ON LYAPUNOV EXPONENTS OF CONTINUOUS SCHRÖDINGER COCYCLES OVER IRRATIONAL ROTATIONS

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Abstract. In this paper we consider continuous, $SL(2, \mathbb{R})$-valued, Schrödinger cocycles over irrational rotations. We prove two generic results on the Lyapunov exponents which improve the corresponding ones contained in a paper by Bjerklöv, Damanik and Johnson.

1. Introduction

Let $\alpha$ be a fixed irrational number and $A : \mathbb{T} \mapsto SL(2, \mathbb{R})$ be a continuous map. Then $A$ generates a continuous, $SL(2, \mathbb{R})$-valued cocycle $\{A(n, \theta)\}$ over the irrational rotations $\theta \mapsto \theta + \alpha$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (or a continuous, quasi-periodic, $SL(2, \mathbb{R})$-valued cocycle with frequency $\alpha$). More precisely, define

$$A(n, \theta) = \begin{cases} A(\theta + (n - 1)\alpha) \ldots A(\theta), & n > 0, \\ Id, & n = 0, \\ A^{-1}(\theta - n\alpha) \ldots A^{-1}(\theta - \alpha), & n < 0. \end{cases}$$

It is clear that $\{A(n, \theta)\}$ satisfies the cocycle property:

$$A(n + m, \theta) = A(n, \theta + m\alpha)A(m, \theta), \quad m, n \in \mathbb{Z}, \quad \theta \in \mathbb{T}.$$

The cocycle admits a well-defined (maximal) Lyapunov exponent given by

$$\Lambda(A) := \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A(n, \theta)\| d\theta = \inf_{n \geq 1} \frac{1}{n} \int_{\mathbb{T}} \log \|A(n, \theta)\| d\theta;$$

i.e., the limit exists and is independent of $\theta$. When $\Lambda(A) > 0$, the corresponding cocycle is said to be uniformly hyperbolic if

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A(n, \theta)\| = \Lambda(A)$$

uniformly in $\theta$ and to be non-uniformly hyperbolic if otherwise.

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In this paper, we pay particular attention to continuous, quasi-periodic, $SL(2, \mathbb{R})$-valued, Schrödinger cocycles with fixed irrational frequency $\alpha$, i.e., a family \( \{A_{f,E}(n, \theta) : E \in \mathbb{R}, f \in C(T) \} \) of quasi-periodic, $SL(2, \mathbb{R})$-valued cocycles with the frequency $\alpha$ which is generated by the continuous, $SL(2, \mathbb{R})$-valued functions
\[
A_{f,E}(\theta) = \left( \begin{array}{cc} E - f(\theta) & -1 \\ 1 & 0 \end{array} \right).
\]
Such cocycles are referred to as Schrödinger cocycles because they arise and play important roles in the study of the spectral problem of the discrete quasi-periodic Schrödinger operator,
\[
[H_f \psi](n) = (\Delta + f(\theta + (n - 1)\alpha)) \psi(n) = E \psi(n),
\]
where $\Delta \psi(n) = \psi(n + 1) + \psi(n - 1)$. For simplicity, we denote $\Lambda_f(E) =: \Lambda(A_{f,E})$, $A_f(n, \cdot) =: A_{f,0}(n, \cdot)$, and $\Lambda_f =: \Lambda_f(0)$.

Related to the spectral problem especially with respect to the non-existence of an absolutely continuous spectrum, one often considers, for a fixed $f$, a two-parameter family \( \{A_{f,E}(n, \theta)\} \) of Schrödinger cocycles, and studies the positivity of the Lyapunov exponents $\Lambda_{\lambda f}(E)$ for $\lambda$ sufficiently large. In particular, when $\alpha$ satisfies appropriate Diophantine conditions, for a certain class of smooth $f$, it is known that $\Lambda_{\lambda f}(E)$ is of scale of $\log \lambda$ as $\lambda \gg 1$ uniformly in $E$ (see, e.g., [2, 6, 11, 12, 16]). However, in a recent work of Bjerklöv, Damanik, and Johnson [3] such uniform bounds are shown to be extremely unstable within the class of continuous functions. More precisely, it is shown in [3] that for every countable set \( \{\lambda_m\}_{m=1}^{\infty} \subset (0, +\infty) \), there exists a residual set of $f \in C(T)$ for which $\inf_{E \in \mathbb{R}} \Lambda_{\lambda_m f}(E) = 0$ for each $m \in \mathbb{N}$.

In this paper, we will show that this result can be improved as follows.

