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MINIMAL VOLUME OF SLABS IN THE COMPLEX CUBE

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Abstract. We study the volume of sections and slabs in the n-dimensional cube for complex scalars. In particular, we investigate the directions of minimal volume for a small width of the slab.

1. Introduction and results

Let $||\cdot||_\infty$ and $|\cdot|$ denote the supremum and the euclidean norm on $\mathbb{K}^n$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For volume considerations, $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ and the $(2n)$-dimensional volume is used. Let

$$B_\infty := \{x \in \mathbb{K}^n \mid ||x||_\infty \leq \alpha\}$$

be the n-dimensional cube of volume 1, i.e. $\alpha = 1/2$ if $\mathbb{K} = \mathbb{R}$ and $\alpha = 1/\sqrt{\pi}$ if $\mathbb{K} = \mathbb{C}$. Given some direction $a \in \mathbb{K}^n$ of length $|a| = 1$ and $t \in \mathbb{K}$, we introduce the *section* of the cube

$$S(a,t) := \{x \in \mathbb{K}^n \mid ||x||_\infty \leq \alpha,\ \langle x, a\rangle = \alpha t\}$$

and for $t > 0$ the *slab* in the cube

$$S\ell(a,t) := \{x \in \mathbb{K}^n \mid ||x||_\infty \leq \alpha,\ |\langle x, a\rangle| \leq \alpha t\}\ .$$

Let

$$A(a,t) := \mathrm{vol}_{n-1}(S(a,t)) \quad , \quad t \in \mathbb{K},$$
$$V(a,t) := \mathrm{vol}_n(S\ell(a,t)) \quad , \quad t > 0,$$

denote the volumes of the section and the slab of B_∞. In the complex case, one has to use vol_{2n-2} and vol_{2n} for A and V. For $a = (a_k)_{k=1}^n$, let a^* denote the decreasing rearrangement of the sequence $(|a_k|)_{k=1}^n$, i.e. $|a_1| \geq \cdots \geq |a_n| \geq 0$. Since the volume is invariant under coordinate permutations and sign changes, which in the complex case means invariant under permutations and rotations of coordinate disks, we have

$$A(a,t) = A(a^*, |t|)\ ,\ V(a,t) = V(a^*, t).$$

We will therefore assume in the following that $a = (a_k)_{k=1}^n$, $a_1 \geq \cdots \geq a_n \geq 0$ and $t \geq 0$.

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Let $f_1 = (1, 0, \dots, 0)$, $f_2 := (1, 1, 0, \dots, 0)/\sqrt{2}$. For central sections $1 = A(f_1, 0) \le A(a, 0) \le A(f_2, 0) = 2^{m/2}$, and where $m = 1$ for $\mathbb{K} = \mathbb{R}$, $m = 2$ for $\mathbb{K} = \mathbb{C}$. The lower bound is due to Hadwiger [Ha], Vaaler [V] and Hensley [He]; the upper bound for $\mathbb{K} = \mathbb{R}$ is a well-known result of K. Ball [B]. The complex case is due to Oleskiewicz-Pełczyński [OP].

For small $t > 0$ it would seem natural that the volume of slabs mimics this behaviour in the sense that for $\mathbb{K} = \mathbb{R}$

$$t = V(f_1, t) \le V(a, t) \le V(f_2, t) = \sqrt{2}t - t^2/2.$$

In fact, it was proved by Barthe-Koldobsky [BK] that the lower bound is true provided that $t \le 3/4$ holds. For $t > 2(\sqrt{2} - 1)$, however, $V(f_1, t) \le V(a, t)$ is false for $a = f_2$, and for t larger but close to this number $2(\sqrt{2} - 1)$, f_2 probably is the direction of minimal slab volume (this is true at least for $n = 2$ and 3). The lower bound

$$t^2 = V(f_1, t) \le V(a, t)$$

remains true in the complex case for small t. This generalization of the result of [BK] to the complex case is the main result of this paper. Formula (2.3) states that

$$V(f_2, t) = 2t^2 - ((1 + t^2)t\sqrt{2 - t^2} + (2t^2 - 1)\arcsin(t\sqrt{2 - t^2}))/\pi,$$

and for $t \ge t_1 \approx 0.867115$, $V(f_2, t) < V(f_1, t)$, and for $t \ge t_1$ but near t_1, t_2 is probably the direction of minimal volume (true for $n = 2$). Our main result states that f_1 is the direction of minimal volume for slabs of width $t \le 4/5$, leaving a gap for width $4/5 < t < t_1$ similar to the real case where the gap is $3/4 < t < 2(\sqrt{2} - 1)$.

