BASIC HYPERGEOMETRIC FUNCTIONS
AND ORTHOGONAL LAURENT POLYNOMIALS

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ABSTRACT. A three-complex-parameter class of orthogonal Laurent polynomials on the unit circle associated with basic hypergeometric or q-hypergeometric functions is considered. To be precise, we consider the orthogonality properties of the sequence of polynomials \( \{ 2\Phi_1(q^{-n}, q^{b+1}; q^{-c+b-n}; q^{-c+d-1}z) \}_{n=0}^{\infty} \)
where \( 0 < q < 1 \) and the complex parameters \( b, c \) and \( d \) are such that \( b \neq -1, -2, \ldots, c - b + 1 \neq -1, -2, \ldots, \Re(d) > 0 \) and \( \Re(c - d + 2) > 0 \). Explicit expressions for the recurrence coefficients, moments, orthogonality and also asymptotic properties are given. By a special choice of the parameters, results regarding a class of Szegő polynomials are also derived.

1. Introduction

Given the double sequence \( \{ \mu_n \}_{n=-\infty}^{\infty} \) of complex numbers, let the linear functional \( M \) on the space of Laurent polynomials be defined by
\[
M[z^{-n}] = \mu_n, \quad n = 0, \pm 1, \pm 2, \ldots .
\]
The functional \( M \) can be referred to as a moment functional.

Let \( D_n, n = 0, 1, \ldots, \) be the associated Toeplitz determinants as given by:
\[
D_{-1} = 1, \quad D_0 = \mu_0 \quad \text{and} \quad D_n = \left| \begin{array}{cccc}
\mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\
\mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_n & \mu_{n-1} & \cdots & \mu_0
\end{array} \right|, \quad n \geq 1.
\]

We consider the sequence of polynomials \( \{ Q_n \} \) that satisfies
\[
M[z^{-s}Q_n(z)] = \rho_n \delta_{n,s}, \quad 0 \leq s \leq n, \quad n \geq 1,
\]
where \( Q_n \), for any \( n \geq 0 \), is a monic polynomial of degree \( n \). If the moment functional \( M \) is such that \( D_n \neq 0, n \geq 0 \), then we will refer to it as a semi-definite
moment functional. In this case it is easily seen that the sequence of polynomials \( \{Q_n\} \) exists uniquely and that

\[
Q_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_n \\
\mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1} & \mu_{n-2} & \cdots & \mu_{-1} \\
1 & z & \cdots & \mu_n \\
\end{vmatrix}
\]

and \( \rho_n = \mathcal{M}[z^{-n}Q_n(z)] = \frac{D_n}{D_{n-1}}. \)

There have been different nomenclatures used with respect to such polynomials in recent years. The polynomials \( Q_n \) are related to orthogonal Laurent polynomials considered by, for example, Hendriksen and van Rossum [11] and Jones and Thron [15], in the sense that the Laurent polynomials

\[
B_{2n}(z) = z^{-n}Q_{2n}(z), \quad B_{2n+1}(z) = z^{-n}Q_{2n+1}(z), \quad n \geq 0,
\]
satisfy the orthogonality relations \( \mathcal{M}[B_n(z)B_m(z)] = \delta_{n,m}\rho_n, \quad n, m = 0, 1, 2, \ldots. \)

With the monic polynomials \( \{Q_n\} \) given by

\[
\hat{Q}_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_n \\
\mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n+1} & \mu_{n+2} & \cdots & \mu_1 \\
1 & z & \cdots & \mu_n \\
\end{vmatrix}, \quad n \geq 1,
\]

we obtain the biorthogonality relations \( \mathcal{M}[\hat{Q}_m(1/z)Q_n(z)] = \delta_{n,m}\rho_n, \quad n, m = 0, 1, 2, \ldots. \) Hence, Zhedanov [32] calls such polynomials Laurent biorthogonal.

With respect to the moment functional \( \mathcal{L}[z^n] = \mathcal{M}[z^{-n}] = \mu_n, \quad n = 0, \pm 1, \pm 2, \ldots, \) the reciprocal polynomials \( Q^*_n(z) = z^nQ_n(1/z) \) satisfy the orthogonality relations \( \mathcal{L}[z^{-n+s}Q^*_n(z)] = \delta_{n,s}\rho_n, \quad 0 \leq s \leq n. \) Polynomials satisfying such orthogonality relations have been referred to as \( L \)-orthogonal polynomials in some earlier contributions, including [1], of one of the present authors. We remark that Zhedanov [32] uses the definition \( \mathcal{L}[z^n] = \mu_n, \quad n = 0, \pm 1, \pm 2, \ldots, \) for his moment functional and \( \mathcal{L}[\hat{Q}_m(z)Q_n(1/z)] = \delta_{n,m}\rho_n, \quad n, m = 0, 1, 2, \ldots. \)

