PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 5, May 2012, Pages 1739–1747
S 0002-9939(2011)11132-8
Article electronically published on September 22, 2011

ON THE EIKONAL EQUATION FOR DEGENERATE ELLIPTIC OPERATORS

PAOLO ALBANO

(Communicated by Matthew J. Gursky)

Abstract. We consider the nonnegative viscosity solution of the homogeneous Dirichlet problem for an eikonal equation associated to an operator sum of squares of vector fields of Grushin type in a symmetric domain. We show that the solution is locally Lipschitz continuous except at the characteristic boundary point. In the characteristic boundary point the solution has a Hölder regularity with exponent related to the Hörmander bracket condition. Finally, the singular set is an analytic stratification given by the characteristic boundary point and a half line.

1. Introduction and statement of the results

Let $M > 0$, let k be a positive integer and define

$$\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y > M|x|^{k+1}\}.$$

We consider the homogeneous Dirichlet problem

$$\begin{cases} |\nabla_x u(x,y)|^2 + |x|^{2k}(\partial_y u(x,y))^2 = 1 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

We observe that

(1) the vector fields $X_i = \partial_{x_i}$ and $X_{n+i} = x_i^k \partial_y$, $i = 1, \dots, n$, satisfy the Hörmander bracket condition;
(2) $\partial\Omega$ is smooth and $(0, 0) \in \partial\Omega$ is a characteristic point.

We recall that a continuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of equation (1.1) if for every function φ of class C^1 such that $u - \varphi$ has a local maximum at $(x, y) \in \Omega$ we have

$$|\nabla_x \varphi(x,y)|^2 + |x|^{2k}(\partial_y \varphi(x,y))^2 \le 1.$$

Furthermore, u is a viscosity supersolution of equation (1.1) if for every function φ of class C^1 such that $u - \varphi$ has a local minimum at $(x, y) \in \Omega$ we have

$$|\nabla_x \varphi(x,y)|^2 + |x|^{2k}(\partial_y \varphi(x,y))^2 \ge 1.$$

Received by the editors June 8, 2010 and, in revised form, December 6, 2010 and January 27, 2011.
2010 *Mathematics Subject Classification.* Primary 35F30, 35F21, 35D40.
Key words and phrases. Viscosity solutions, eikonal equation, degenerate equations, singularities.

If a function u is both a viscosity subsolution and supersolution of equation (1.1), we say that u is a viscosity solution of equation (1.1).

The regularity in the interior of a solution of (1.1) can be studied as in [10]. Following such an approach one may obtain local Hölder regularity of the exponent $1/(k+1)$ (see [11]). On the other hand, we recall that in the set $\Omega \setminus \{(0,y) : y > 0\}$, since the equation is (locally) nondegenerate, u is locally Lipschitz continuous. We prove

Theorem 1.1. *The nonnegative viscosity solution of the Dirichlet problem* (1.1) *is locally Lipschitz continuous in* Ω. *Furthermore,* u *is Hölder continuous of the exponent* $1/(k+1)$ *at* $(0,0)$.

Remark 1.1. (i) We point out that the exponent $1/(k+1)$ in Theorem 1.1 is optimal.

(ii) In [7] the authors proved that if Δ is a fat distribution in $\mathbb{R}^{n+1}$, then for every $x_0 \in \mathbb{R}^{n+1}$ the sub-Riemannian distance function, $d_{SR}(x_0, \cdot)$, is locally semiconcave (hence locally Lipschitz continuous) in $\mathbb{R}^{n+1} \setminus \{x_0\}$. In the special case $k = 1$, the fact that the nonnegative viscosity solution of (1.1) is locally Lipschitz continuous in Ω can be deduced from Corollary 6.1 of [7].

In other words, the degeneracy of the equation reflects on the regularity of the solution near the characteristic boundary point. We define

$$\Sigma(u) = \{(x,y) \in \overline{\Omega} \mid u \text{ is not differentiable at } (x,y)\}.$$

We recall that, in the case of a noncharacteristic smooth boundary, the solution is smooth near the boundary (see e.g. [1]). We have

Theorem 1.2. *Let* u *be the nonnegative viscosity solution of the Dirichlet problem* (1.1). *Then,* $\Sigma(u) = \{(0,y) \mid y \geq 0\}$.