Theorem 1. *Let $\mathbb{K} = \mathbb{C}$, $a \in \mathbb{C}^n$ with $|a| = 1$ and $0 \le t \le 4/5$. Then*

$$t^2 = V(f_1, t) \le V(a, t). \tag{1.1}$$

Note that

$$V(a, t) = 2 \int_0^t A(a, s) s \, ds, \; 1 = \mathrm{vol}_{2n}(B_\infty) = 2 \int_0^\infty A(a, s) s \, ds.$$

As in [BK] the main technical tool is an integral inequality of independent interest, to be applied to $f = 2A(a, \cdot)$:

Proposition 2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be in $L_1(\mathbb{R}_+, sds)$, non-increasing and log-concave. Let $t > 0$. Assume that*

$$\int_0^t f(s)\, s \, ds \le \left(\frac{4}{5}\right)^2 \int_0^\infty f(s)\, s \, ds. \tag{1.2}$$

Then

$$2 \left(\int_0^\infty f(s)\, s^3 \, ds \right) \left(\int_0^t f(s)\, s \, ds \right) \ge t^2 \left(\int_0^\infty f(s)\, s \, ds \right)^2 . \tag{1.3}$$

Concerning the direction of the maximal volume of slabs, in the case of *complex* scalars, different from the case of the maximal volume of sections [OP], $V(a, t) \le V(f_2, t)$ is false in general for *any* width $t > 0$: for large $n \in \mathbb{N}$, we show $V(f_n, t) > V(f_2, t)$, where $f_n := (1, \dots, 1)/\sqrt{n} \in S^{n-1}$:

Proposition 3. *Let $0 < t < 4/9$ and $n \geq 20$. Then for $\mathbb{K} = \mathbb{C}$ and any $n \geq 1/t$,*

$$V(f_n, t) > V(f_2, t).$$

Contrary to this, in the case of *real* scalars $\mathbb{K} = \mathbb{R}$, $V(f_n, t) < V(f_2, t)$ holds provided that $t < t_0$, where t_0 is approximately $1/14$.

2. Volume formulas

We mention and recall some volume formulas for sections and slabs of the cube which yield some statements and formulas before the formulation of Theorem 1.

Proposition 4. *Let $a \in \mathbb{R}^n_+$, $|a| = 1$, $t \geq 0$. Then*

$$A(a,t) = \left\{ \begin{array}{ll} \frac{2}{\pi} \int\limits_0^\infty \prod\limits_{k=1}^n \frac{\sin(a_k s)}{a_k s} \cos(ts)\, ds & \mathbb{K} = \mathbb{R} \\ \frac{1}{2} \int\limits_0^\infty \prod\limits_{k=1}^n j_1(a_k s)\, J_0(ts)\, s\, ds & \mathbb{K} = \mathbb{C} \end{array} \right\} \tag{2.1}$$

and

$$V(a,t) = \left\{ \begin{array}{ll} t\ \frac{2}{\pi} \int\limits_0^\infty \prod\limits_{k=1}^n \frac{\sin(a_k s)}{a_k s} \frac{\sin(ts)}{ts}\, ds & \mathbb{K} = \mathbb{R} \\ t^2\ \frac{1}{2} \int\limits_0^\infty \prod\limits_{k=1}^n j_1(a_k s)\, j_1(ts)\, s\, ds & \mathbb{K} = \mathbb{C} \end{array} \right\}, \tag{2.2}$$

where J_0, J_1 are the first Bessel functions and $j_1(x) = 2\ J_1(x)/x$.