In a recent manuscript [17], \( \{Q_n\} \) has been called a sequence of monic Szegő type polynomials when \( \mathcal{M} \) is such that \( D_n \neq 0 \) and \( \mu_{-n} = \mu_n \) for \( n \geq 0. \) In this case the Zhedanov [32] biorthogonality can be written as \( \mathcal{M}[Q_m(1/z)Q_n(z)] = \delta_{n,m}\rho_n, \quad n, m = 0, 1, 2, \ldots. \)

However, if \( \mathcal{M} \) is such that \( D_n > 0 \) and \( \mu_{-n} = \mu_n, \quad n \geq 0, \) then this moment functional is known as a positive definite moment functional and the sequence of polynomials \( \{S_n\} = \{Q_n\} \) are known as monic Szegő polynomials. Now we must have \( \mathcal{M}[f] = \int_C f(z)d\mu(z), \) where \( \mu(z) = \mu(e^{i\theta}) \) is a positive measure on the unit circle \( C = \{z = e^{i\theta}: 0 \leq \theta \leq 2\pi\}. \) Since the integration is along the unit circle, \( \int_C z^{-j}S_n(z)d\mu(z) = \int_C \overline{z}^jS_n(z)d\mu(z) \) and the associated sequence of monic Szegő polynomials \( \{S_n\} \) are usually defined by

\[
\int_C \overline{S_n(z)}S_m(z)d\mu(z) = \int_C \overline{S_n(e^{i\theta})}S_m(e^{i\theta})d\mu(e^{i\theta}) = \kappa_{n,j}^{-2}\delta_{n,j}, \quad m, n = 0, 1, 2, \ldots,
\]

where \( \kappa_n^{-2} = \|S_n\|^2 = \int_C |S_n(z)|^2d\mu(z). \)

With his publications [23] and [29], Szegő introduced these orthogonal polynomials on the unit circle in the early 20th century. Many interesting results on these
orthogonal Laurent polynomials can be found in his classical book [30], the first edition of which was published in 1939. Since then, these polynomials which bear the name of Szegő were extensively studied by many. We cite, for example, [5], [6], [7], [10], [19], [20], [22], [23] and [27] as some of the very recent contributions. The recent publications of the two excellent volumes [23] and [24] by Simon have given a boost to the interest in studying these polynomials. We also cite the recent book [13] by Ismail containing a nice chapter on these orthogonal polynomials on the unit circle.

Some information on the Szegő polynomials with respect to the measure \( d\mu(e^{i\theta}) = [e^{-\theta}]\lvert\sin^2(\theta/2)\rvert^\lambda d\theta \), defined for \( \eta, \lambda \in \mathbb{R} \) and \( \lambda > -1/2 \), are provided in [27]. It was shown that these Szegő polynomials are constant multiples of the hypergeometric functions \( \mathcal{F}_2(-n, b + 1; b + b + 1; 1 - z) \), \( n \geq 1 \), where \( \eta = \Im m(b) \) and \( \lambda = \Re e(b) \).

Results used in [27] have an important root in the paper [11] of Hendriksen and van Rossum, where these authors look at T-fractions and orthogonal Laurent polynomials originating from three-term contiguous relations satisfied by the hypergeometric functions \( \mathcal{F}_2(a, b; c; z) \).

In this paper, using a three-term contiguous relation satisfied by \( \mathcal{F}_3(q^{-b+1}; q^{b+1}; q; q, q, q^{c+d-1} - z) \), with 0 < \( q < 1 \) and the three complex parameters \( b, c \) and \( d \) are such that \( b \neq -1, -2, \ldots, c - b + 1 \neq -1, -2, \ldots \), \( \Re e(d) > 0 \) and \( \Re e(c + 2 - d) > 0 \). The orthogonality is with respect to the semi-definite moment functional \( \mathcal{M}^{(b, c, d)} \) given by

\[
\mathcal{M}^{(b, c, d)}[f(z)] = \frac{\tau^{(b, c)}}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \left( \frac{q^{b+1} e^{i\theta}}{q^{b} e^{-i\theta}} \right)_\infty \left( \frac{q^{c+d} e^{i\theta}}{q^{c+d-2} e^{-i\theta}} \right)_\infty d\theta.
\]

Here the constant \( \tau^{(b, c)} \), defined in Theorem 5.3, is such that \( \mathcal{M}^{(b, c, d)}[1] = 1 \). By considering separately the real and imaginary parts of \( b, c \) and \( d \), and neglecting \( \Im m(d) \), which induces only a rotation, we can also consider \( \{ z^{-\lfloor n/2 \rfloor} Q_n^{(b, c, d)}(z) \}_n \) as a five-real-parameter class of orthogonal Laurent polynomials.