**Theorem 1.** For a residual set of $f \in C(T)$,
\[
\inf_{E \in \mathbb{R}} \Lambda_{\lambda f}(E) = 0
\]
for any $\lambda > 0$.

For general quasi-periodic, continuous, $SL(2, \mathbb{R})$-valued cocycles, it is shown in [4] that there is a residual set $R \subset C(T, SL(2, \mathbb{R}))$ such that for $A \in R$, either $A$ is uniformly hyperbolic or $\Lambda(A) = 0$ (see [9, 10] for similar results that hold for a generic set of pairs $(\alpha, f)$; see also [11]). The same is also shown to hold for Schrödinger cocycles with $E = 0$ ([3, 5]).

Our next result proves the same phenomenon for the parametrized Schrödinger cocycles with $E = 0$.

**Theorem 2.** The set
\[
\{ f \in C(T) : A_{\lambda f}(n, \cdot) \text{ is uniformly hyperbolic or } \Lambda_{\lambda f} = 0 \text{ for any } \lambda \in (0, \infty) \}
\]
is residual.

The rest of this paper is devoted to the proof of Theorems 1 and 2. Our proofs essentially follow the approaches of [3] with necessary modifications.
2. Proofs of theorems

Throughout the rest of the paper, we let $\alpha$ be a fixed irrational number. For a Schrödinger operator $H_f$ of the form \((1.1)\) with $\theta \in \mathbb{T}$ and $f \in L^1(\mathbb{T})$, it is well known that the spectrum $\sigma(H_f)$ is independent of $\theta \in \mathbb{T}$ almost everywhere, and if $f \in C(\mathbb{T})$, then $\sigma(H_f)$ is completely independent of $\theta$. Uniform and non-uniform hyperbolicities of the corresponding (measurable) Schrödinger cocycles $A_{f,E}(n,\cdot)$ can be defined similarly to the continuous case.

As in \cite{3}, the following result will play an important role in the proofs of the theorems.

**Theorem 2.1.** Suppose $f : \mathbb{T} \to \mathbb{R}$ is of the form

\[
f(\theta) = \sum_{m=1}^{M} f_m \chi_{[\beta_{m-1}, \beta_m]}(\theta),
\]

where $0 = \beta_0 < \beta_1 < \cdots < \beta_M = 1$ are rational numbers and $f_1, \ldots, f_M$ are real. Then $\sigma(H_f) = \{E : \Lambda_f(E) = 0\}$.

**Proof.** See \cite{7,8}. $\square$

A crucial step in proving the above result is to show that for any $f$ having the form \((2.1)\),

\[
\lim_{n \to \infty} \frac{1}{n} \log \|A_{f,E}(n,\theta)\| = \Lambda_f(E)
\]

for every $E \in \mathbb{R}$ uniformly in $\theta \in \mathbb{T}$ \((7,8,14)\). The result then follows from the following.

**Theorem 2.2.** For any $f \in L^1(\mathbb{T})$,

\[
\sigma(H_f) = \{E : \Lambda_f(E) = 0 \text{ or } A_{f,E}(n,\cdot) \text{ is non-uniformly hyperbolic}\}.
\]

**Proof.** See \cite{15,13}. $\square$

**Lemma 2.3.** For any non-empty compact subset $K \subseteq (0, +\infty)$, the set

\[
M_{K,0} := \{f \in C(\mathbb{T}) : \inf_{E \in \mathbb{R}} \Lambda_{f,E}(E) = 0 \text{ for any } \lambda \in K\}
\]

is residual in $C(\mathbb{T})$.

**Proof.** Let $K \subseteq (0, +\infty)$ be a non-empty compact subset. We consider the family of sets

\[
M_{K,\delta} = \{f \in C(\mathbb{T}) : \forall \lambda \in K \exists E_\lambda \in \mathbb{R} \text{ such that } \Lambda_{f,E}(E_\lambda) < \delta\}, \quad \delta > 0.
\]

We will show that each $M_{K,\delta}$ is open and dense, and hence $M_{K,0} = \bigcap_{\delta > 0} M_{K,\delta}$ is residual.