We remark that Theorem 1.2 is in accordance with the results proved in [16] and [5]. Indeed, the viscosity solution of the Dirichlet problem for an eikonal type equation with real analytic data (and suitable additional conditions which are satisfied in the case of equation (1.1)) is expected to be a subanalytic function. Then, using the general result in [15] one obtains that $\Sigma(u)$ is an analytic stratification. Theorem 1.2 yields that in the case of equation (1.1), due to the fact that the domain Ω is symmetric, there are only two strata: the point $(0,0)$ and the half line $\{(0,y) \mid y > 0\}$.

2. Proof of Theorem 1.1

We begin with an existence and uniqueness result.

Theorem 2.1. *There exists a unique nonnegative viscosity solution* $u \in C(\overline{\Omega})$ *of* (1.1). *Furthermore, we have*

(*i*) $u(x,y) = u(Rx,y)$, *for every* $(x,y) \in \Omega$ *and for every orthogonal matrix* R;
(*ii*) $\lambda u(x,y) = u(\lambda x, \lambda^{k+1} y)$, *for every* $(x,y) \in \Omega$ *and for every* $\lambda > 0$.

Proof. The existence of a nonnegative continuous viscosity solution of (1.1) can be proved using the Perron-Ishii method. For this purpose consider the functions

$$v_1(x,y) = \frac{M}{\sqrt{1+M^2}} \left(-|x| + \left(\frac{y}{M}\right)^{\frac{1}{k+1}} \right) \qquad ((x,y) \in \overline{\Omega})$$

and

$$v_2(x,y) = -|x| + \left(\frac{y}{M}\right)^{\frac{1}{k+1}} \qquad ((x,y) \in \overline{\Omega}).$$

A direct computation shows that v_1 is a nonnegative viscosity subsolution of equation (1.1) and v_2 is a nonnegative viscosity supersolution. Furthermore $v_1 = v_2 = 0$ on $\partial\Omega$. Defining

$$u(x,y) = \sup\{v(x,y) \,:\, v \text{ is a viscosity subsolution of (1.1)} \\ \text{and } v_1 \le v \le v_2 \text{ in } \Omega\},$$

it is a standard argument of the theory of viscosity solutions to conclude that u is a continuous viscosity solution of (1.1) (see e.g. [12]).

Let us prove the uniqueness of the nonnegative viscosity solution of problem (1.1). Applying the Kružkov transformation ($w(x,y) = 1 - e^{-u(x,y)}$), it suffices to show that if $w_1 \colon \Omega \to [0,1[$ and $w_2 \colon \Omega \to [0,1[$ are a viscosity subsolution and supersolution respectively of the Dirichlet problem

$$w(x,y) + \sqrt{|D_x w(x,y)|^2 + |x|^{2k}(\partial_y w(x,y))^2} = 1 \quad \text{in } \Omega \qquad w\Big|_{\partial\Omega} = 0$$

and $w_1 = w_2 = 0$ on $\partial\Omega$, then $w_1 \le w_2$ in Ω. For $\delta > 0$, set

$$M_\delta = \max_{(x,y)\in\Omega} \Big\{ w_1(x,y) - w_2(x,y) - \delta y^2 \Big\}.$$

We note that $M_\delta \in [0,2]$. We claim that

$$\limsup_{\delta\downarrow 0} M_\delta = 0. \tag{2.1}$$

We point out that (2.1) yields the conclusion. Indeed,

$$w_1(x,y) \le w_2(x,y) + \delta y^2 + M_\delta \qquad \forall (x,y) \in \Omega$$

and, for fixed $(x,y) \in \Omega$, taking the limit, as $\delta \downarrow 0$, in the above inequality we deduce that $w_1 \le w_2$ in Ω.