The class of polynomials considered here is somewhat different and broader than the class of orthogonal Laurent polynomials \( \{ z^{-\lfloor n/2 \rfloor} P_n(-z, \alpha, \beta) \}_n \) that follows from Pastro [21], where

\[
P_n(z, \alpha, \beta) = \mathcal{F}_3(q^{-b+1}; q^{b+1}; q; q^{-1}q^{-\alpha}; q^{-1}q^{-\beta}; q^{-1}q^{-\beta+3/2} - z),
\]

with \( \lvert q \rvert < 1 \), \( \alpha > 1/2 \) and \( \beta > 1/2 \). Pastro shows that the polynomials \( P_n(-z, \alpha, \beta) \) are the Laurent biorthogonal polynomials with respect to the semi-definite moment functional given by

\[
\mathcal{M}^{(\alpha, \beta)}[f(z)] = \int_0^{2\pi} f(e^{i\theta}) \left( \frac{q^{1/2} e^{i\theta}}{q^{1/2} e^{-i\theta}} \right)_\infty \left( \frac{q^{1/2} e^{-i\theta}}{q^{1/2} e^{i\theta}} \right)_\infty d\theta.
\]

These Pastro polynomials can be considered as belonging to a class with the three real parameters \( \alpha, \beta \) and, say, \( \bar{\beta} \) if one assumes the \( \bar{q} \) to be such that \( \bar{q} = |q| e^{i\theta} \). An
explicit expression for the moments \( \tilde{M}^{(\alpha, \beta)}[z^{-n}] \) is found in Vinet and Zhedanov [31].

Note that the moment functional \( \tilde{M}^{(\alpha, \beta)} \) can only be made positive definite with the choice \(-1 < \tilde{q} < 1\) and \(\alpha = \beta > 1/2\). Thus with this choice, Pastro [21] recovers the class of real Szegő polynomials previously described by Askey [4] p. 806).

By a special choice of the parameters \(b, c\) and \(d\) we also obtain in the present manuscript information regarding the class of (complex and real) Szegő polynomials \(S_n^{(\lambda, \eta, \phi)}\) characterized by the reflection coefficients

\[
q_n^{(\lambda, \eta, \phi)} = \frac{(q^{\lambda+\eta}; q)_n}{(q^{\lambda+1-i\eta}; q)_n}, \quad n \geq 1,
\]

where \(\lambda, \eta, \phi \in \mathbb{R}\) and \(\lambda > -1/2\). The parameter \(\phi\), which comes from \(Im(d)\), induces only a rotation and can be made equal to zero without any loss of generality.

The polynomials obtained by taking \(\eta = \phi = 0\) and \(\lambda > -1/2\), for example, coincide with the real Szegő polynomials of [21] and [4], obtained when \(0 < \tilde{q} < 1\) and \(\alpha = \beta > 1/2\).

The paper is organized as follows. In Section 2 we present some fundamental results on three-term recurrence relations, continued fractions and basic hypergeometric functions, which we will be using in later sections. In Section 3 we define the monic \(q\)-hypergeometric polynomials \(Q_n^{(b, c, d)}(z)\) and obtain their orthogonality and asymptotic properties. In Section 4, in addition to discussing when the polynomials \(Q_n^{(b, c, d)}(z)\) coincide with the Szegő polynomials \(S_n^{(\lambda, \eta, \phi)}\) mentioned above, we also obtain explicitly the associated Szegő function.

2. SOME PRELIMINARY RESULTS

Let \(\{Q_n\}\) be the sequence of polynomials given by the three-term recurrence relation

\[
Q_{n+1}(z) = (z + \beta_{n+1})Q_n(z) - \alpha_{n+1}zQ_{n-1}(z), \quad n \geq 1,
\]

with \(Q_0(z) = 1\) and \(Q_1(z) = z + \beta_1\).

**Lemma 2.1.** Let \(\beta_1 \neq 0\) and \(\alpha_{n+1} \neq 0\) for \(n \geq 1\). Given any sequence \(\{h_n\}\) of arbitrary complex numbers \(h_n\) (or complex functions \(h_n(z)\)), let the sequence of functions \(\{G_n(h_n; z)\}\) be such that \(G_1(h_1; z) = \frac{\beta_1}{z + \beta_1 - h_1}\) and

\[
G_n(h_n; z) = \frac{\beta_1}{z + \beta_1} - \frac{\alpha_2 z}{z + \beta_2} - \cdots - \frac{\alpha_n z}{z + \beta_n - h_n}, \quad n \geq 2.
\]

Then

\[
G_n(h_n; z) - G_n(0; z) = \frac{\beta_1 \alpha_2 \alpha_3 \cdots \alpha_n h_n z^{n-1}}{Q_n(z)Q_n(z) - h_nQ_{n-1}(z)}.
\]

**Proof.** Let the sequence of polynomials \(\{R_n\}\) be such that

\[
R_{n+1}(z) = (z + \beta_{n+1})R_n(z) - \alpha_{n+1}zR_{n-1}(z), \quad n \geq 1,
\]

with \(R_0(z) = 0\), \(R_1(z) = \beta_1\). Then from basic results on continued fractions (see, for example, [14], [15])

\[
G_n(h_n; z) - G_n(0; z) = \frac{R_n(z) - h_nR_{n-1}(z)}{Q_n(z) - h_nQ_{n-1}(z) - R_n(z)} - \frac{R_n(z)}{Q_n(z)}, \quad n \geq 1.
\]
where $Q_n$ be defined by

\[
0 = G_0(z) = \sum_{\alpha_2}^{\alpha_n} a_{\alpha_2} \cdots a_{\alpha_n} \cdot Q_n(z) = \beta_1 \alpha_2 \alpha_3 \cdots \alpha_n z^{n-1}.
\]

