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# ON THE NORMS OF DISCRETE ANALOGUES OF CONVOLUTION OPERATORS

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Abstract. We consider a discrete analogue of convolution operator $T(f) = K * f$ from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$: $T_{dis}(g) = K_{dis} * g$ from $\ell^p(\mathbb{Z}^d)$ to $\ell^p(\mathbb{Z}^d)$ where $K_{dis} = K|_{\mathbb{Z}^d}$ and $\hat{K}$ is supported in the fundamental cube. We show that the estimate $\|T_{dis}\|_p \leq C^d \|T\|_p$ with $C > 1$ cannot be improved for a certain range of $p$.

We consider a convolution operator $T(f) = K * f$ from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with a distribution kernel $K$ whose Fourier transform $\hat{K}(\xi)$ is supported in the fundamental cube $Q = \{\xi = (\xi_1, \xi_2, ..., \xi_d) : -\frac{1}{2} < \xi_i \leq \frac{1}{2}, i = 1, 2, ..., d\} = (-\frac{1}{2}, \frac{1}{2}]^d$. The Fourier transform $(T(f))^\wedge = \hat{K}(\xi)\hat{f}(\xi)$. It is well known that if $T(f) = K * f$ acts from $L^p$ to $L^p$, then it also acts from $L^{p'}$ to $L^{p'}$ with the same norm. Thus we restrict ourselves to the case $1 \leq p \leq 2$. In particular, we can say that $T(f) = K * f$ is an operator from $L^r$ to $L^r$ where $p \leq r \leq 2$. We will be interested in the norms of $T$ from $L^p$ to $L^p$. If $p = 1$, then $\|T\|_1 = \|K\|_1$, and if $p = 2$, then $\|T\|_2 = \|\hat{K}\|_\infty$. Interpolating, we get $\|T\|_p \leq \|K\|_1^{\frac{2}{p}-1} \|\hat{K}\|_\infty^{2-\frac{2}{p}}$ for $1 < p < 2$.

As we mentioned before, $T$ is also a bounded operator from $L^2$ to $L^2$ and therefore $\hat{K}(\xi)$ is bounded, which makes $K(x) = \int_{\mathbb{R}^d} \hat{K}(\xi) e^{i2\pi\xi x} d\xi$ an infinitely differentiable function extendable to an entire function of exponential type. We define $K_{dis} = K|_{\mathbb{Z}^d}$ and $T_{dis}(g) = K_{dis} * g$ as an operator on $\mathbb{Z}^d$:

$$T_{dis}(g)(n) = \sum_{m \in \mathbb{Z}^d} K(n - m) g(m).$$

Since $\hat{K}$ is supported in the fundamental cube $Q$, we can view $K(n)$ as its Fourier coefficients and thus $\hat{K}(\xi) = \sum_{n \in \mathbb{Z}^d} K(n) e^{i2\pi n\xi} \chi_Q(\xi)$; i.e., $K_{dis}$ and $K$ are determined by each other. We will be interested in norms of $T_{dis}$ from $\ell^p(\mathbb{Z}^d)$ to $\ell^p(\mathbb{Z}^d)$. What was said above about $\|T\|_p$ is also true for $\|T_{dis}\|_p$. If $p = 1$, then $\|T_{dis}\|_1 = \|K_{dis}\|_1$, and if $p = 2$, then $\|T_{dis}\|_2 = \|\hat{K}\|_\infty$. Interpolating, we get $\|T_{dis}\|_p \leq \|K_{dis}\|_1^{\frac{2}{p}-1} \|\hat{K}\|_\infty^{2-\frac{2}{p}}$ for $1 < p < 2$. We will find the following estimate useful.

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**Proposition 1.**

$$\|T_{dis}\|_p \geq \|K_{dis}\|_p. \tag{1}$$

To show this, just take $g(n) = \delta_0(n)$.

The main question we are interested in is how $\|T_{dis}\|_p$ is controlled by $\|T\|_p$. As we said earlier, we restrict ourselves to the case $1 \leq p \leq 2$ since $\|T_{dis}\|_{p'} = \|T_{dis}\|_p$ and $\|T\|_{p'} = \|T\|_p$. The following result is due to Magyar, Stein and Wainger [1, Proposition 2.1]: *If $T$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, then $T_{dis}$ is a bounded operator from $\ell^p(\mathbb{Z}^d)$ to $\ell^p(\mathbb{Z}^d)$ and*

$$\|T_{dis}\|_p \leq C(d)\|T\|_p \tag{2}$$

*where $C$ depends only on the dimension $d$ but not on $p$.*

The actual statement is more general. Magyar, Stein and Wainger raised the following question in their paper (see Remark 1 after Proposition 2.1 in [1]): *Is it possible to take $C$ independently of the dimension $d$ and, in particular, if $C = 1$?*

We will answer this question negatively.

**Theorem 1.** *There is $p_0$ with $1 < p_0 < 2$ such that for all $p$ with $1 \leq p < p_0$ the optimal $C(d)$ in (2) has a lower bound $C(d) \geq B^d$ where $B > 1$.*

Actually the proof of Proposition 2.1 in [1] shows that $C(d) \leq A^d$, and thus this estimate of $C(d)$ cannot be improved. If $p = 2$, then $\|T\|_2 = \|T_{dis}\|_2 = \|\hat{K}\|_\infty$. Therefore, $C(d) = 1$ for $p = 2$. If $K \geq 0$, then $\|T\|_p = \|T_{dis}\|_p = \|\hat{K}\|_\infty = \|K\|_1$. Thus $C(d) \geq 1$.

