PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 5, May 2012, Pages 1697–1704
S 0002-9939(2011)11044-X
Article electronically published on December 27, 2011

SINGULAR ORDINARY DIFFERENTIAL EQUATIONS HOMOGENEOUS OF DEGREE 0 NEAR A CODIMENSION 2 SET

D. BRESCH, B. DESJARDINS, AND E. GRENIER

(Communicated by Walter Craig)

Abstract. This paper deals with an example of a class of ordinary differential equations which are singular near a codimension 2 set with a homogeneous singularity of degree 0. Under some structural assumptions, we prove that for almost all initial data there exists a unique global solution.

1. Introduction

This paper deals with a simple example of a class of ordinary differential equations which are singular on a manifold of codimension 2, the singularity being homogeneous of degree 0 near this singularity.

Let H be a Hilbert space. Let Π denote the orthogonal projection on a given two-dimensional plane P. Let $x_h = \Pi x$ and let $x_v = x - \Pi x$. Note that $x = x_h + x_v$. By a slight abuse of notation we shall say that $x = (x_h, x_v)$, x_h and x_v referring to the "horizontal" and "vertical" components of x. We then identify the plane P with $\mathbb{C}$ by choosing arbitrarily an orthonormal basis on P. This allows us to use polar coordinates on P and thus to define the modulus r and argument θ of x_h and to use the notation $x_h = r\exp(i\theta)$.

Let ϕ be a smooth function defined on $H \times \mathbb{S}^1$ ($\mathbb{S}^1$ being the unit circle). Then

$$\dot{x} = \phi\Big(x, \frac{x_h}{|x_h|}\Big) \tag{1}$$

is a dynamical system, singular on $P^\perp$, orthogonal of P, which is of codimension 2. Moreover, this system is homogeneous of degree 0 in any direction orthogonal to $P^\perp$.

Under some assumptions on ϕ, we show global existence and uniqueness of a solution for almost every initial data. Note that we only take care of the behavior of $t \mapsto x(t)$ near $P^\perp$ and by a slight abuse of language we shall say that a solution is global if it does not reach the singularity $P^\perp$ in finite time.

2. Motivation

The toy model (1) mimics phenomena occurring for instance in the low Mach number limit problem for non-isentropic flows governed by the Euler equations in a periodic box. Such flows are described through a velocity field u, a density ρ

Received by the editors March 25, 2009 and, in revised form, February 4, 2010 and January 20, 2011.

2010 *Mathematics Subject Classification.* Primary 37N10, 35A05, 74H35.

Key words and phrases. Singular ODE's, codimension 2 singularity, global existence and uniqueness, low Mach number limit.

implicitly given by a state law depending on the entropy S and the pressure p; see [4]. After some change of variables, namely introducing q defined by $p = \overline{p}e^{\varepsilon q}$ with $\overline{p}$ a prescribed constant and ε the low Mach number, the non-dimensional system reads

$$a(\partial_t q + u \cdot \nabla q) + \frac{1}{\varepsilon}\mathrm{div} u = 0, \tag{2}$$

$$r(\partial_t u + u \cdot \nabla u) + \frac{1}{\varepsilon}\nabla q = 0, \tag{3}$$

$$\partial_t S + u \cdot \nabla S = 0, \tag{4}$$

where a and r are two smooth functions of S and $\varepsilon\, q$. The singular limit consists in letting the Mach number ε go to 0. Equations (2)–(3) may be written under the compact form

$$\partial_t U^\varepsilon + Q(S^\varepsilon)(U^\varepsilon, U^\varepsilon) + \frac{1}{\varepsilon}A(S^\varepsilon)U^\varepsilon = 0,$$

where $U^\varepsilon = (q, u)$ with $Q(S^\varepsilon)(\cdot, \cdot)$ a quadratic form (in the concrete Euler case (2)–(3), $Q(S^\varepsilon)(U^\varepsilon, U^\varepsilon) = u \cdot \nabla U^\varepsilon$), $A(S^\varepsilon)$ a matrix operator which is given by (6) and S^ε a scalar unknown given through a PDE of the form

$$\partial_t S^\varepsilon = F(S^\varepsilon, U^\varepsilon).$$

In the ill-prepared data case, namely when the initial data do not satisfy the condition