Therefore, the lemma follows from $R_n(z)Q_{n-1}(z) - Q_n(z)R_{n-1}(z) = \beta_1 \alpha_2 \alpha_3 \cdots \alpha_n z^{n-1}$. □

As a particular case of this lemma, if one takes $h_n = \alpha_n z/(z + \beta_n)$, then

\[
G_{n+1}(0; z) - G_n(0; z) = \frac{\beta_1 \alpha_2 \alpha_3 \cdots \alpha_{n+1}}{Q_n(z)Q_{n+1}(z)} z^n, \quad n \geq 1.
\]

**Lemma 2.2.** In the three-term recurrence relation (2.1), if

\[
\beta_n \neq 0 \quad \text{and} \quad \alpha_{n+1} \neq 0, \quad n \geq 1,
\]

then there exists a semi-definite moment functional $\mathcal{M}$ such that the polynomials $Q_n$ satisfy

\[
\mathcal{M}[z^{-s} Q_{n}(z)] = \delta_{n,s} \rho_n, \quad 0 \leq s \leq n, \quad n \geq 1,
\]

where $\rho_n = \frac{\alpha_2 \cdots \alpha_{n+1}}{\beta_2 \cdots \beta_{n+1}}$. Moreover, the associated moments $\mu_n = \mathcal{M}[z^{-n}]$, $n = 0, \pm 1, \pm 2, \ldots$ are such that $L_0(z) = \sum_{j=0}^{\infty} \mu_j z^j$, $L_\infty(z) = -\sum_{j=1}^{\infty} \mu_j z^{-j}$, where

\[
L_0(z) - G_n(0; z) = \rho_n \frac{1}{Q_n(0)} z^n + O(z^{n+1}),
\]

\[
L_\infty(z) - G_n(0; z) = \rho_n Q_{n+1}(0) \frac{1}{z^{n+1}} + O(\frac{1}{z^{n+2}}).
\]

**Proof.** First note that $Q_n(0) = \beta_1 \beta_2 \cdots \beta_n \neq 0$. Now from (2.2) by considering the expansions of $G_n(0; z)$ about the origin and infinity there exist power series $L_0(z) = \sum_{j=0}^{\infty} \mu_j z^j$ and $L_\infty(z) = -\sum_{j=1}^{\infty} \mu_j z^{-j}$ such that (2.3) holds.

With respect to these power series coefficients, if we define the moment functional $\mathcal{M}$ by (1.1), then the lemma follows from the linear system on the coefficients of $Q_n$ and $R_n$ obtained from (2.3).

For $a, b, c \in \mathbb{C}$, $c \neq -1, -2, \ldots$ and $0 < |q| < 1$, the $2\Phi_1$ $q$-hypergeometric or the $2\Phi_1$ basic hypergeometric function (hypergeometric function with base $q$) may be defined by

\[
2\Phi_1(a; q^\alpha; q^\beta; q; z) = \sum_{n=0}^{\infty} \frac{(q^\alpha; q)_n (q^\beta; q)_n}{(q^\gamma; q)_n (q; q)_n} z^n,
\]

for $|z| < 1$ and by analytic continuation for other values of $z \in \mathbb{C}$. Here, $(q^\alpha; q)_0 = 1$ and $(q^\alpha; q)_n = (1-q^\alpha)(1-q^{\alpha+1}) \cdots (1-q^{\alpha+n-1})$, $n \geq 1$.

For more information regarding $q$-hypergeometric functions, we refer to, for example, Andrews, Askey and Roy [2], Gasper and Rahman [8], Koekoek and Swarttouw [10] and Slater [20].

Two “distinct” $q$-hypergeometric functions $2\Phi_1(a_1; q^{a_1}; q^{a_2}; q; z)$ and $2\Phi_1(a_1, q^{a_2}; q, z)$ may be called contiguous if $|a_i - \bar{a}_i| = 0$ or 1 for at least one $i \in \{1, 2, 3\}$. There are interesting relations between contiguous $q$-hypergeometric functions called contiguous relations.
Lemma 2.3. If \( c \neq 0, -1, -2, \ldots \), then
\[
2\Phi_1(q^a, q^{b+1}; q^c; q, z) = (1 + \frac{1 - q^{a-b}}{1 - q^c}) \Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z)
\]
\[
\frac{(1 - q^{a+1}) (1 - q^{c-b})}{(1 - q^c) (1 - q^{c+1})} q^b z 2\Phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, z).
\]