First we show that $M_{K,\delta}$ is open, i.e., $C(\mathbb{T}) \setminus M_{K,\delta}$ is closed. Let $\{f_n\} \subseteq C(\mathbb{T}) \setminus M_{K,\delta}$ be such that $\|f_n - f\|_\infty \to 0$. Then for each $n \in \mathbb{N}$ there exists a $\lambda_n \in K$ with $\Lambda_{\lambda_n, f_n}(E) \geq \delta$ for all $E \in \mathbb{R}$. Since $K$ is compact, there exists a subsequence $\{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ such that $\lim_{i \to \infty} \lambda_{n_i} = \lambda_0$ for some $\lambda_0 \in K$. It follows from the upper-semicontinuity of Lyapunov exponents $\Lambda_{f,E}(\lambda)$ in $\lambda$ that

\[
\Lambda_{\lambda_0,f}(E) \geq \limsup_{i \to \infty} \Lambda_{\lambda_{n_i}, f_{n_i}}(E) \geq \delta
\]

for any $E \in \mathbb{R}$. Hence $f \in C(\mathbb{T}) \setminus M_{K,\delta}$. This shows that $C(\mathbb{T}) \setminus M_{K,\delta}$ is closed.
Next we show that $M_{K,\delta}$ is dense. Let $\epsilon > 0$ and $g \in C(T)$ be given. In the $\frac{\epsilon}{2}$-neighborhood of $g$ with respect to the $L^\infty$ topology, we choose a step function $s$ of the form \( \ell \); i.e., $s$ has finitely many points of discontinuity, all of which are rational, and the jumps of $s$ are bounded by $\frac{\epsilon}{2}$. It then follows from Theorem 2.3 that for any $\lambda \in K$, $\Lambda_{\lambda}$ vanishes on the spectrum $\sigma(H_{\lambda'} \sigma)$; i.e., there exists an $E_{\lambda} \in \sigma(H_{\lambda'} \sigma)$ such that $\Lambda_{\lambda}(E_{\lambda}) = 0$. By the upper-semicontinuity of Lyapunov exponents, there exists a $\delta_\lambda > 0$, for each $\lambda \in K$, such that $\Lambda_{u_\lambda}(E_{\lambda}) < \delta$ for any $u \in B(\lambda, \delta_\lambda) := \{ t \in \mathbb{R} : |t - \lambda| < \delta_\lambda \}$. As $K$ is compact, there exist $u_1, \ldots, u_\ell \in K$ such that $K \subseteq \bigcup_{i=1}^\ell B(u_i, \frac{\delta_\lambda}{2})$. Then

$$\Lambda_{\lambda}(E_{u_i}) < \delta$$

for all $1 \leq i \leq \ell$ and $\lambda \in B(u_i, \delta_\lambda) \cap K$.

Let $\{ f_n \} \subset C(T)$ be such that $\int_0^\pi |s(\theta) - f_n(\theta)| d\theta < \frac{1}{n}$ and $\|s - f_n\|_\infty < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$. We claim that there exists an $n_\delta \in \mathbb{N}$ such that $\Lambda_{u_{i_n}}(E_{u_{i_n}}) < \delta$ for all $1 \leq i \leq \ell$ and $\lambda \in B(u_{i_n}, \frac{\delta_\lambda}{2}) \cap K$; i.e., $\lambda = f_{i_n}$ has the desired properties that $f \in M_{K,\delta}$ and $\|f - g\|_\infty < \epsilon$.

If the claim is not true, then for each $n \in \mathbb{N}$ there exist $i_n \in \{ 1, 2, \ldots, \ell \}$ and $\lambda_n \in B(u_{i_n}, \frac{\delta_{u_{i_n}}}{2}) \cap K$ such that $\Lambda_{u_{i_n}}(E_{u_{i_n}}) > \delta$. Without loss of generality, we assume that $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$ for some $\lambda_0 \in K$ and $i_n \equiv i_0 \equiv 1 \in \{ 1, 2, \ldots, \ell \}$ for all $n \in \mathbb{N}$. It is clear that $\lambda_0 \in B(u_{i_0}, \delta_{u_{i_0}}) \cap K$ and $\lim_{\theta \rightarrow \infty} \int_0^\pi |\lambda_0 s(\theta) - \lambda_k f_{i_0}(\theta)| d\theta = 0$. Hence by the upper semi-continuity of Lyapunov exponents, we have

$$\delta > \Lambda_{\lambda_0}(E_{u_{i_0}}) \geq \limsup_{n \rightarrow \infty} \Lambda_{\lambda_k}(E_{u_{i_0}}) \geq \delta,$$

a contradiction. \( \square \)

**Proof of Theorem 1.** Let $K_n = [\frac{1}{n}, n], n \in \mathbb{N}$. Then by Lemma 2.3

$$\{ f \in C(T) : \inf_{E \in \mathbb{R}} \Lambda_f(E) = 0 \text{ for any } \lambda > 0 \} = \bigcap_{n=1}^{\infty} M_{K_n,0}$$

is residual. \( \square \)