$$\mathrm{div}\, u^0 = 0, \qquad \nabla p^0 = 0, \tag{5}$$

an oscillatory limit with changing eigenvalues occurs. Indeed the eigenvalues and the spectrum of the singular operator A, where

$$A = \begin{pmatrix} 0 & a^{-1}\mathrm{div} \\ r^{-1}\nabla & 0 \end{pmatrix}, \tag{6}$$

will depend on the solution itself. This leads to a complex problem, since eigenvalues may cross. The wave equation, related to the operator A, may also be written under the form

$$\varepsilon \partial_{tt}\psi - \mathrm{div}\,(S^{-1}\nabla\psi) = 0 \tag{7}$$

where S is the entropy quantity; see [4] for more details.

For periodic in space boundary conditions, the filtering method (a clever "change of variable") has been pioneered by S. Schochet (see for instance [5]) for isentropic compressible Euler equations, namely the case $S^\varepsilon \equiv 0$, to justify the asymptotics $\varepsilon \to 0$ in the ill-prepared case. This allows us to get strong compactness on the new unknown. However this approach fails in the case of non-isentropic fluids, since the singular operator A depends on the solution $S^\varepsilon(t)$ itself. After filtration, namely after an appropriate change of variable $v^\varepsilon = \mathcal{L}^\varepsilon_{\mathrm{app}}(-t)U^\varepsilon$ through an approximate resolvent $\mathcal{L}^\varepsilon_{\mathrm{app}}$ of the wave equation, the system may only be written under the form

$$\mathcal{L}^\varepsilon_{\mathrm{app}}(t)\partial_t v^\varepsilon = Q(S^\varepsilon)(\mathcal{L}^\varepsilon_{\mathrm{app}}(t)v^\varepsilon, \mathcal{L}^\varepsilon_{\mathrm{app}}(t)v^\varepsilon) + \mathcal{E}^\varepsilon(t)\mathcal{L}^\varepsilon_{\mathrm{app}}(t)v^\varepsilon$$

where $\mathcal{E}^\varepsilon$ is an error term which is well defined away from the crossing eigenvalues set and singular on it; see [2] for more details. The main difficulty in the general study is then to prove, after defining appropriate infinite-dimension measures, that for almost all initial data, the limit flow does not meet double eigenvalues and crosses the resonance set transversally. Note that such a double eigenvalue set has been proven to be of codimension 2 in previous works. The previous equation has the same form as the ODE

$$\partial_t \phi + Q(\phi) = R(x, \frac{x - \Pi x}{\|x - \Pi x\|}),$$

where Π is the orthogonal projection on a codimension 2 variety. This explains the necessity of studying ODEs of the form (1). Our result will be used to treat oscillatory limits with changing eigenvalues and particularly the low Mach number limit for non-isentropic flows. The interested reader is referred to [2] for a complete strategy explanation on such a topic. Note that another type of singular ordinary differential equation occurring in fluid mechanics has been studied recently; see for instance [1].

In the very special case of only one spatial dimension, the limit $\varepsilon \to 0$ for non-isentropic flow can be both calculated completely and justified; see [4]. In the multi-dimensional case the formal calculation of the extra term in the limit, which once again involves the spectral decomposition of the fast operator, assumes that the spectrum of that fast operator is simple and non-resonant; see [3] for the viscous case and [4] for the inviscid one. For certain finite-dimensional truncations of the equations those assumptions can be shown to be generic and to ensure convergence to the limit equations. This has been done in the paper [4].

3. Setup and main result

3.1. **Notation.** Let us first introduce some notation. We define $\psi(x_h, x_v, \theta)$ as the argument of $\Pi\phi\Big((x_h, x_v), \exp(i\theta)\Big)$ and $\tilde{\psi}(x_h, x_v, \theta)$ its modulus in such a way that

$$\Pi\phi\Big((x_h, x_v), \exp(i\theta)\Big) = \tilde{\psi}(x_h, x_v, \theta) \exp\Big(i\psi(x_h, x_v, \theta)\Big).$$

Structural properties. We will assume that there exist an integer $N_0 > 0$ and N_0 smooth functions of x_h and x_v, $\Theta_1(x_h, x_v)$, $\ldots$, $\Theta_{N_0}(x_h, x_v)$, satisfying the following properties:

(H1) For all $x = (x_h, x_v) \in H$, the equation in θ

$$\psi(x_h, x_v, \theta) \in \theta + \pi\mathbb{Z}$$

has exactly N_0 solutions $\Theta_1(x_h, x_v) \ldots \Theta_{N_0}(x_h, x_v)$.

