onto the disc $|\zeta| \leq 1/2$, that is,

$$G(z, \zeta) = \log \left| \frac{1 - \bar{\zeta}z}{z - \zeta} \right|.$$

Then $w \leq v$ and $w(z_1) = v(z_1) = 0$, which implies that

$$\left| \frac{\partial v}{\partial n}(z_1) \right| \geq \left| \frac{\partial w}{\partial |z|}(z_1) \right|.$$

Minimizing $|\partial G / \partial |z||$ over $|z| = 1$ and $|\zeta| = 1/2$, we obtain $1/3$, which proves the lemma. □

PROOF OF THE THEOREM

We may assume without loss of generality that f_0 has infinitely many zeros. Indeed, we can compose f with an automorphism of \mathbf{P}^n ; for example replace f_0 by $f_0 + cf_1$, $c \in \mathbf{C}$, and leave all other f_j unchanged. This transformation changes neither the n omitted hyperplanes nor the rate of growth of $T(r, f)$ and multiplies the spherical derivative by a bounded factor.

Let $u_j = \log |f_j|$ and

$$u^* = \max_{1 \leq j \leq n} u_j.$$

Here and in what follows \max denotes the pointwise maximum of subharmonic functions.

Proposition 1. *Suppose that at some point z_1 we have*

$$u_m(z_1) = u_k(z_1) \geq u_j(z_1)$$

for some $m \neq k$ and all j where $m, k, j \in \{0, \dots, n\}$. Then

$$\|f'(z_1)\| \geq (n + 1)^{-1} |\nabla u_m(z_1) - \nabla u_k(z_1)|.$$

Proof.

$$\|f'(z_1)\| \geq \frac{|f'_m(z_1)f_k(z_1) - f_m(z_1)f'_k(z_1)|}{|f_0(z_1)|^2 + \dots + |f_n(z_1)|^2} \geq (n + 1)^{-1} \left| \frac{f'_m(z_1)}{f_m(z_1)} - \frac{f'_k(z_1)}{f_k(z_1)} \right|,$$

and the conclusion of the proposition follows since $|\nabla \log |f|| = |f'/f|$. □

Proposition 2. *For every $\epsilon > 0$, we have*

$$u(z) \leq u^*(z) + K(2 + \epsilon)^{\sigma+1}(n + 1)|z|^{\sigma+1}$$

for all $|z| > r_0(\epsilon)$.

Proof. If $u_0(z) \leq u^*(z)$ for all sufficiently large $|z|$, then there is nothing to prove. Suppose that $u_0(a) > u^*(a)$, and consider the largest disc $B(a, R)$ centered at a where the inequality $u_0(z) > u^*(z)$ persists. If z_0 is the zero of the smallest modulus of f_0 , then $R \leq |a| + |z_0| < (1 + \epsilon)|a|$ when $|a|$ is large enough.

There is a point $z_1 \in \partial B(a, R)$ such that $u_0(z_1) = u^*(z_1)$. This means that there is some $k \in \{1, \dots, n\}$ such that $u_0(z_1) = u_k(z_1) \geq u_m(z_1)$ for all $m \in \{1, \dots, n\}$. Applying Proposition 1 we obtain

$$|\nabla u_k(z_1) - \nabla u_0(z_1)| \leq (n + 1)\|f'(z_1)\|.$$

Now $u_0(z) > u^*(z) \geq u_k(z)$ for $z \in B(a, R)$, so we can apply Lemma 1 to $v = u_0 - u_k$ in the disc $B(a, R)$. This gives

$$u_0(a) - u_k(a) \leq 2R|\nabla u_k(z_1) - \nabla u_0(z_1)| \leq 2R(n + 1)\|f'(z_1)\|.$$

Now $R < (1 + \epsilon)|a|$ and $|z_1| \leq (2 + \epsilon)|a|$, so

$$u_0(a) \leq u^*(a) + K(2 + \epsilon)^{\sigma+1}(n + 1)|a|^{\sigma+1},$$

and the result follows because $u = \max\{u_0, u^*\} + O(1)$. □

Next we study the Riesz measure of the subharmonic function

$$u^* = \max\{u_1, \dots, u_n\}.$$

We begin with the maximum of two harmonic functions. Let u_1 and u_2 be two harmonic functions in \mathbf{C} of the form $u_j = \text{Re } P_j$ where $P_j \neq 0$ are polynomials. Suppose that $u_1 \neq u_2$. Then the set $E = \{z \in \mathbf{C} : u_1(z) = u_2(z)\}$ is a proper real-algebraic subset of $\overline{\mathbf{C}}$ without isolated points. Apart from a finite set of ramification points, E consists of smooth curves. For every smooth point $z \in E$, we denote by $J(z)$ the jump of the normal (to E) derivative of the function $w = \max\{u_1, u_2\}$ at the point z . This jump is always positive and the Riesz measure μ_w is given by the formula

$$(7) \quad d\mu_w = \frac{J(z)}{2\pi}|dz|,$$

which means that μ_w is supported by E and has a density $J(z)/2\pi$ with respect to the length element $|dz|$ on E .