For $\varepsilon, \delta > 0$ we define

$$\Phi_{\delta,\varepsilon}(x,y,z,t) = w_1(x,y) - w_2(z,t) - \frac{\delta}{2}y^2 - \frac{\delta}{2}t^2 - \frac{|x-z|^2 + (y-t)^2}{2\varepsilon}.$$

We have that there exists $(x_\varepsilon, y_\varepsilon), (z_\varepsilon, t_\varepsilon) \in \overline{\Omega}$ such that

$$\max_{(x,y),(z,t)\in\overline{\Omega}} \Phi_{\delta,\varepsilon}(x,y,z,t) = \Phi_{\delta,\varepsilon}(x_\varepsilon, y_\varepsilon, z_\varepsilon, t_\varepsilon).$$

We point out that

$$\Phi_{\delta,\varepsilon}(x_\varepsilon, y_\varepsilon, z_\varepsilon, t_\varepsilon) \ge M_\delta. \tag{2.2}$$

We remark that the points $(x_\varepsilon, y_\varepsilon)$ and $(z_\varepsilon, t_\varepsilon)$ are contained in a compact set of $\overline{\Omega}$ (depending on the parameter δ and independent of ε). Furthermore, there exists C such that

$$|x_\varepsilon - z_\varepsilon|^2 + (y_\varepsilon - t_\varepsilon)^2 \le C\varepsilon. \tag{2.3}$$

Possibly taking a subsequence $\varepsilon_j \downarrow 0$, we may assume that there exists $(\hat{x}, \hat{y})$ such that

$$\lim_{\varepsilon\downarrow 0}(x_\varepsilon, y_\varepsilon) = (\hat{x}, \hat{y}). \tag{2.4}$$

Hence, using the inequality $\Phi_{\delta,\varepsilon}(x_\varepsilon, y_\varepsilon, z_\varepsilon, t_\varepsilon) \geq \Phi_{\delta,\varepsilon}(\hat{x}, \hat{y}, \hat{x}, \hat{y})$, (2.3) and (2.4), we deduce that

$$\lim_{\varepsilon \downarrow 0} \frac{|x_\varepsilon - z_\varepsilon|^2 + (y_\varepsilon - t_\varepsilon)^2}{2\varepsilon} = 0. \tag{2.5}$$

If there exists a sequence ε_j such that $\varepsilon_j \downarrow 0$ and $(x_{\varepsilon_j}, y_{\varepsilon_j}) \in \partial\Omega$, then, by (2.2), $M_\delta \leq 0$, and the conclusion follows. Analogously, if there exists a sequence ε_j such that $\varepsilon_j \downarrow 0$ and $(z_{\varepsilon_j}, t_{\varepsilon_j}) \in \partial\Omega$, then passing to the limit in the inequality (2.2) the conclusion follows. Let us suppose that $(x_\varepsilon, y_\varepsilon), (z_\varepsilon, t_\varepsilon) \in \Omega$. Using the definition of viscosity subsolution and supersolution we obtain that

$$\begin{aligned} M_\delta + \frac{|x_\varepsilon - z_\varepsilon|^2 + (y_\varepsilon - t_\varepsilon)^2}{2\varepsilon} + \frac{\delta}{2} y_\varepsilon^2 + \frac{\delta}{2} t_\varepsilon^2 &\leq w_1(x_\varepsilon, y_\varepsilon) - w_2(z_\varepsilon, t_\varepsilon) \\ &\leq \sqrt{\left|\frac{x_\varepsilon - z_\varepsilon}{\varepsilon}\right|^2 + |z_\varepsilon|^{2k}\Big(-\delta t_\varepsilon + \frac{y_\varepsilon - t_\varepsilon}{\varepsilon}\Big)} \\ &\quad - \sqrt{\left|\frac{x_\varepsilon - z_\varepsilon}{\varepsilon}\right|^2 + |x_\varepsilon|^{2k}\Big(\delta y_\varepsilon + \frac{y_\varepsilon - t_\varepsilon}{\varepsilon}\Big)}. \end{aligned}$$

An elementary algebraic computation yields that there exist a positive number $\gamma < 1$ and a constant C_1 independent of ε such that

$$M_\delta \leq C_1 \left(\delta^\gamma + \frac{|x_\varepsilon - z_\varepsilon|\ |y_\varepsilon - t_\varepsilon|}{\varepsilon}\right).$$

Hence, taking the limit in the above inequality, as $\varepsilon \downarrow 0$, using (2.5) and then sending $\delta \downarrow 0$, we conclude that $\lim_{\delta \downarrow 0} M_\delta = 0$ and the uniqueness result follows.