Let us note that (2.1), $\mathbb{K} = \mathbb{R}$ is a well-known result of Polya [P]. For $\mathbb{K} = \mathbb{C}$ and $t = 0$, it is found in [OP]. Formulas (2.2) follow from (2.1) by integrating over t, using $J_0(s)s = (J_1(s)s)'$ in the complex case. The standard way to prove (2.1) is to write (with α as in the introduction)

$$A(a,t) = \int\limits_{\langle z,a\rangle = \alpha t} \chi(||z||_\infty) dz,$$

where χ is the indicator function of $[0, \alpha]$, and to take the Fourier transform in the t-variable, $\hat{A}(a, \cdot)$, which by coordinate independence is a constant times a product of scaled (by a_k) Fourier transforms of the ball $B = \{z \in \mathbb{K} \mid |z| \leq \alpha\}$. For $\mathbb{K} = \mathbb{C}$ this is where J_1 appears. Taking the inverse transform then gives (2.1).

In particular, for $a = f_1$ and $a = f_2$, (2.2) yields

(2.3)

$$V(f_1, t) = t \quad (\mathbb{K} = \mathbb{R})\ ,\ V(f_1, t) = t^2\ (\mathbb{K} = \mathbb{C}),$$

$$V(f_2, t) = \sqrt{2}t - t^2/2 \quad (\mathbb{K} = \mathbb{R}),$$

$$V(f_2, t) = 2t^2 - ((1 + t^2)t\sqrt{2 - t^2} + (2t^2 - 1)\arcsin(t\sqrt{2 - t^2}))/\pi \quad (\mathbb{K} = \mathbb{C})$$

so that $V(f_1, t_1) = V(f_2, t_1)$ for $t_1 = 2(\sqrt{2} - 1)$ in the real case and $t_1 \simeq 0.867115$ in the complex case, as mentioned in the introduction. For $0 < t < t_1$, $V(f_1, t) < V(f_2, t)$.

Using these formulas, we now give the

Proof of Proposition 3. (a) For $f_n = (1,\dots,1)/\sqrt{n}$, (2.2) implies for complex scalars and $t>0$:

$$V(f_n,t) = \frac{t^2}{2}\int_0^\infty (j_1(s/\sqrt{n}))^n j_1(ts)\, s\, ds\ .$$

Since $\lim_{u\to 0} j_1(u)\, e^{u^2/8} = 1$, $\lim_{n\to\infty} j_1(s/\sqrt{n})^n = e^{-s^2/8}$ for any $s\in\mathbb{R}$, and we find using the dominated convergence theorem that

$$\begin{aligned}\lim_{n\to\infty} V(f_n,t) &= \frac{t^2}{2}\int_0^\infty e^{-s^2/8} j_1(ts)\, s\, ds \\ &= t\int_0^\infty e^{-s^2/8} J_1(ts)\, ds = 1-e^{-2t^2}\ . \end{aligned} \tag{2.4}$$

(b) For the properties of Bessel functions, cf. Watson [W].

We will use the following estimates for the modified Bessel functions $j_1(t) = 2J_1(t)/t$:

$$|j_1(t)| \le \min(1, 1.66/t^{3/2}), \qquad 0<t, \tag{2.5}$$

$$|j_1(t)| \ge \exp(-t^2/8 - t^4/316), \quad 0\le t\le 2. \tag{2.6}$$

As for (2.5), $\sqrt{t}\,|J_1(t)|$ has a decreasing sequence of maxima, tending to $\sqrt{2/\pi}$ by the asymptotic formula for $J_1(t)$, with the first maximum near $t_0\approx 2.17$ bounded by 0.83. To prove (2.6), consider $f(t) := j_1(t)\exp(t^2/8+t^4/316)$. Then

$$g(t) := f'(t)\exp(-t^2/8-t^4/316) = J_1(t)(1/2+2/79\,t^2) - 2J_2(t)/t.$$

In $[0,2]$, g is positive for $t<t_1\approx 1.67$ and negative for $t>t_1$, as can be checked by using the alternating power series expansion of J_1 and J_2 to approximate g by polynomials. Thus f is first increasing and then decreasing. Since $f(0)=1$ and $f(2)>0$, (2.6) holds for $t\in[0,2]$.