Now let $E_{i,j} = \{z : u_i(z) = u_j(z) \geq u_k(z), 1 \leq k \leq n\}$, and let $E = \bigcup E_{i,j}$ where the union is taken over all pairs $1 \leq i, j \leq n$ for which $u_i \neq u_j$. Then E is a proper real semi-algebraic subset of $\overline{\mathbf{C}}$ and ∞ is not an isolated point of E . For the elementary properties of semi-algebraic sets that we use here, see, for example,

[1, 5]. There exists $r_0 > 0$ such that $\Gamma = E \cap \{r_0 < |z| < \infty\}$ is a union of finitely many disjoint smooth simple curves,

$$\Gamma = \bigcup_{k=1}^m \Gamma_k.$$

This union coincides with the support of μ_{u^*} in $\{z : r_0 < |z| < \infty\}$.

Consider a point $z_0 \in \Gamma$. Then $z_0 \in \Gamma_k$ for some k . As Γ_k is a smooth curve, there is a neighborhood D of z_0 which does not contain other curves Γ_j , $j \neq k$, and which is divided by Γ_k into two parts, D_1 and D_2 . Then there exist i and j such that $u^*(z) = u_i(z)$, $z \in D_1$ and $u^*(z) = u_j(z)$, $z \in D_2$, and $u^*(z) = \max\{u_i(z), u_j(z)\}$, $z \in D$. So the restriction of the Riesz measure μ_{u^*} on D is supported by $\Gamma_k \cap D$ and has density $J(z)/(2\pi)$ where

$$|J(z)| = |\partial u_i / \partial n - \partial u_j / \partial n|(z) = |\nabla(u_i - u_j)|(z)$$

and $\partial/\partial n$ is the derivation in the direction of a normal to Γ_k . Taking into account that $u_j = \text{Re } P_j$ where P_j are polynomials, we conclude that there exist positive numbers c_k and b_k such that

$$(8) \quad J(z)/(2\pi) = (c_k + o(1))|z|^{b_k}, \quad z \rightarrow \infty, \quad z \in \Gamma_k.$$

Let $b = \max_k b_k$, and among those curves Γ_k for which $b_k = b$ choose one with maximal c_k (which we denote by c_0). We denote this chosen curve by Γ_0 and fix it for the rest of the proof.

Proposition 3. *We have*

$$b \leq \sigma \quad \text{and} \quad c_0 \leq 3 \cdot 4^\sigma K(n + 1).$$

Proof. We consider two cases.

Case 1. There is a sequence $z_n \rightarrow \infty$, $z_n \in \Gamma_0$, such that $u_0(z_n) \leq u^*(z_n)$. Then (1) and Proposition 1 imply that

$$J(z_n) \leq (n + 1)K|z_n|^\sigma,$$

and comparison with (8) shows that $b \leq \sigma$ and $c_0 \leq K(n + 1)/(2\pi)$.

Case 2. $u_0(z) > u^*(z)$ for all sufficiently large $z \in \Gamma_0$. Let a be a point on Γ_0 , $|a| > 3r_0$, and $u_0(a) > u^*(a)$. Let $B(a, R)$ be the largest open disc centered at a in which the inequality $u_0(z) > u^*(z)$ holds. Then

$$(9) \quad R \leq |a| + O(1), \quad a \rightarrow \infty,$$

because we assume that f_0 has zeros, so $u_0(z_0) = -\infty$ for some z_0 .

In $B(a, R)$ we consider the positive superharmonic function $v = u_0 - u^*$. Let us check that it satisfies the conditions of Lemma 2. The existence of a point $z_1 \in \partial B(a, R)$ with $v(z_1) = 0$ follows from the definition of $B(a, R)$. The Riesz measure of μ_v is estimated using (7), (8):

$$|\mu_v(B(a, R/2))| \geq |\mu_v(\Gamma_0 \cap B(a, R/2))| \geq c_0 R(|a| - R/2)^b.$$

Now Lemma 2 applied to v in $B(a, R)$ implies that

$$(10) \quad |\nabla v(z_1)| \geq (c_0/3)(|a| - R/2)^b.$$

On the other hand (1) and Proposition 1 imply that

$$|\nabla v(z_1)| \leq K(n + 1)(|a| + R)^\sigma.$$

Combining these two inequalities and taking (9) into account, we obtain $b \leq \sigma$ and $c_0 \leq 3 \cdot 4^\sigma K(n+1)$, as required. \square

We denote

$$T^*(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^*(re^{i\theta}) d\theta - u^*(0).$$

This is the characteristic of the “reduced curve” (f_1, \dots, f_n) .

Proposition 4.

$$T^*(r) \leq 6 \cdot 4^\sigma K \frac{n(n+1)^2}{\sigma+1} r^{\sigma+1}.$$

Proof. By Jensen’s formula,

$$T^*(r) = \int_0^r \nu(t) \frac{dt}{t},$$

where $\nu(t) = \mu_{u^*}(\{z : |z| \leq t\})$. The number of curves Γ_k supporting the Riesz measure of u^* is easily seen to be at most $2n(n-1)(\sigma+1)$. The density of the Riesz measure μ_{u^*} on each curve Γ_k is given by (8), where $c_k \leq c_0$ and $b_k \leq b$ and the parameters c_0 and b are estimated in Proposition 3. Combining all these data, we obtain the result. \square

It remains to combine Propositions 2 and 4 to obtain the final result.

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