Let us verify that u satisfies conditions (i) and (ii) in the statement of the theorem. Let u be the nonnegative viscosity solution of (1.1), let R be an orthogonal matrix and let λ be a positive number. We have that the functions $u(Rx, y)$ and $\lambda^{-1} u(\lambda x, \lambda^{1/(k+1)} y)$ satisfy (1.1). Hence, by uniqueness, we deduce that $u(x, y) = u(Rx, y)$ and $u(x, y) = \lambda^{-1} u(\lambda x, \lambda^{1/(k+1)} y)$. This completes our proof. □

Let us show that u is Hölder continuous of the exponent $1/(k+1)$ (and not better) at the point $(0, 0)$. We recall that, by construction,

$$0 \leq \frac{M}{\sqrt{1 + M^2}} \left(-|x| + \left(\frac{y}{M}\right)^{\frac{1}{k+1}}\right) \leq u(x, y) \leq -|x| + \left(\frac{y}{M}\right)^{\frac{1}{k+1}}.$$

Hence, for every $(x, y) \in \Omega$, we obtain the Hölder estimate

$$|u(x, y) - u(0, 0)| = u(x, y) \leq -|x| + \left(\frac{y}{M}\right)^{\frac{1}{k+1}} \leq \frac{(|x|^2 + y^2)^{\frac{1}{2(k+1)}}}{M^{1/(k+1)}}.$$

Moreover, since on the y axis

$$\frac{M}{\sqrt{1 + M^2}} \left(\frac{y}{M}\right)^{\frac{1}{k+1}} \leq u(0, y),$$

we deduce that our Hölder estimate is optimal.

Let us show that u is locally Lipschitz continuous in Ω. If $(x, y) \in \Omega$ with $x \neq 0$, the equation (1.1) is nondegenerate near (x, y), and it is well known that every (sub)solution of a nondegenerate eikonal equation is locally Lipschitz continuous (see e.g. [6]).

It remains to consider the case of a point of the form $(0, y)$ with $y > 0$.

First we show that u is locally Lipschitz w.r.t. the variable x. More precisely, let $\Omega' \Subset \Omega'' \Subset \Omega$. We want to show that

$$|u(x,y) - u(x',y)| \le c|x - x'| \qquad \forall (x,y),(x',y) \in \Omega' . \tag{2.6}$$

This part of the proof follows the ideas introduced in [10]. For this purpose, let us consider the upper ε-envelope of u (in Ω''), i.e.

$$u^\varepsilon(x,y) = \sup_{(z,t)\in\Omega''} \left\{ u(z,t) - \frac{|x-z|^2 + (y-t)^2}{\varepsilon} \right\} . \tag{2.7}$$

The following properties of u^ε are well known (see e.g. [13]):

(1) $u^\varepsilon \ge u$ in Ω'';
(2) u^ε is Lipschitz continuous in Ω', and there exists C independent of ε such that, for every $(x,y),(x',y') \in \Omega'$,
$$|u^\varepsilon(x,y) - u^\varepsilon(x',y')| \le \frac{C}{\varepsilon}(|x-x'| + |y-y'|);$$
(3) $\lim_{\varepsilon\to 0} \sup_{(x,y)\in\Omega'} |u^\varepsilon(x,y) - u(x,y)| = 0$.

