ON THE NORMS OF DISCRETE ANALOGUES 
OF CONVOLUTION OPERATORS

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Abstract. We consider a discrete analogue of convolution operator \( T(f) = K \ast f \) from \( L^p(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \): \( T_{dis}(g) = K_{dis} \ast g \) from \( \ell^p(\mathbb{Z}^d) \) to \( \ell^q(\mathbb{Z}^d) \) where \( K_{dis} = K|_{\mathbb{Z}^d} \) and \( K \) is supported in the fundamental cube. We show that the estimate \( \|T_{dis}\|_p \leq C^d\|T\|_p \) with \( C > 1 \) cannot be improved for a certain range of \( p \).

We consider a convolution operator \( T(f) = K \ast f \) from \( L^p(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \) with a distribution kernel \( K \) whose Fourier transform \( \hat{K}(\xi) \) is supported in the fundamental cube \( Q = \{ \xi = (\xi_1, \xi_2, \ldots, \xi_d) : -\frac{1}{2} < \xi_i \leq \frac{1}{2}, i = 1, 2, \ldots, d \} = (-\frac{1}{2}, \frac{1}{2})^d \). The Fourier transform \( (T(f))^\wedge = \hat{K}(\xi)\hat{f}(\xi) \). It is well known that if \( T(f) = K \ast f \) acts from \( L^p \) to \( L^q \), then it also acts from \( L^p(\mathbb{Z}^d) \) to \( L^q(\mathbb{Z}^d) \) with the same norm. Thus we restrict ourselves to the case \( 1 \leq p \leq 2 \). In particular, we can say that \( T(f) = K \ast f \) is an operator from \( L^r \) to \( L^r \) where \( p \leq r \leq 2 \). We will be interested in the norms of \( T \) from \( L^p \) to \( L^p \). If \( p = 1 \), then \( \|T\|_1 = \|K\|_1 \), and if \( p = 2 \), then \( \|T\|_2 = \|K\|_\infty \).

Interpolating, we get \( \|T\|_p \leq \|K\|_1^{\frac{1}{p}-1} \|K\|_\infty^{\frac{1}{p}} \) for \( 1 < p < 2 \).

As we mentioned before, \( T \) is also a bounded operator from \( L^2 \) to \( L^2 \) and therefore \( \hat{K}(\xi) \) is bounded, which makes \( K(x) = \int_{\mathbb{R}^d} \hat{K}(\xi)e^{i2\pi\xi x}d\xi \) an infinitely differentiable function extendable to an entire function of exponential type. We define \( K_{dis} = K|_{\mathbb{Z}^d} \) and \( T_{dis}(g) = K_{dis} \ast g \) as an operator on \( \mathbb{Z}^d \):

\[
T_{dis}(g)(n) = \sum_{m\in\mathbb{Z}^d} K(n - m)g(m).
\]

Since \( \hat{K} \) is supported in the fundamental cube \( Q \), we can view \( K(n) \) as its Fourier coefficients and thus \( \hat{K}(\xi) = \sum_{n\in\mathbb{Z}^d} K(n)e^{i2\pi\xi n} \chi_Q(\xi) \); i.e., \( K_{dis} \) and \( K \) are determined by each other. We will be interested in norms of \( T_{dis} \) from \( \ell^p(\mathbb{Z}^d) \) to \( \ell^q(\mathbb{Z}^d) \). What was said above about \( \|T\|_p \) is also true for \( \|T_{dis}\|_p \). If \( p = 1 \), then \( \|T_{dis}\|_1 = \|K_{dis}\|_1 \), and if \( p = 2 \), then \( \|T_{dis}\|_2 = \|K\|_\infty \). Interpolating, we get \( \|T_{dis}\|_p \leq \|K_{dis}\|_1^{\frac{1}{p}-1} \|K\|_\infty^{\frac{1}{p}} \) for \( 1 < p < 2 \). We will find the following estimate useful.

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Proposition 1.