(H2) For every j, the following sign condition holds for all $x = (x_h, x_v) \in H$:

$$\partial_\theta \psi(x_h, x_v, \Theta_j(x_h, x_v)) < 1.$$

Note that this implies that the solutions Θ_j are all simple.

(H3) $\tilde{\psi}$ does not vanish.

Note that, for $\widetilde{\psi} = 1$, (H1) implies that $\Big\{(\tau \exp(i\Theta_j(0, x_v)), x_v), \quad \tau > 0\Big\}$ is a trajectory for

$$\dot{x}_h = \Pi\phi\Big((0, x_v), \frac{x_h}{|x_h|}\Big). \tag{8}$$

This trajectory goes to the singularity or leaves it, depending on its orientation. In particular for some initial data we reach the singularity in finite time. The flow is not defined everywhere, and we can only hope almost everywhere results.

3.2. **Main result.** The main result of this paper is

Theorem 3.1 (Stable and unstable manifolds)**.** *Let us assume that* (H1), (H2) and (H3) *hold true. Let* $x_0 \in P^\perp$ *and let* $\rho > 0$*. There exists a finite number of manifolds* V_k*, of codimension* 1*, with boundary* Σ*, such that:*

- *For any initial data* x_1 *in one of the manifolds* V_k*, the corresponding solution of* (1) *reaches* Σ *in finite time (in the past or in the future).*
- *For any initial data* x_1 *in* $B(x_0, \rho)$ *outside all these manifold* V_k*, the corresponding solution of* (1) *reaches the boundary of* $B(x_0, \rho)$ *before* Σ*.*

With this result one can define the flow $\Psi(t)$ of this equation, flow which is defined everywhere except on the manifolds V_k. The outline of the paper is as follows. First we study a simplified two-dimensional equation

$$\dot{y} = \phi\Big(\frac{y}{|y|}\Big) \tag{9}$$

and then extend it by perturbation arguments to equations of the general form (1).

4. Study of systems of the form (9)

We first prove Theorem 3.1 in the particular case of systems of the form (9). Note that (9) may be reduced to the study of a two-dimensional dynamical system, with $x \in \mathbb{C}$. First we turn to polar coordinates, define $y = r\exp(i\theta)$ and make the change of time defined by $d\tau/dtE = 1/r$ to get

$$\frac{dr}{d\tau} = r\tilde{\psi}(\theta)\cos(\psi(\theta) - \theta), \tag{10}$$

$$\frac{d\theta}{d\tau} = \tilde{\psi}(\theta)\sin(\psi(\theta) - \theta). \tag{11}$$

Remark that $\dot{r}$ has constant sign close to each side of Θ_j. Note that r as a function of t may vanish in finite time, whereas r as a function of τ never vanishes, as is clear from (10). Moreover (11) does not involve r, which greatly simplifies the analysis of (9). Note also that $\tilde{\psi}$ is always positive. Therefore the dynamics of (11) is given by assumption (H1). There exist N_0 fixed points $\Theta_j(x_v)$ which are stable provided

$$(\psi'(\Theta_j) - 1)\cos(\psi(\Theta_j) - \Theta_j) < 0$$

and unstable if $(\psi'(\Theta_j) - 1)\cos(\psi(\Theta_j) - \Theta_j) > 0$ (the null case being ruled out by (H2)).

Moreover the Θ_j are the only fixed points of (11) and the dynamics of solutions $t \mapsto \theta(t)$ of (11) is very simple: $t \mapsto \theta(t)$ goes in a monotonic way from some $\Theta_j(x_v)$ (unstable equilibrium, limit value as τ goes to $-\infty$) to $\Theta_{j-1}(x_v)$ or $\Theta_{j+1}(x_v)$ as τ goes to $+\infty$ (stable equilibria). All the solutions of (11) are global in the τ variable and go from an unstable Θ_j to a close stable one.