Proof. From contiguous relations obtained by Heine (see [8, p. 22]), we consider the following:
\[
2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z) = 2\Phi_1(q^{a+1}, q^{b}; q^{c}; q, z)
\]
\[
+ \frac{(1 - q^{a+1}) (1 - q^{c-b})}{(1 - q^c) (1 - q^{c+1})} q^b z 2\Phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, z),
\]
\[
2\Phi_1(q^{a+1}, q^{b}; q^{c}; q, z) = 2\Phi_1(q^a, q^{b+1}; q^{c}; q, z)
\]
\[
- \frac{(1 - q^{a-b})}{(1 - q^c)} q^b z 2\Phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, z),
\]
which hold for \( c \neq 0, -1, -2, \ldots \). Substitution for \( 2\Phi_1(q^{a+1}, q^{b}; q^{c}; q, z) \) in the first relation using the other gives the required result. \( \square \)

We will be using the \( q \)-binomial theorem (see [16, Eq. (0.5.2)]),
\[
(2.4) \quad 2\Phi_1(q^a, q^{c}; q; q, z) = 1 \Phi_0(q^a; q, z) = \frac{(q^a z; q)_\infty}{(z; q)_\infty},
\]
which holds for \( c \neq 0, -1, -2, \ldots \) and \( |z| < 1 \), and the Heine transformation formula (see [16, Eq. (0.6.3)]),
\[
(2.5) \quad 2\Phi_1(q^{a+c}, q^{c+1}; q^{a+c}; q, z)
\]
\[
(2.6) \quad \frac{(q^b; q)_n}{(q^c; q)_n} q^{-n(n+1)/2} (-z)^n 2\Phi_1(q^{-n}, q^{c-n+1}; q^{b-n+1}; q, q^{c-b+n+1} z^{n-1}),
\]
for \( n \geq 0, \) which hold when \( c \neq 0, -1, -2, \ldots \) and \( b \neq -n + 1, -n + 2, -n + 3, \ldots \).

3. \( q \)-ORTHOGONAL LAURENT POLYNOMIALS

From now on we restrict the value of \( q \) to be such that \( 0 < q < 1 \). Then for any \( b \in \mathbb{C} \) we have
\[
q^b = \hat{q}^b \quad \text{and} \quad |q^b| = q^{\Re(b)}.
\]

With \( b, c, d \in \mathbb{C} \) and \( c - b + 1 \neq 0, -1, -2, \ldots \), let
\[
F_n^{(b,c,d)}(z) = \frac{2\Phi_1(q^{n+1}, q^{-b}; q^{c-b+n+2}; q, q^d z)}{2\Phi_1(q^n, q^{-b}; q^{c-b+n+1}; q, q^d z)} , \quad n \geq 0.
\]
Then from Lemma 2.3
\[
F_n^{(b,c,d)}(z) = \frac{1}{1 + y_n^{(b,c,d)} z - f_n^{(b,c,d)} z^2 f_n^{(b,c,d)}(z)} , \quad n \geq 0,
\]
where
\[
y_n^{(b,c,d)} = \frac{y^{(b,c,d)}}{n+1} z - \frac{f_n^{(b,c,d)}}{n+2} z^2 f_n^{(b,c,d)}(z).
\]
with respect to the semi-definite moment functional

\[ M_{n+1}(\beta) = (1 - q^\beta) (1 - q^{\beta + 1}) q^{-b+d-1}, \]

for \( n \geq 1 \). This leads to the continued fraction expansion

\[ F_n(z) = \frac{1}{1 + g_n(z)} \frac{f_n(z)}{1 + g_{n+1}(z) F_n(z)}. \]

Also assuming \( b \neq -1, -2, \ldots \), this can be written in the equivalent form

\[ F_n(z) = \frac{\beta_n}{z + \beta_n} - \frac{\alpha_n}{z + \beta_n} - \cdots - \frac{\alpha_n}{z + \beta_n} \frac{F_n(z)}{\beta_n}, \]

where \( \beta_n = 1/g_n(z) \) and \( \alpha_n = f_n(z) / (g_n(z)) \), \( n \geq 1 \).

**Theorem 3.1.** With \( b \neq -1, -2, \ldots \) and \( c - b + 1 \neq -1, -2, \ldots \), let the sequence of monic polynomials \( \{Q_n(z)\} \) be given by

\[ Q_n(z) = (z + \beta_n) Q_{n+1}(z) - \alpha_n z Q_{n-1}(z), \quad n \geq 1, \]

with \( Q_0(z) = 1 \) and \( Q_1(z) = z + \beta_1 \), where

\[ \beta_n = \frac{1 - q^{b+n}}{1 - q^{b+n}} q^{b+d+1}, \quad \alpha_n = \frac{(1 - q^n) (1 - q^{n+1})}{(1 - q^{b+n}) (1 - q^{d+n})} q^{b+d+1}, \quad n \geq 1. \]

Then the polynomials \( Q_n(z) \) satisfy the orthogonality relations

\[ \mathcal{M}(b,c,d) \left[ z^{-s} Q_n(z) \right] = \delta_{s,n} \rho_n, \quad 0 \leq s \leq n, \quad n \geq 1, \]

with respect to the semi-definite moment functional

\[ \mathcal{M}(b,c,d) \left[ z^{-j} \right] = \frac{(q^b; q)_j}{(q^{c+b+2}; q)_j} q^{j d}, \quad j = 0, \pm 1, \pm 2, \ldots. \]