(c) By (2.2) and (2.4), we have for any $t>0$

$$V(f_n,t) = (1-e^{-2t^2}) - t^2/2\int_0^\infty (\, e^{-s^2/8} - j_1(s/\sqrt{n})^n\,)\, j_1(ts)\, s\, ds. \tag{2.7}$$

Using (2.6) and $|j_1(u)|\le 1$, we find

$$\begin{aligned} I_1 &:= t^2/2\int_0^{2\sqrt{n}} |\, e^{-s^2/8} - j_1(s/\sqrt{n})^n|\, j_1(ts)\, s\, ds \\ &\le t^2/2\int_0^{2\sqrt{n}} e^{-s^2/8}(1-e^{-s^4/(316\,n)})\, s\, ds \\ &\le t^2/2\int_0^\infty e^{-s^2/8} s^5/(316\,n)\, ds = 64/79\; t^2/n. \end{aligned}$$

For $s>2\sqrt{n}$, we estimate both terms in the integrand in (2.7) separately:

$$I_2 := t^2/2\int_{2\sqrt{n}}^\infty e^{-s^2/8}\,|j_1(ts)|\, s\, ds \le t^2/2\int_{2\sqrt{n}}^\infty s\, e^{-s^2/8}\, ds = 2/e^{n/2}\, t^2$$

and, using (2.5),

$$I_3 := t^2/2 \int_{2\sqrt{n}}^{\infty} |\, j_1(s/\sqrt{n})|^{\,n} |j_1(ts)|\, s\, ds = t^2\, n/2 \int_2^{\infty} |\, j_1(u)|^{\,n} |j_1(\sqrt{n}\, t\, u)|\, u\, du$$
$$\leq t^2\, n/2 \int_2^{\infty} (1.66/u^{3/2})^{\,n}\, u\, du = 4n/(3n-4)\, (0.83/\sqrt{2})^{\,n}\, t^2 \leq 2/e^{n/2}\, t^2, \quad n \geq 4.$$

Therefore $I := I_1 + I_2 + I_3 \leq (64/79 + 4n/e^{n/2})\, t^2/n \leq 0.815 t^2/n$, provided that $n \geq 20$. Using formula (2.3) for $V(f_2, t)$, elementary estimates give that $V(f_2, t) \leq 2t^2 - t^3$ for any $t \in [0, 1]$. Therefore $V(f_2, t) < V(f_n, t)$ will follow from $0.815 + 2nt^2 < nt$. It is elementary to check that this holds for $1/n < t < 4/9$ and $n \geq 20$. □

3. Slabs of minimal volume

We now prove Theorem 1 by first using Proposition 2 and then giving the (independent) proof of Proposition 2.

Proof of Theorem 1 *from Proposition* 2. Let $a \in S^{n-1}(\mathbb{C})$ and $0 \leq t \leq 4/5$. For the volume of the slab and the unit cube we have in the complex case by polar integration

$$V(a, t) = 2 \int_0^t A(a, s)\, s\, ds, \;\; 1 = 2 \int_0^{\infty} A(a, s)\, s\, ds\ .$$

Assume that the direction a satisfies $V(a, t) \leq V(f_1, t) = t^2$. We claim that in this case $V(a, t) = V(f_1, t)$; i.e. f_1 is a minimal volume direction for the complex slab. The assumption in Proposition 2 is satisfied for $f = 2A(a, \cdot)$ since

$$V(a, t) = \int_0^t f(s)\, s\, ds \leq t^2 \leq \left(\frac{4}{5}\right)^2 = \left(\frac{4}{5}\right)^2 \int_0^{\infty} f(s)\, s\, ds\ .$$

Further, for the circle $B = \{z \in \mathbb{C} \mid |z| \leq \alpha = 1/\sqrt{\pi}\}$ of area 1,

$$\begin{aligned}
\int_0^{\infty} f(s)\, s^3\, ds &= 2 \int_0^{\infty} A(a, s)\, s^3\, ds \\
&= 2\pi^2 \int_0^{\infty} A(a, u/\alpha)\, u^3\, du \\
&= \pi \int_B A(a, |v|/\alpha)\;\, |v|^2\, dv \\
&= \pi \int_B \left(\int_{\langle z, a\rangle = |v|} dz \right) |v|^2\, dv \\
&= \pi \int_{B_\infty} |\langle w, a\rangle|^2\, dw = \frac{1}{2},
\end{aligned}$$

where the last equality follows from

$$\int_{B^2} w_i \overline{w}_j \, dw_i dw_j = 0 \ , \ \int_B |w_i|^2 dw_i = \frac{1}{2\pi} \ .$$