It then remains to solve (10). The behavior depends on the sign of

$$\tilde{\psi}(\theta)\cos(\psi(\theta) - \theta).$$

As $\tilde{\psi}(\theta) > 0$ and as $\psi'(\Theta_j) < 1$, $\tilde{\psi}(\theta)\cos(\psi(\theta) - \theta)$ is positive if Θ_j is stable and negative if Θ_j is unstable.

Therefore solutions of (9) are global (except if θ constantly equals some of the Θ_j where the solution goes to the singularity in finite time in the future or in the past) and are asymptotic in $+\infty$ to some stable Θ_j and in $-\infty$ to some instable Θ_j.

The phase portrait can be described as follows:

- There exist N_0 particular solutions which are straight lines, going to or coming from the origin in finite positive or negative times.
- All the other trajectories are global in time and are asymptotic to two of the particular solutions as time goes to $+\infty$ or $-\infty$.

Theorem 3.1 is then straightforward. □

Note that hypothesis (H2) is crucial. If we assume $\psi'(\Theta') > 1$, then the conclusion is completely changed: all the trajectories come from the singularity and go back to the singularity in finite time, except for $\theta = \Theta_j$. In this case, almost all the trajectories blow up in finite time.

5. Trajectories near the singular set

In this section we will describe the behavior of solutions near the singular set. Let $x_0 = (0, x_v^0)$ be a point of $P^\perp$. Locally the geometry of the flow is described by the angles $\Theta_j(x_v^0)$ which split the space into angular sectors

$$\Omega_j = \{\Theta_j(x_v^0) < \theta < \Theta_{j+1}(x_v^0)\}.$$

If there were no x_h dependence of the flow, the angles Ω_j would be invariant under the flow as in the previous section. This is not the case here, and we have to be more precise in the spatial description.

5.1. **Domain decomposition.** Let $\alpha > 0$ and $\eta > 0$ be small. Then by continuity there exist angles θ_j^+ and θ_j^- such that

$$\left|\tilde{\psi}(x_h, x_v, \theta)\sin(\psi(x_h, x_v, \theta) - \theta)\right| \geq \alpha \tag{12}$$

when $|x_h| + |x_v - x_v^0| < 2\eta$ and $\Theta_j(x_v^0) < \theta_j^+ < \theta < \theta_{j+1}^- < \Theta_{j+1}(x_v^0)$.

Let us now introduce the following sets, for ε chosen later on with $\varepsilon < \eta$:

$$\Omega_j(\varepsilon, \eta) = \Big\{(x_h, x_v, \theta) \quad | \quad |x_h| < \varepsilon, \quad |x_v - x_v^0| < \eta, \quad \theta_j^- < \theta < \theta_j^+\Big\},$$
$$\Sigma_j(\varepsilon, \eta) = \Big\{(x_h, x_v, \theta) \quad | \quad |x_h| < \varepsilon, \quad |x_v - x_v^0| < \eta, \quad \theta_j^+ < \theta < \theta_{j+1}^-\Big\},$$

and

$$\Omega_v(\varepsilon, \eta) = \Big\{(x_h, x_v, \theta) \quad | \quad |x_h| < \varepsilon, \quad |x_v - x_v^0| < \eta\Big\}.$$

Note that $\Omega_v(\varepsilon, \eta)$ is the union of the $\Omega_j(\varepsilon, \eta)$ and $\Sigma_j(\varepsilon, \eta)$ and of their boundaries. The $\Omega_j(\varepsilon, \eta)$ will be the neighborhood of the stable and unstable manifolds.

5.2. **Study of the trajectory.** Let us fix the ideas that Θ_j is unstable and Θ_{j+1} is stable (the discussion is similar if the stabilities are interchanged). Let $x(t)$ be a trajectory with $x(0) \in \Sigma_j(\varepsilon, \eta/2)$. This subsection will be divided into three parts. In a first part, we prove the following:

Claim. If ε is small enough, there exist $m > 0$ and t_+ such that if $t \in [t_+, t_+ + m[$, then $x(t) \in \Omega_{j+1}(\eta, \eta)$. Analogously, there exists t_- such that if $t \in]t_- - m, t_-]$, then $x(t) \in \Omega_j(\eta, \eta)$. Moreover, for $|t|$ large enough, the trajectory $\Omega_v(\eta, \eta)$ exits in finite time.