(1) \[ \|T_{\text{dis}}\|_p \geq \|K_{\text{dis}}\|_p. \]

To show this, just take \( g(n) = \delta_0(n) \).

The main question we are interested in is how \( \|T_{\text{dis}}\|_p \) is controlled by \( \|T\|_p \). As we said earlier, we restrict ourselves to the case \( 1 \leq p \leq 2 \) since \( \|T_{\text{dis}}\|_{p'} = \|T_{\text{dis}}\|_p \) and \( \|T\|_{p'} = \|T\|_p \). The following result is due to Magyar, Stein and Wainger [1, Proposition 2.1]: If \( T \) is a bounded operator from \( L^p(\mathbb{R}^d) \) to \( L^p(\mathbb{R}^d) \), then \( T_{\text{dis}} \) is a bounded operator from \( L^p(\mathbb{Z}^d) \) to \( L^p(\mathbb{Z}^d) \) and

(2) \[ \|T_{\text{dis}}\|_p \leq C(d)\|T\|_p \]

where \( C \) depends only on the dimension \( d \) but not on \( p \).

The actual statement is more general. Magyar, Stein and Wainger raised the following question in their paper (see Remark 1 after Proposition 2.1 in [1]): Is it possible to take \( C \) independently of the dimension \( d \) and, in particular, if \( C = 1 \)?

We will answer this question negatively.

Theorem 1. There is \( p_0 \) with \( 1 < p_0 < 2 \) such that for all \( p \) with \( 1 \leq p < p_0 \) the optimal \( C(d) \) in [2] has a lower bound \( C(d) \geq Bd \) where \( B > 1 \).

Actually the proof of Proposition 2.1 in [1] shows that \( C(d) \leq A^d \), and thus this estimate of \( C(d) \) cannot be improved. If \( p = 2 \), then \( \|T\|_2 = \|T_{\text{dis}}\|_2 = \|K\|_{\infty} \).

Therefore, \( C(d) = 1 \) for \( p = 2 \). If \( K \geq 0 \), then \( \|T\|_p = \|T_{\text{dis}}\|_p = \|K\|_{\infty} = \|K\|_1 \).

Thus \( C(d) \geq 1 \).

Proof of Theorem 1. First we consider the case when the dimension \( d = 1 \) and \( p = 1 \). We will construct a kernel \( K \) such that \( \|T_{\text{dis}}\|_1 = \sum_{n \in \mathbb{Z}} |K(n)| > \int_{\mathbb{R}} |K(x)|dx = \|T\|_1 \). Let \( \phi(x) \) be a nonnegative Schwartz function whose Fourier transform \( \hat{\phi}(\xi) \) is also nonnegative and supported in \( [-\frac{1}{2}, \frac{1}{2}] \), for example, \( \phi(x) = \psi(x) \cdot \hat{\psi}(x) \) where \( \psi(\xi) \) is nonnegative and supported in \( [-\frac{1}{4}, \frac{1}{4}] \). Define \( \hat{K}(\xi) = \hat{\phi}(2(\xi - \frac{1}{2})) + \hat{\phi}(2(\xi + \frac{1}{2})) \). Then \( K(\xi) \) is supported in \( [-\frac{1}{2}, \frac{1}{2}] \) and

\[ K(x) = \cos \left( \frac{\pi x}{2} \right) \phi \left( \frac{x}{2} \right). \]

Therefore

\[ |K(n)| = \begin{cases} \phi \left( \frac{n}{2} \right) & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases} \]

Hence,

\[ \sum_{n \in \mathbb{Z}} |K(n)| = \sum_{m \in \mathbb{Z}} \phi(m) = \hat{\phi}(0). \]
Now we will estimate $\|K\|_1$:

$$
\int |K(x)| dx = \int \phi\left(\frac{x}{2}\right) |\cos \frac{\pi x}{2}| dx
= 2 \int \phi(x) |\cos \pi x| dx
> 2 \int \phi(x) \cos^2 \pi x dx
= \int \phi(x) (1 + \cos 2\pi x) dx
= \hat{\phi}(0) + \frac{1}{2} \hat{\phi}(1) + \frac{1}{2} \hat{\phi}(-1) = \hat{\phi}(0).
$$