Here,

\[ \rho_n = \frac{\alpha_2 \cdots \alpha_n}{\beta_2 \cdots \beta_n} = \frac{(q; q)_n (q^{c+2}; q)_n}{(q^{b+1}; q)_n (q^{c-b+2}; q)_n}. \]

**Proof.** We first prove the theorem for \( c - b + 1 \neq 0, -1, -2, \ldots \) and \( b \neq -1, -2, \ldots \). With these restrictions \( \beta_n \neq 0 \) and \( \alpha_n \neq 0 \), \( n \geq 1 \), and hence from Lemma 2.2 there exists a semi-definite moment functional such that (3.4) holds.

To obtain the values of \( \mu_j \) \( \beta_n \neq 0 \) and \( \alpha_n \neq 0 \), \( n \geq 1 \), and hence from Lemma 2.2 there exists a semi-definite moment functional such that (3.4) holds.

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Then from Lemma 2.1 and the continued fraction expansion (3.2),

\[ F_0^{(b,c,d)}(z) - G_n^{(b,c,d)}(z) = \frac{\beta_1^{(b,c,d)} \alpha_2^{(b,c,d)} \cdots \alpha_n^{(b,c,d)} \alpha_{n+1}^{(b,c,d)} z^n F_n^{(b,c,d)}(z)}{Q_n^{(b,c,d)}(z) \left[ \beta_{n+1}^{(b,c,d)} G_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)} z F_n^{(b,c,d)}(z) Q_n^{(b,c,d)}(z) \right]} \]

\[ = \rho_n^{(b,c,d)} \frac{1}{Q_n^{(b,c,d)}(0)} z^n + O(z^{n+1}), \quad \text{for } n \geq 1. \]

Since \( F_0^{(b,c,d)}(z) = 2 \Phi_1(q, q^{-b}; q^{c-b+2}; q, q^{c+2-d} z) \), from the latter part of Lemma 2.2

\[ \mu_j^{(b,c,d)} = \frac{(q^{-b}; q)_j}{(q^{c-b+2}; q)_j} q^{j(d-2)}, \quad j = 0, 1, 2, \ldots, \]

thus giving the results for the positive moments.

From (3.1), by realizing that \( q_n^{(b,c,c+2-d)} = \beta_n^{(b,c,d)} \) and \( f_{n+1}^{(b,c,c+2-d)} = \alpha_{n+1}^{(b,c,d)} \), \( n \geq 1 \), we also obtain the continued fraction expansion

\[ \frac{\beta_1^{(b,c,d)}}{z} F_0^{(c-b,c,c+2-d)}(z^{-1}) = \frac{\beta_1^{(b,c,d)} z \alpha_2^{(b,c,d)} \cdots \alpha_n^{(b,c,d)} z \alpha_n^{(b,c,d)} z}{z + \beta_1^{(b,c,d)} + \alpha_2^{(b,c,d)} + \cdots + \alpha_n^{(b,c,d)} + 1} \frac{\alpha_n^{(b,c,d)} z}{z + \beta_1^{(b,c,d)}} \frac{F_n^{(c-b,c,c+2-d)}(z^{-1})}{1}. \]

Hence, again from Lemma 2.1

\[ \frac{\beta_1^{(b,c,d)}}{z} F_0^{(c-b,c,c+2-d)}(z^{-1}) - G_n^{(b,c,d)}(z) = \frac{\beta_1^{(b,c,d)} \alpha_2^{(b,c,d)} \cdots \alpha_n^{(b,c,d)} \alpha_{n+1}^{(b,c,d)} z^{n+1} F_n^{(c-b,c,c+2-d)}(z^{-1})}{Q_n^{(b,c,d)}(z) \left[ \beta_{n+1}^{(b,c,d)} G_n^{(b,c,d)}(z) - \alpha_{n+1}^{(b,c,d)} z F_n^{(b,c,d)}(z) Q_n^{(b,c,d)}(z) \right]} \]

\[ = \rho_n^{(b,c,d)} Q_n^{(b,c,d)}(0) \frac{1}{z^{n+1}} + O\left(\frac{1}{z^{n+2}}\right), \quad \text{for } n \geq 1. \]

Since \( F_0^{(c-b,c,c+2-d)}(z^{-1}) = 2 \Phi_1(q, q^{-b}; q^{b+2}; q, q^{c+2-d} z^{-1}) \), from the latter part of Lemma 2.2

\[ \mu_j^{(b,c,d)} = \frac{(q^{-c+b+1}; q)_j}{(q^{b+2}; q)_j} q^{j(d-2)}, \quad j = 1, 2, 3, \ldots. \]

Thus, using \((a, q)_n = (a; q)_\infty / (aq^n; q)_\infty\), for \( n = 0, \pm 1, \pm 2, \ldots \), we also obtain the results for the negative moments. This concludes the theorem for \( c - b + 1 \neq 0, -1, -2, \ldots \) and \( b \neq -1, -2, \ldots \).