*Proof of Theorem* 1. First we consider the case when the dimension $d = 1$ and $p = 1$. We will construct a kernel $K$ such that $\|T_{dis}\|_1 = \sum_{n\in\mathbb{Z}} |K(n)| > \int_{\mathbb{R}} |K(x)|dx = \|T\|_1$. Let $\phi(x)$ be a nonnegative Schwartz function whose Fourier transform $\hat{\phi}(\xi)$ is also nonnegative and supported in $[-\frac{1}{2}, \frac{1}{2}]$, for example, $\phi(x) = \psi(x) \cdot \bar{\psi}(x)$ where $\hat{\psi}(\xi)$ is nonnegative and supported in $[-\frac{1}{4}, \frac{1}{4}]$. Define $\hat{K}(\xi) = \hat{\phi}(2(\xi - \frac{1}{4})) + \hat{\phi}(2(\xi + \frac{1}{4}))$. Then $\hat{K}(\xi)$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$ and

$$K(x) = \cos\left(\frac{\pi x}{2}\right)\phi\left(\frac{x}{2}\right).$$

Therefore

$$|K(n)| = \begin{cases} \phi(\frac{n}{2}) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Hence,

$$\sum_{n\in\mathbb{Z}} |K(n)| = \sum_{m\in\mathbb{Z}} \phi(m) = \hat{\phi}(0).$$

Now we will estimate $\|K\|_1$:

$$\begin{aligned}\int |K(x)|dx &= \int \phi(\frac{x}{2})|\cos\frac{\pi x}{2}|dx \\ &= 2\int \phi(x)|\cos \pi x|dx \\ &> 2\int \phi(x)\cos^2 \pi x dx \\ &= \int \phi(x)(1+\cos 2\pi x)dx \\ &= \hat{\phi}(0)+\frac{1}{2}\hat{\phi}(1)+\frac{1}{2}\hat{\phi}(-1)=\hat{\phi}(0).\end{aligned}$$

Thus we obtain $\int |K(x)|dx > \sum_{n\in\mathbb{Z}} |K(n)|$. This is the opposite of what we need to get. Now consider a family of kernels $K_t(x) = K(x+t)$ where $0 \le t \le 1$. We have $\|K_t\|_1 = \|K\|_1$. On the other hand $\int_0^1 \sum_{n\in\mathbb{Z}} |K(n+t)|dt = \int |K(x)|dx$. Since for $t = 0$ we have $\int |K(x)|dx > \sum_{n\in\mathbb{Z}} |K(n)|$, there exists $t$ such that $\int |K(x)|dx < \sum_{n\in\mathbb{Z}} |K_t(n)|$. For this kernel $K_t$ we have $\|T_{dis}\|_1 > \|T\|_1$. Basically, the idea is to show that $h(t) = \sum_{n\in\mathbb{Z}} |K(n+t)|$ is not a constant. This can also be done by calculating Fourier coefficients $\hat{h}(m) = |K|^\wedge(m)$ and showing that they do not vanish; take for example $K(x) = \left(\frac{\sin\frac{\pi x}{3}}{\pi x}\right)^3$.

Now we consider the case $p > 1$. Suppose $T$ is a bounded operator from $L^1$ to $L^1$. Then $T_{dis}$ is also a bounded operator from $\ell^1$ to $\ell^1$. As we said earlier, $\|T\|_\infty = \|T\|_1$ and $\|T_{dis}\|_\infty = \|T_{dis}\|_1$. Interpolating between $p = 1$ and $p = \infty$, we have that $T$ and $T_{dis}$ are bounded operators from $L^p$ to $L^p$ and $\log \|T\|_p$ and $\log \|T_{dis}\|_p$ are convex functions. Thus $\|T\|_p$ and $\|T_{dis}\|_p$ are continuous functions for $p \ge 1$. Let $K$ be a kernel such that $\|T_{dis}\|_1 > \|T\|_1$. Then $\|T_{dis}\|_p > \|T\|_p$ for some range of $p$: $1 \le p < p_0$. We can prove this in a different way which also uses the continuous dependence of $\|K_{dis}\|_p$ on $p$. Recall from (1) that $\|T_{dis}\|_p \ge \|K_{dis}\|_p$ and $\|T\|_p \le \|K\|_1$. If $K$ is a kernel such that $\sum_{n\in\mathbb{Z}} |K(n)| > \int |K(x)|dx$, then

$$\begin{aligned}\|T_{dis}\|_p &\ge \left(\sum_{n\in\mathbb{Z}} |K(n)|^p\right)^{\frac{1}{p}} \\ &> \int |K(x)|dx \ge \|T\|_p\end{aligned}$$

for some range of $p$: $1 \le p < p_0$.

Now we will do the case when the dimension $d > 1$. Let $T$ be a convolution operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ with kernel $K$. Define a convolution operator $\tilde{T}$ from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with kernel $\tilde{K}(x) = K(x_1)K(x_2)\cdots K(x_d)$. It is easy to show that $\|\tilde{T}\|_p = \|T\|_p^d$. The same is true for discrete analogues. Let $K$ be such a

kernel that $\frac{\|T_{dis}\|_p}{\|T\|_p} = B > 1$. Note that $\tilde{K}_{dis} = (\tilde{K})_{dis}$. Then

$$\frac{\|(\tilde{T})_{dis}\|_p}{\|\tilde{T}\|_p} = \frac{\|T_{dis}\|_p^d}{\|T\|_p^d} = B^d.$$

It would be interesting to know if there are kernels $K$ with $\hat{K}$ supported in $[-\frac{1}{2}, \frac{1}{2}]$ such that $\|T_{dis}\|_p > \|T\|_p$ for $1 \leq p < 2$, not just for $1 \leq p < p_0$.

## References

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