Thus we obtain $\int |K(x)| dx > \sum_{n \in \mathbb{Z}} |K(n)|$. This is the opposite of what we need to get. Now consider a family of kernels $K_t(x) = K(x + t)$ where $0 \leq t \leq 1$. We have $\|K_t\|_1 = \|K\|_1$. On the other hand $\int_0^1 \sum_{n \in \mathbb{Z}} |K(n+t)| dt = \int |K(x)| dx$. Since for $t = 0$ we have $\int |K(x)| dx > \sum_{n \in \mathbb{Z}} |K(n)|$, there exists $t$ such that $\int |K(x)| dx < \sum_{n \in \mathbb{Z}} |K_t(n)|$.

For this kernel $K$, we have $\|T_{dis}\|_1 > \|T\|_1$. Basically, the idea is to show that $h(t) = \sum_{n \in \mathbb{Z}} |K(n+t)|$ is not a constant. This can also be done by calculating Fourier coefficients $\hat{h}(m) = |K|^\wedge(m)$ and showing that they do not vanish; take for example $K(x) = \left(\frac{\sin \frac{x}{\pi}}{\pi x}\right)^3$.

Now we consider the case $p > 1$. Suppose $T$ is a bounded operator from $L^1$ to $L^1$. Then $T_{dis}$ is also a bounded operator from $\ell^1$ to $\ell^1$. As we said earlier, $\|T\|_\infty = \|T\|_1$ and $\|T_{dis}\|_\infty = \|T_{dis}\|_1$. Interpolating between $p = 1$ and $p = \infty$, we have that $T$ and $T_{dis}$ are bounded operators from $L^p$ to $L^p$ and $\log \|T\|_p$ and $\log \|T_{dis}\|_p$ are convex functions. Thus $\|T\|_p$ and $\|T_{dis}\|_p$ are continuous functions for $p \geq 1$. Let $K$ be a kernel such that $\|T_{dis}\|_1 > \|T\|_1$. Then $\|T_{dis}\|_p > \|T\|_p$ for some range of $p$: $1 \leq p < p_0$. We can prove this in a different way which also uses the continuous dependence of $\|K_{dis}\|_p$ on $p$. Recall from (4) that $\|T_{dis}\|_p \geq \|K_{dis}\|_p$ and $\|T\|_p \leq \|K\|_1$. If $K$ is a kernel such that $\sum_{n \in \mathbb{Z}} |K(n)| > \int |K(x)| dx$, then

$$
\|T_{dis}\|_p \geq \left(\sum_{n \in \mathbb{Z}} |K(n)|^p\right)^{\frac{1}{p}}
> \int |K(x)| dx \geq \|T\|_p
$$

for some range of $p$: $1 \leq p < p_0$.

Now we will do the case when the dimension $d > 1$. Let $T$ be a convolution operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ with kernel $K$. Define a convolution operator $\tilde{T}$ from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with kernel $\tilde{K}(x) = K(x_1)K(x_2)\cdots K(x_d)$. It is easy to show that $\|\tilde{T}\|_p = \|T\|_p$. The same is true for discrete analogues. Let $K$ be such a
kernel that \( \|T_{dis}\|_p = B > 1 \). Note that \( \hat{K}_{dis} = (\hat{K})_{dis} \). Then
\[
\frac{\|T_{dis}\|_p}{\|T\|_p} = \frac{\|T_{dis}\|_p^d}{\|T\|_p^d} = B^d.
\]

It would be interesting to know if there are kernels \( K \) with \( \hat{K} \) supported in \([-\frac{1}{2}, \frac{1}{2}]\) such that \( \|T_{dis}\|_p > \|T\|_p \) for \( 1 \leq p < 2 \), not just for \( 1 \leq p < p_0 \).

References


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