We claim that there exists $c > 0$ such that u^ε is a viscosity subsolution of the equation

$$|\nabla_x u^\varepsilon(x,y)| + |x|^k |\partial_y u^\varepsilon(x,y)| = c \quad \text{ in } \Omega'. \tag{2.8}$$

Let φ be a function of class C^1 such that $u^\varepsilon - \varphi$ has a local maximum at $(x_0,y_0) \in \Omega'$. Let (x_0', y_0') be a point given by the formula

$$u^\varepsilon(x_0,y_0) = u(x_0',y_0') - \frac{|x_0 - x_0'|^2 + |y_0 - y_0'|^2}{\varepsilon}.$$

Furthermore, we have

$$|x_0 - x_0'|^2 + |y_0 - y_0'|^2 \le \varepsilon \text{ osc }_{\Omega''} u. \tag{2.9}$$

Here, as usual, $\text{osc }_{\Omega''} u = \sup_{\Omega''} u - \inf_{\Omega''} u$.

We remark that (x_0', y_0') is in the interior of Ω''. Indeed, denoting by ω a modulus of continuity for u in $\overline{\Omega''}$, we have that for every $(x,y) \in \Omega'$,

$$u^\varepsilon(x,y) = \sup_{(x',y')\in A} \left\{ u(x',y') - \frac{|x-x'|^2 + |y-y'|^2}{\varepsilon} \right\}$$

with

$$A = \Big\{ (x',y') \in \Omega'' \mid \Big(|x-x'|^2 + |y-y'|^2 \Big) \le 2\|u\|_{L^\infty(\Omega'')}\varepsilon \quad \text{and} \quad \Big(|x-x'|^2 + |y-y'|^2 \Big) \le \varepsilon\omega(|x-y|) \Big\}.$$

Then, for every (x,y) and (x',y') close to (x_0,y_0) and (x_0',y_0') respectively, we have that

$$u(x',y') - \frac{|x-x'|^2 + |y-y'|^2}{\varepsilon} - \varphi(x,y) \le u(x_0',y_0') - \frac{|x_0 - x_0'|^2 + |y_0 - y_0'|^2}{\varepsilon} - \varphi(x_0,y_0).$$

Hence, taking $(x,y) = (x_0 + x' - x_0', y_0 + y' - y_0')$ in the above formula, we deduce that

$$u(x',y') - \varphi(x_0 + x' - x_0', y_0 + y' - y_0') \le u(x_0',y_0') - \varphi(x_0,y_0);$$

i.e. $u-\varphi$ has a local maximum at (x_0', y_0') and

$$D\varphi(x_0,y_0)=\frac{2}{\varepsilon}(x_0'-x_0, y_0'-y_0).$$

Since u is a viscosity subsolution of equation (1.1), we have that

$$\begin{aligned}1 \geq \sqrt{|\nabla_x\varphi(x_0,y_0)|^2+|x_0'|^{2k}(\partial_y\varphi(x_0,y_0))^2}\\ \geq (|\nabla_x\varphi(x_0,y_0)|+|x_0'|^k\ |\partial_y\varphi(x_0,y_0)|)/\sqrt{2}.\end{aligned}$$

Hence, using (2.9), we conclude that there exists $c>0$ dependent on Ω'' and $\operatorname{osc}_{\Omega''}u$ such that

$$|\nabla_x\varphi(x_0,y_0)|+|x_0|^k\ |\partial_y\varphi(x_0,y_0)|\leq c;$$

i.e. (2.8) holds.

In particular, since u^ε is locally Lipschitz continuous, we have that

$$|\nabla_x u^\varepsilon(x,y)|+|x|^k\ |\partial_y u^\varepsilon(x,y)|\leq c \qquad \text{a.e. in } \Omega'.$$

Hence, we deduce that

$$|u^\varepsilon(x,y)-u^\varepsilon(x',y)|\leq c|x-x'| \qquad \forall (x,y),(x',y)\in\Omega',$$

and, taking the limit as $\varepsilon\downarrow 0$ in the above inequality, we conclude that the partial Lipschitz estimate (2.6) holds. Let (x,y') be a point near $(0,y)$; then

$$\begin{aligned}|u(x,y')-u(0,y)|\leq |u(x,y')-u(0,y')|+|u(0,y')-u(0,y)|\\ \leq 2|x|+u(0,1)|(y')^{1/(k+1)}-(y)^{1/(k+1)}|.\end{aligned}$$

In the last line we used the homogeneity of u,

$$\lambda u(0,1)=u(0,\lambda^{k+1})\,.$$

Hence, provided that $|y|$ and $|y'|$ are away from 0 (uniformly bounded below by a positive constant), we deduce that

$$|u(x,y')-u(0,y)|\leq C[|x|+|y-y'|]\,.$$

This completes the proof of the local Lipschitz regularity of u.