In the second part we give bounds on $r(t)$ allowing us in the last part to study the dynamic on Ω_j as a standard dynamical system.

Proof of the Claim. Let $x(t)$ be a solution of (1), such that $x(0) = (x_h(0), x_v(0))$ with $x_h(0) = r(0)\exp(i\theta(0))$ with $r(0) > 0$. Using polar coordinates and introducing again the change of time variable, we get

$$\dot{x}_v = r(\mathrm{Id} - \Pi)\phi\Big((x_h, x_v), \frac{x_h}{|x_h|}\Big), \tag{13}$$

with

$$\dot{r} = r\tilde{\psi}(x_h, x_v, \theta)\cos(\psi(x_h, x_v, \theta) - \theta), \tag{14}$$

$$\dot{\theta} = \tilde{\psi}(x_h, x_v, \theta)\sin(\psi(x_h, x_v, \theta) - \theta). \tag{15}$$

Let $]a, b[$ be the maximal time interval containing 0 such that $x(t) \in \Omega_v(\eta, \eta)$ for $t \in]a, b[$. By definition of θ_j^+ and θ_{j+1}^-, $\theta(t)$ is increasing as long as it belongs to $\Omega_v(\eta,\eta) - \Omega_j(\eta,\eta) - \Omega_{j+1}(\eta,\eta)$. Let $]a_1, b_1[$ be the maximum time interval containing 0 such that $\theta_j^+ < \theta(t) < \theta_{j+1}^-$. Of course $a \leq a_1 < b_1 \leq b$.

If $a < a_1$, then for $t = a_1$, $x(t) \in \Omega_j \cap \Omega_v$ and therefore $x(t) \in \Omega_j \cap \Omega_v$ for any $a < t < a_1$. Similarly, if $b > b_1$, then for any $b_1 < t < b$, $x(t) \in \Omega_{j+1} \cap \Omega_v$.

Note that, using (15) and (12), we can bound $b_1 - a_1$ by

$$b_1 - a_1 \leq \frac{\theta_3 - \theta_2}{\alpha}, \tag{16}$$

where θ_2 and θ_3 correspond respectively to the angle at time $t = a_1$ and $t = b_1$. This implies that b_1 and $-a_1$ are less than $(\theta_3 - \theta_2)/\alpha$. Moreover

$$\frac{d}{d\tau}\log r = \tilde{\psi}(x_h, x_v, \theta)\cos(\psi(x_h, x_v, \theta) - \theta). \tag{17}$$

Note that on Ω_v, $\tilde{\psi}$ is bounded by some constant C_0. Hence for $a_1 < \tau < b_1$, integrating (17) with respect to τ and using (16), we get

$$\exp(-C_0\alpha^{-1}(\theta_3 - \theta_2)) \leq \frac{r(\tau)}{r(0)} \leq \exp(C_0\alpha^{-1}(\theta_3 - \theta_2)). \tag{18}$$

Therefore if

$$\varepsilon < \frac{\eta}{2}\exp(-2C_0\alpha^{-1}(\theta_3 - \theta_2)), \tag{19}$$

using the condition that $r(0) < \varepsilon$ and (18), $r(\tau)$ remains smaller than η for $a_1 < \tau < b_1$. Hence:

The solution cannot leave Ω_v at $\tau = a_1$ or $\tau = b_1$
at the boundary $|x_h| = \eta$.

Using now (18)–(19), $r(\tau)$ remains smaller than $\frac{\eta}{2}\exp(-C_0\alpha^{-1}(\theta_3 - \theta_2))$. Thus, using (13), since $|x_v - x_v^0| < \eta$:

The trajectory cannot leave Ω_v at its horizontal boundary.