Now to extend the results for \( c - b + 1 \neq -1, -2, \ldots \) and \( b \neq -1, -2, \ldots \), we need to prove the theorem for \( c - b + 1 = 0 \) and \( b \neq -1, -2, \ldots \).

If \( b \neq -1, -2, \ldots \), then \( \beta_1^{(b,b-1,d)} = 0 \) and \( \beta_n^{(b,b-1,d)} = \alpha_n^{(b,b-1,d)} \neq 0 \) for \( n \geq 1 \).

Hence, \( Q_n^{(b,b-1,d)}(z) = z^n, n \geq 0 \) and

\[ M^{(b,b-1,d)}[z^{-s} Q_n^{(b,b-1,d)}(z)] = M^{(b,b-1,d)}[z^{-s}] = \frac{(q^{-b}; q)_{-n+s}}{(q; q)_{-n+s}} q^{(-n+s)d}. \]

Since

\[ \frac{(q^{-b}; q)_{-n+s}}{(q; q)_{-n+s}} q^{(-n+s)d} = 0 \quad \text{if } s < n \quad \text{and} \quad \rho_n^{(b,b-1)} = \frac{(q^{-b}; q)_0}{(q; q)_0} q^{(0)d} = 1, \]
the validity of the theorem when \( c - b + 1 = 0 \) and \( b \neq -1, -2, \ldots \) is confirmed. This concludes the theorem.

The same explicit expression for the moments, when the moment functional is considered as in Pastro [21], is obtained in [31].

From the three-term recurrence relation (3.3) it follows that

\[
Q_n^{(b,c,d)}(0) = \beta_1^{(b,c,d)} \beta_2^{(b,c,d)} \cdots \beta_n^{(b,c,d)} = \frac{(q^{-b+1};q)_n}{(q^{b+1};q)_n} q^{n(b-d+1)}, \quad n \geq 1.
\]

**Theorem 3.2.** Let \( b \neq -1, -2, \ldots \) and \( c - b + 1 \neq -1, -2, \ldots \). Then

\begin{align*}
\text{a)} & \quad \lim_{n \to \infty} \beta_n^{(b,c,d)} = q^{b-d+1}, \quad \lim_{n \to \infty} a_n^{(b,c,d)} = q^{b-d+1}, \\
\text{b)} & \quad \lim_{n \to \infty} q^{-n(b-d+1)} Q_n^{(b,c,d)}(0) = (1-q)^{-c+2b} \frac{\Gamma_q(b+1)}{\Gamma_q(c-b+1)}.
\end{align*}

**Proof.** Part a) of this theorem is clear. To obtain parts b) and c) we use the definition

\[
\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}
\]

of the \( q \)-gamma function.

**Theorem 3.3.** Let \( b \neq -1, -2, \ldots \) and \( c - b + 1 \neq -1, -2, \ldots \). Then the monic polynomials \( Q_n^{(b,c,d)} \), \( n \geq 0 \), given by the recurrence relation (3.3) have the explicit representation

\[
(3.5) \quad Q_n^{(b,c,d)}(z) = \frac{(q^{-b+1};q)_n}{(q^{b+1};q)_n} q^{n(b-d+1)} 2\Phi_1(q^{-n},q^{b+1};q^{-c+b-n},q,q^{-c+d-1}z).
\]

**Proof.** From (3.3) it is easily verified that the reciprocal (or inverse) polynomials

\[
Q_n^{*,(b,c,d)}(z) = z^n Q_n^{(b,c,d)}(1/z) \quad \text{and} \quad Q_n^{*,(b,c,d)}(z) = z^n Q_n^{(b,c,d)}(1/z), \quad n \geq 0,
\]

satisfy the three-term recurrence relations

\[
(3.6) \quad Q_{n+1}^{*,(b,c,d)}(z) = (1 + \beta_n^{(b,c,d)} z) Q_n^{*,(b,c,d)}(z) - \alpha_n^{(b,c,d)} z Q_n^{*,(b,c,d)}(z), \quad n \geq 1,
\]

with \( Q_0^{*,(b,c,d)}(z) = Q_0^{(b,c,d)}(z) = 1 \), \( Q_1^{*,(b,c,d)}(z) = 1 + \beta_1^{(b,c,d)} z \) and \( Q_1^{*,(b,c,d)}(z) = 1 + \beta_1^{(b,c,d)} z \). This means that

\[
Q_n^{*,(b,c,d)}(z) = 2\Phi_1(q^{-n},q^{c-b+1};q^{-b-n},q,q^{-d+1}z), \quad n \geq 1,
\]

which we can easily verify from Lemma 2.3. Hence, application of the transformation (2.6) in \( Q_n^{*,(b,c,d)} \), for example, gives the required results of the theorem. \( \square \)
Note that by the application of the transform \(2.4\) in \(Q_n^{(b,c,d)}\), for example, we can also write that
\[
\frac{(q^{-d+1}z; q)_\infty}{(q^{d+2}z; q)_\infty} Q_n^{(b,c,d)}(z) = 2\Phi_1(q^{-b}; q^{-c-n-1}, q^{-b-n}; q, q^{-d+2}z), \quad n \geq 1,
\]
provided that \(|z| < q^{-\Re(c-d+2)}\). This can also be directly verified from Lemma 2.3 and (3.6).