3. Proof of Theorem 1.2

We want to show that, for every $y>0$, u is not differentiable at $(0,y)$. We argue by contradiction assuming that u is differentiable at $(0,y)$, for a suitable $y>0$. Then equation (1.1) implies that

$$|\nabla_x u(0,y)|=1\,.$$

On the other hand, by (i) in Theorem 2.1, we have that

$$\nabla_x u(0,y)=0.$$

From this contradiction we deduce that u is not differentiable in the set $\{(0,y) : y>0\}$, i.e. $\{(0,y) : y>0\}\subset\Sigma(u)$. We remark that the point $(0,0)$ cannot be a point of differentiability for u (otherwise u would be better than Hölder continuous at $(0,0)$, in contradiction with Theorem 1.1).

It remains to show that all the singularities of u are in the set $\{(0,y)\mid y\geq 0\}$.

We point out that the points in $\partial\Omega\setminus\{(0,0)\}$ are points of differentiability for u. Indeed, let $(x_0,y_0)\in\partial\Omega\setminus\{(0,0)\}$. Then there exists a neighbourhood of (x_0,y_0),

B, such that $x \neq 0$ for every $(x,y) \in B$. Hence, the Hamiltonian $H(x,p,q) = |p|^2 + |x|^{2k}q^2$ is nondegenerate and smooth. We recall the following characterization (see [2] and [4]): let $\mathcal{O} \subset \mathbb{R}^n$ be a convex set, let $\Gamma \subset \partial\mathcal{O}$ and let u be a continuous viscosity solution of the eikonal equation

$$\begin{cases} \langle A(z)Du(z), Du(z)\rangle = 1 & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Assume that $A(\cdot)$ is positive definite and that $z \mapsto A(z)$ is of class $C^{1,1}$. Then, u is differentiable at $z_0 \in \Gamma$ if and only if Γ is differentiable at z_0.

Since Ω is convex the characterization above applies to u, and we deduce that $(x_0, y_0) \notin \Sigma(u)$ since (x_0, y_0) is a point of differentiability for $\partial\Omega$.

We argue by contradiction assuming that there exists a singular point $(x_1, y_1) \in \Omega$ with $x_1 \neq 0$. We may assume that $x_1 > 0$. Furthermore, since u is rotationally invariant w.r.t. the variables x, it suffices to consider the case of $n = 1$. Condition (ii) in Theorem 2.1 implies that

$$\Lambda := \{(\lambda x_1, \lambda^{k+1} y_1) \mid \lambda > 0\} \subset \Sigma(u).$$

Our proof is completed if we show that $\Lambda = \emptyset$.

We recall that the nonnegative viscosity solution of (1.1) can be represented as the minimum time function of the following problem. Consider the state equation

$$\begin{cases} x'(t) = \alpha(t), \quad t > 0, \\ y'(t) = [x(t)]^k \beta(t), \quad t > 0, \\ (x(0), y(0)) = (x,y) \in \mathbb{R}^2. \end{cases} \tag{3.1}$$

Here, $(\alpha, \beta)\colon [0, +\infty[\to \overline{B}_1(0)$, the control, is a measurable function taking values in the closed unit ball of $\mathbb{R}^2$ with center at the origin. We denote by $\mathcal{A}$ the set of all the control functions. We consider the target $\mathcal{K} = \partial\Omega$. As usual we define the arrival time to the target $\mathcal{K}$ of the trajectory starting at $(x,y) \in \Omega$ as

$$\tau((x,y),(\alpha,\beta)) = \inf\{t \geq 0 \mid (x(t), y(t)) \in \partial\Omega\} \in [0, \infty].$$