We thus conclude by Proposition 2

$$2 \left(\int_0^\infty f(s)\, s^3 \, ds \right) \left(\int_0^t f(s)\, s \, ds \right) \geq t^2 \left(\int_0^\infty f(s)\, s \, ds \right)^2 ;$$

i.e. $2 \cdot \frac{1}{2} \cdot V(a,t) \geq t^2 \cdot 1^2, \ V(a,t) \geq t^2 = V(f_1, t)$. □

Proof of Proposition 2. Let $t > 0$ and $0 < v < V$. Consider

$$F_{t,v,V} := \big\{ f : \mathbb{R}_+ \to \mathbb{R}_+ \mid f \text{ non-increasing, log-concave, } \int_0^\infty f(s)\, s \, ds = V, \ \int_0^t f(s)\, s \, ds = v \big\}.$$

We want to prove that under the assumption (1.2) on f,

$$\inf \left\{ \int_0^\infty f(s)\, s^3 \, ds \mid f \in F_{t,v,V} \right\} \geq \frac{t^2 V^2}{2v} \ . \tag{3.1}$$

Using the monotonicity and log-concavity of f, one proves similarly as in [BK] that

(3.2)

$$\inf \left\{ \int_0^\infty f(s)\, s^3 \, ds \mid f \in F_{t,v,V} \right\}$$
$$= \inf \left\{ \int_0^d e^{-ps}\, s^3 \, ds \mid p \geq 0, \ d \geq t \geq 0, \ \int_0^d e^{-ps}\, s \, ds = V, \ \int_0^t e^{-ps}\, s \, ds = v \right\}.$$

Thus we only have to consider functions $f(s) = \chi_{[0,d]}(s) e^{-ps}$. The constraints in (3.2) mean that

$$V = \int_0^d e^{-ps} s ds = \frac{1}{p^2} \int_0^x e^{-u} u du, \ v = \int_0^t e^{-ps} s ds = \frac{1}{p^2} \int_0^y e^{-u} u du,$$

with $x := pd \geq y := pt$. Moreover,

$$\int_0^d e^{-ps} s^3 ds = \frac{1}{p^4} \int_0^x e^{-u} u^3 du \ .$$

Assumption (1.2) for these functions is thus equivalent to

$$\int_0^y e^{-u} u du \leq \left(\frac{4}{5} \right)^2 \int_0^x e^{-u} u du, \ 0 \leq y \leq x \ . \tag{3.3}$$

Therefore, using (3.2), inequality (1.3) of Proposition 2 will follow provided we can show that for any $0 \le y \le x$ satisfying (3.3) we have

$$\frac{2\int_0^x e^{-u}u^3du}{\left(\int_0^x e^{-u}udu\right)^2} \ge \frac{y^2}{\left(\int_0^y e^{-u}udu\right)}. \tag{3.4}$$

The function g, $g(y) := y^2/\left(\int_0^y e^{-u}udu\right)$ is increasing since

$$2\int_0^y e^{y-u}udu \ge 2\int_0^y udu = y^2 \ ,$$

i.e.

$$2y\int_0^y e^{-u}udu \ge e^{-y}y^3 \ .$$

Define $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by the requirement

$$\int_0^{\varphi(x)} e^{-u}udu = \left(\frac{4}{5}\right)^2 \int_0^x e^{-u}udu \ . \tag{3.5}$$

Hence by assumption $y \le \varphi(x) \le x$, and (3.4) will follow from the stronger claim

$$\frac{2\int_0^x e^{-u}u^3du}{\left(\int_0^x e^{-u}udu\right)^2} \ge \frac{\varphi(x)^2}{\left(\int_0^{\varphi(x)} e^{-u}udu\right)} = \left(\frac{5}{4}\right)^2 \frac{\varphi(x)^2}{\left(\int_0^x e^{-u}udu\right)} \ ,$$

i.e.

$$\varphi(x)^2 \le \frac{32}{25}\left(\int_0^x e^{-u}u^3du \Big/ \int_0^x e^{-u}udu\right) =: \psi(x) \ . \tag{3.6}$$