**Proof of Theorem 2.** It is sufficient to show that for any non-empty compact set $K \subseteq (0, \infty)$, the set

$$N_K = \{ f \in C(T) : \exists \lambda \in K \text{ s.t. } A_{\lambda f}(n, \cdot) \text{ is non-uniformly hyperbolic} \}$$

is a meagre set, i.e., a countable union of nowhere-dense sets. This will follow once we prove that

$$N_{K,\gamma} = \{ f \in C(T) : \exists \lambda \in K \text{ s.t. } A_{\lambda f}(n, \cdot) \text{ is non-uniformly hyperbolic and } \Lambda_{\lambda} \geq \gamma \}$$

is nowhere dense for every $\gamma > 0$.

Let $\gamma > 0$ be given. We first show that $N_{K,\gamma}$ is closed. Let $\{ f_i \} \subset N_{K,\gamma}$ and $f_0 \in C(T)$ be such that $\lim_{i \rightarrow \infty} \|f_i - f_0\|_\infty = 0$. Then for each $i \in \mathbb{N}$, there exists a $\lambda_i \in K$ such that $A_{\lambda_i f_i}(n, \cdot)$ is non-uniformly hyperbolic and $\Lambda_{\lambda_i} \geq \gamma$. Without loss of generality, we assume that $\lim_{i \rightarrow \infty} \lambda_i = \lambda_0$ for some $\lambda_0 \in K$. Then $\lim_{i \rightarrow \infty} \|\lambda_i f_i - \lambda_0 f_0\|_\infty = 0$, and hence $\Lambda_{\lambda_0 f_0} \geq \limsup_{i \rightarrow \infty} \Lambda_{\lambda_i f_i} \geq \gamma$ according to the upper semi-continuity of Lyapunov exponents. Since uniform hyperbolicity is an open property, $A_{\lambda_0 f_0}(n, \cdot)$ is non-uniformly hyperbolic. This shows that $f_0 \in N_{K,\gamma}$. Hence $N_{K,\gamma}$ is closed.
Next we show that $N_{K,\gamma}$ has no interior. This amounts to showing that for any given $f \in N_{K,\gamma}$ and $\epsilon > 0$ there exists a function $g \in C(\mathbb{T})$ such that $\|f - g\|_\infty < \epsilon$ and $g \notin N_{K,\gamma}$. For the given $f \in N_{K,\gamma}$, we let $\lambda_\ast \in K$ be such that $A_{\lambda,f}(n,\cdot)$ is non-uniformly hyperbolic and $\Lambda_{\lambda,f} \geq \gamma$. Also let $\{s_m\}$ be a sequence of step functions of the form (2.1) in the $\frac{\epsilon}{4}$-neighborhood of $f$ that converge to $f$ in the $L^\infty$ topology. Then for each $\theta \in \mathbb{T}$, the operators $H_m = \triangle + \lambda_m s_m(\alpha + \theta)$ converge strongly to $H = \triangle + \lambda_s(\alpha + \theta)$. Since $A_{\lambda,f}(n,\cdot)$ is non-uniformly hyperbolic, we have by Theorem 2.2 that $0 \in \sigma(H)$. By the strong convergence of $H_m$, we also have a sequence $\{E_m\} \subset \sigma(H_m)$ such that $E_m \to 0$. Now let $m \to 1$ be fixed such that $|E_m| < \frac{\epsilon}{2}$. Then $s = s_m - E_m$ is a step function of the form (2.1) in the $\frac{\epsilon}{2}$-neighborhood of $f$ such that $0$ belongs to the spectrum of $\overline{H} = \triangle + \lambda_s(\alpha + \theta)$. It follows from Theorem 2.2 that $\Lambda_{\lambda,s} = 0$. Now consider a sequence of continuous functions $\{g_i\} \subset C(\mathbb{T})$ with $\int_{\mathbb{T}} |s(\theta) - g_i(\theta)| d\theta < \frac{\epsilon}{4}$ and $\|s - g_i\|_\infty < \frac{\epsilon}{4}$ for all $i \in \mathbb{N}$. We have by the upper semi-continuity of Lyapunov exponents that

$$0 \leq \Lambda_{\lambda,s} \geq \lim_{i \to \infty} \Lambda_{\lambda,s,g_i}.$$ 

Hence we can choose a $k \in \mathbb{N}$ such that the function $g = g_k$ has the desired properties that $\Lambda_{\lambda,s} < \gamma$ and $\|f - g\|_\infty < \epsilon$. \hfill \square

References


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