Then the minimum time function is given by

$$T(x,y) = \inf_{(\alpha,\beta)\in\mathcal{A}} \tau((x,y),(\alpha,\beta)).$$

Let $(x,y) \in \Omega$; if (α,β) is a control function realizing the above infimum, then we say that (α,β) is an optimal control. Furthermore, the solution of the state equation (3.1) associated to such a control function is called an optimal trajectory. Since the vector fields ∂_x and $x^k\partial_y$ satisfy the Hörmander bracket condition, we have that T is continuous. Furthermore, T is a viscosity solution of equation (1.1). Hence, by uniqueness, $u(x,y) = T(x,y)$ for $(x,y) \in \overline{\Omega}$.

We claim that there exist a point (x_0, y_0), with $x_0 > 0$, and a time optimal trajectory starting at (x_0, y_0), $(x(\cdot), y(\cdot))$ such that

$$y_0 > \frac{y_1}{x_1^{k+1}} x_0^{k+1} \tag{3.2}$$

and

$$(x(T(x_0,y_0)), y(T(x_0,y_0))) \in \partial\Omega \setminus \{(0,0)\}. \tag{3.3}$$

In order to prove our claim we argue by contradiction assuming that for every (x, y), with $x > 0$ and $y > y_1 x^{k+1}/x_1^{k+1}$, and for every time optimal trajectory starting at (x, y), $(x(\cdot), y(\cdot))$, we have that

$$(x(T(x,y)), y(T(x,y))) = (0,0). \tag{3.4}$$

We denote by $d(x, y)$ the sub-Riemannian distance of (x, y) from $(0, 0)$ associated to the vector fields ∂_x and $x^k \partial_y$. We recall that d is a continuous viscosity solution of the equation

$$(\partial_x d(x,y))^2 + x^{2k}(\partial_y d(x,y))^2 = 1 \quad \text{in} \quad \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3.5}$$

Since d is a (super-)solution of equation (3.5) and $d \geq u$ on $\partial\Omega$, by comparison, we deduce that

$$d(x,y) \geq u(x,y) \qquad \forall (x,y) \in \Omega. \tag{3.6}$$

If (x, y) is a point such that $y > y_1 x^{k+1}/x_1^{k+1}$, by (3.4), we deduce that

$$u(x,y) = d(x,y)$$

and a time optimal trajectory starting at (x, y) coincides with a geodesic between (x, y) and $(0, 0)$. We recall that a geodesic is a solution of the Hamiltonian system

$$\begin{cases} z'(t) = \dfrac{1}{2}\zeta(t), \\ y'(t) = \dfrac{1}{2}\eta(t)[z(t)]^{2k}, \\ \zeta'(t) = -\dfrac{k}{2}[\eta(t)]^2\ [z(t)]^{2k-1}, \\ \eta'(t) = 0 \end{cases} \tag{3.7}$$

with the boundary conditions

$$(z(0), y(0)) = (x, y) \qquad \text{and} \qquad (z(1), y(1)) = (0,0).$$

Taking the time derivative of the first equation in (3.7) and using the third equation we find that

$$z''(t) = -\frac{k}{4}\eta^2[z(t)]^{2k-1} \qquad t \in [0,1].$$

Now, an elementary computation yields that

$$t = \int_x^z \left([z'(1)]^2 - \frac{\eta^2 s^{2k}}{4}\right)^{-1/2} ds.$$

Hence, using z as a parameter, we deduce that

$$\frac{dy}{dz} = \frac{\eta}{2} z^{2k}\left([z'(1)]^2 - \frac{\eta^2 z^{2k}}{4}\right)^{-1/2}. \tag{3.8}$$

Set $f(z) = Mz^{k+1}$. The fact that the geodesic $(z(\cdot), y(\cdot))$ belongs to the set Ω implies that there exists $\delta > 0$ such that

$$y(z) > f(z) \qquad \forall z \in]0, \delta[,$$

but this is in contradiction with the fact that, by (3.8), the function y vanishes of higher order than the function f at $z = 0$. Hence our claim (3.3) follows.