Indeed

$$|x_v - x_v^0| \leq \frac{\eta\, C_0}{2} b_1 \exp(-C_0\alpha^{-1}(\theta_3 - \theta_2))$$

and

$$b_1 \leq (\theta_3 - \theta_2)/\alpha.$$

Therefore

$$\textit{The trajectory goes from } \Omega_j(\eta,\eta) \textit{ to } \Omega_{j+1}(\eta,\eta).$$

This ends the proof of the Claim. □

Bounds on $r(t)$. Let us go back to the genuine time t. On $\Omega_j(\eta,\eta)$ and $\Omega_{j+1}(\eta,\eta)$, provided α is small enough, $|\cos(\psi(x_h,x_v,\theta)-\theta)|$ is larger than $1/2$. Therefore

$$\frac{d}{dt}|x_h(t)|^2 = 2r(t)\dot r(t)$$

is bounded away from 0 by $r(t)\min\widetilde{\psi}/2$. Therefore, as long as $r(t)$ remains in Ω_v, for $t > b_1$ we have

$$r(b_1) + \gamma_1(t-b_1) \le r(t) \le r(b_1) + \gamma_2(t-b_1)$$

for some non-negative constants γ_1, γ_2. A similar result is true for $t < a_1$.

Dynamics on Ω_j. As $\widetilde{\psi} \neq 0$ and $\cos(\psi(x_h,x_v,\theta)-\theta) \neq 0$, we change time on (13)–(15) through $d\widetilde{\tau}/d\tau = \widetilde{\psi}\cos(\psi(x_h,x_v,\theta)-\theta)$ to get

$$\dot x_v = \frac{r}{\widetilde{\psi}\cos(\psi(x_h,x_v,\theta)-\theta)}(1-\Pi)\phi\Big(x_h,x_v,r\Big), \tag{20}$$

$$\dot r = r, \tag{21}$$

$$\dot\theta = \tan(\psi(x_h,x_v,\theta)-\theta) \tag{22}$$

with $x_h = (r\cos\theta, r\sin\theta)$. Note that now, due to the previous part (see the Claim and its proof), the dynamics takes place in Ω_j, where equations (20)–(22) are not singular. Points of the form $(x_v, 0, \Theta_j(0,x_v))$ are equilibrium points of (20)–(22) and form codimension 2 unstable equilibrium points (or stable, depending on the parity of j). Therefore by classical arguments, (20)–(22) admit a related unstable manifold. Using the description of the trajectory detailed in the previous paragraphs, we see that this unstable manifold is exactly what we wanted. Note that now since $\dot r > 0$ (see bounds on r previously given), we get the dynamics on Ω_j as a standard dynamical system.

Acknowledgments

The first author would like to thank the French ANR-08-BLANC-0301-01 Project named "MathOcean" (2008-2011) managed by D. Lannes. The authors thank the referees for their numerous important comments.

References

[1] S. Bianchini, L.V. Spinolo, *Invariant Manifolds for a Singular Ordinary Differential Equations*, J. Differential Equations **250** (2011), no. 4, 1788–1827 MR2763556.

[2] D. Bresch, B. Desjardins, E. Grenier, Oscillatory Limits with Changing Eigenvalues: A Formal Study. New Directions in Mathematical Fluid Mechanics, The Alexander V. Kazhikhov Memorial Volume A.V. Fursikov, G.P. Galdi, V.V. Pukhnachev, Eds. Adv. Math. Fluid Mech. (2010), 91–104.

[3] D. Bresch, B. Desjardins, E. Grenier, C.–K. Lin, *Low Mach Number Limit of Viscous Polytropic Flows: Formal Asymptotics in the Periodic Case*, Stud. Appl. Math. **109** (2002), no. 2, 125–149. MR1917042 (2003g:76091)

[4] G. Métivier, S. Schochet, *Averaging Theorems for Conservative Systems and the Weakly Compressible Euler Equations*, J. Differential Equations **187** (2003), no. 1, 106–183. MR1946548 (2003i:76077)

[5] S. Schochet, *Fast Singular Limits of Hyperbolic PDEs*, J. Differential Equations **114** (1994), no. 2, 476–512. MR1303036 (95k:35131)

LAMA, UMR5127 CNRS, Université de Savoie, 73376 Le Bourget du lac, France
E-mail address: Didier.bresch@univ-savoie.fr

ENS Ulm, D.M.A., 45 rue d'Ulm, 75230 Paris cedex 05, France – and – Modélisation Mesures et Applications S.A., 66 avenue des Champs Elysées, 75008 Paris, France
E-mail address: Benoit.Desjardins@mines.org

U.M.P.A., École Normale Supérieure de Lyon, 46, allée d'Italie, 69364 Lyon Cedex 07, France
E-mail address: egrenier@umpa.ens-lyon.fr