Let $x \in [0,4]$. Since $\varphi(0) = 0$, $\lim_{x\searrow 0}\psi(x) = 0$, to prove (3.6) in this range of x it suffices to show that $(\varphi(x)^2)' \le \psi'(x)$. We have by (3.5)

$$(\varphi(x)^2)' = 2\varphi(x)\varphi'(x) = \frac{32}{25}e^{\varphi(x)}x\ e^{-x}$$

and

$$\psi'(x) = \frac{32/25}{\left(\int_0^x e^{-u}udu\right)^2}\left[\left(\int_0^x e^{-u}udu\right)x^3e^{-x} - xe^{-x}\left(\int_0^x e^{-u}u^3du\right)\right].$$

Hence $(\varphi(x)^2)' \le \psi'(x)$ holds provided that

$$e^{\varphi(x)}\left(\int_0^x e^{-u}udu\right)^2 \le \int_0^x (x^2u - u^3)e^{-u}du$$

or equivalently

$$\begin{aligned} e^{\varphi(x)} &\leq f(x)/g(x),\ f(x) := x^2(e^x+2) - 6(e^x - (1+x)), \\ g(x) &= (e^x - (1+x))(1-(1+x)e^{-x}). \end{aligned} \tag{3.7}$$

We claim that $\varphi(x) \leq \frac{4}{5}x - \frac{4}{75}x^2 =: \gamma(x)$ holds. This will follow from

$$\int_0^{\gamma(x)} e^{-u} u du \geq \left(\frac{4}{5}\right)^2 \int_0^x e^{-u} u du \tag{3.8}$$

by definition of φ. (3.8) will follow from the stronger inequality for the derivatives

$$\gamma'(x) e^{-\gamma(x)} \gamma(x) \geq \left(\frac{4}{5}\right)^2 e^{-x} x \ ,$$

i.e.

$$(1 - \frac{2}{15}x)(1 - \frac{1}{15}x) e^{x/5 + 4x^2/75} \geq 1$$

or

$$\frac{x}{5} + \frac{4}{75}x^2 + \ln(1 - \frac{2}{15}x) + \ln(1 - \frac{1}{15}x) \geq 0 \ . \tag{3.9}$$

For $x \leq 4$, $\frac{2}{15}x \leq \frac{3}{5}$ and for $y \leq \frac{3}{5}$, $\ln(1-y) \geq -y - y^2$, which implies

$$\ln(1 - \frac{2}{15}x) + \ln(1 - \frac{1}{15}x) \geq -\frac{x}{5} - \frac{x^2}{45} \ .$$

Hence (3.9) and (3.8) hold for $x \leq 4$. We conclude that

$$e^{\varphi(x)} \leq e^{4/5x - 4x^2/75} \leq 1 + \frac{4}{5}x + \frac{4}{15}x^2 + \frac{16}{375}x^3 + \frac{8}{5625}x^4 \ , \tag{3.10}$$

the last inequality coming from the series development of the middle term. As for f and g in (3.7), one calculates the Taylor expansions

$$\begin{aligned} f(x) &= \frac{x^4}{4} \sum_{n=0}^{\infty} \frac{4(n+1)(n+6)}{(n+4)!} x^n \\ g(x) &= \frac{x^4}{4} \sum_{n=0}^{\infty} (-1)^n \cdot 4 \cdot \frac{(n+4)^2 - 3(n+4) + 1 + (-1)^n}{(n+4)!} x^n \ . \end{aligned}$$

The Taylor polynomial for $f(x)/g(x)$ of order 3 yields a lower bound

$$1 + \frac{4}{5}x + \frac{13}{45}x^2 + \frac{11}{189}x^3 \leq f(x)/g(x) \ . \tag{3.11}$$

Inequality (3.7) now follows from (3.10), (3.11) and

$$\frac{4}{15}x^2 + \frac{16}{375}x^3 + \frac{8}{5625}x^4 \leq \frac{13}{45}x^2 + \frac{11}{189}x^3,$$

which is true since $1 + \frac{367}{525}x - \frac{8}{125}x^2 \geq 0$ for $0 \leq x \leq 4$ holds.

For $x \geq 4$, inequality (3.9) is verified more easily: φ and ψ are increasing functions with

$$\lim_{x\to\infty} \psi(x) = \frac{192}{25} = 7.68 \ , \ \lim_{x\to\infty} \varphi(x)^2 \simeq 4,74 \ .$$

For $x \geq 4$

$$\psi(x) \geq \psi(4) \simeq 4.79 \geq \varphi(\infty)^2 \geq \varphi(x)^2 \ . \qquad \square$$

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