We observe that the Hamiltonian, $H(x,p,q) = p^2 + x^{2k}q^2$, is smooth and strictly convex w.r.t. (p,q) if $x \neq 0$. Hence, by [3], u is a semiconcave function in every bounded subset of $\Omega \setminus \{(0,y) \mid y > 0\}$ (i.e. can be written as the sum of a smooth function with a concave function).

Now, let (x_0, y_0) be a point such that conditions (3.2) and (3.3) are fulfilled. Then, following backwards $(x(\cdot), y(\cdot))$, we find that there exists $t_* \in]0, T(x_0,y_0)[$ such that

$$(x(T(x_0,y_0) - t_*), y(T(x_0,y_0) - t_*)) \in \Lambda = \{(\lambda x_1, \lambda^{k+1} y_1) \mid \lambda > 0\} \subset \Sigma(u).$$

This fact is in contradiction with the property that if the minimum time function, $T(\cdot)$, is a semiconcave function and the Hamiltonian is strictly convex w.r.t. the variables p, then $T(\cdot)$ is differentiable along the optimal trajectories (possibly excluding the endpoints); see [8]. We deduce that $\Lambda = \emptyset$.

This completes our proof.

References

[1] P. Albano, *On the regularity near the boundary of viscosity solutions of eikonal equations*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (2006), 177–188. MR2273092 (2007h:35024)

[2] P. Albano, *The singularities of the distance function near convex boundary points*, NoDEA Nonlinear Differential Equations Appl. 16 (2009), 273–281. MR2497333 (2011b:35069)

[3] P. Albano, *On the local semiconcavity of the solutions of the eikonal equation*, Nonlinear Analysis 73 (2010), 458–464. MR2650829

[4] P. Albano, *On the extension of the solutions of Hamilton-Jacobi equations*, Nonlinear Analysis 74 (2011), 1421–1425. MR2746820

[5] P. Albano and A. Bove, *Analytic stratifications and the cut locus of a class of distance functions*, Israel J. Math. 154 (2006), 61–91. MR2254534 (2007i:35019)

[6] G. Barles, Solutions de viscosité des équations de Hamilton-Jacobi. Mathématiques & Applications, 17, Springer-Verlag, Paris, 1994. MR1613876 (2000b:49054)

[7] P. Cannarsa and L. Rifford, *Semiconcavity results for optimal control problems admitting no singular minimizing controls*, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), 773–802. MR2436793 (2009e:49043)

[8] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Progress in Nonlinear Differential Equations and Their Applications, 58, Birkhäuser Boston, 2004. MR2041617 (2005e:49001)

[9] M.G. Crandall, H. Ishii and P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27 (1992), 1–67. MR1118699 (92j:35050)

[10] L.C. Evans and M.R. James, *The Hamilton-Jacobi-Bellman equation for time-optimal control*, SIAM J. Control Optim. 27 (1989), 1477–1489. MR1022439 (90j:49020)

[11] D.A. Gomes, *Hamilton-Jacobi methods for vakonomic mechanics*, NoDEA Nonlinear Differential Equations Appl. 14 (2007), 233–257. MR2364892 (2009a:37121)

[12] H. Ishii, *Perron's method for Hamilton-Jacobi equations*, Duke Math. J. 55 (1987), 369–384. MR894587 (89a:35053)

[13] R. Jensen, P.L. Lions and P.E. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*, Proc. Amer. Math. Soc. 102 (1988), 975–978. MR934877 (89f:35027)

[14] P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics, 69, Pitman, Boston, Mass.-London, 1982. MR667669 (84a:49038)

[15] M. Tamm, *Subanalytic sets in the calculus of variation*, Acta Math. 146 (1981), 167–199. MR611382 (82h:32012)

[16] E. Trélat, *Global subanalytic solutions of Hamilton-Jacobi type equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), 363–387. MR2217656 (2007b:35048)

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40127 Bologna, Italy

E-mail address: albano@dm.unibo.it