**Theorem 3.4.** Let \(b \neq -1, -2, \ldots, c - b + 1 \neq -1, -2, \ldots\) and
\[
\sigma = \min\{q^{-\Re(c-d+2)}, q^{-\Re(b-d+1)}\}.
\]
Then uniformly on compact subsets of \(|z| < \sigma\),
\[
\lim_{n \to \infty} Q_n^{(b,c,d)}(z) = \frac{(q^{d-2}z; q)_\infty}{(q^{d+1}z; q)_\infty}.
\]

**Proof.** Since
\[
\lim_{n \to \infty} \frac{(q^{-c-n-1}; q)_j q^{(d-2)j}}{(q^{-b-n}; q)_j} = q^{(b-d+1)j},
\]
using Lebesgue’s dominated convergence theorem and then (2.4), we obtain
\[
\lim_{n \to \infty} 2\Phi_1(q^{-b}; q^{-c-n-1}, q^{-b-n}; q, q^{-d+2}z) = 1\Phi_0(q^{-b}; q, q^{-b+d+1}z) = \frac{(q^{-d+1}z; q)_\infty}{(q^{d+1}z; q)_\infty},
\]
uniformly on compact subsets of \(|z| < \sigma\). Thus, the result of the theorem follows.

**Theorem 3.5.** In addition to \(b \neq -1, -2, \ldots\) and \(c - b + 1 \neq -1, -2, \ldots\), if one also assumes that
\[
\Re(c + 2) > \Re(d) > 0,
\]
then the polynomials \(Q_n^{(b,c,d)}\), \(n \geq 0\), given by \(3.5\), satisfy the orthogonality relations
\[
\frac{x^{(b,c)}}{2\pi i} \int_{C} z^{-s} Q_n^{(b,c,d)}(z) \frac{(q^{-b+d}z; q)_\infty q^{b-d+1}z; q)_\infty}{(q^d; q)_\infty (q^{c-d+2}; q)_\infty} \frac{1}{z} \, dz = \delta_{n,s} \rho_n^{(b,c)}, \quad 0 \leq s \leq n.
\]
Here, \(\rho_n^{(b,c)}\) are as in Theorem 3.1 and
\[
\tau^{(b,c)} = \frac{(q; q)_\infty (q^{c+2}; q)_\infty}{(q^{c-b+2}; q)_\infty (q^{b+1}; q)_\infty}.
\]

**Proof.** Let us consider the following identity of Ramanujan:
\[
\sum_{n=0}^{\infty} \frac{(\alpha; q)_n \delta^x}{(\beta; q)_n} = \frac{(q; q)_\infty (\alpha^2; q)_\infty (\alpha \delta^x; q)_\infty (\frac{\alpha^2}{\delta^2}; q)_\infty}{(\beta; q)_\infty (\frac{q}{\alpha}; q)_\infty (\frac{q^2}{\delta^2}; q)_\infty (\delta x; q)_\infty},
\]
which holds for \(|\beta \alpha | < |x| < 1\). Simple proofs of this identity can be found in [8] and [12].

In our case, since \(0 < q < 1\), with the assumptions of the theorem if we take \(x = q^{d}z\), \(\alpha = q^{-b}\) and \(\beta = q^{-b+2}\), then
\[
2\Phi_1(q^{-b}; q^{-c-n-1}, q^{-b-n}; q, q^{-d+2}z) = \tau^{(b,c)} \frac{(q^{-d+1}z; q)_\infty q^{b-d+1}z; q)_\infty}{(q^d; q)_\infty (q^{c-d+2}; q)_\infty},
\]
which holds for \(|q^{c+2-d} | < |z| < |q^{-d}|\), where \(|q^{c+2-d} | < 1\) and \(|q^{-d}| > 1\).
Hence, multiplying by $z^{-j-1}$ and integrating along the unit circle we obtain from Laurent’s theorem
\[
\frac{(q^{-b}; q)_j}{(q^{c-b+2}; q)_j} q^{jd} = \frac{\tau(b,c)}{2\pi i} \int_c z^{-j-1} \left( \frac{q^{b-d+1}/z; q}{(q^d/z; q)\infty} \right) dz, \quad j = 0, \pm 1, \pm 2, \ldots.
\]
Thus, the moment functional in Theorem 3.1 satisfies
\[
\mathcal{M}^{(b,c,d)}[z^{-j}] = \frac{\tau(b,c)}{2\pi i} \int_c z^{-j-1} \left( \frac{q^{b-d+1}/z; q}{(q^d/z; q)\infty} \right) dz,
\]
for $j = 0, \pm 1, \pm 2, \ldots$, which completes the proof of the theorem. \(\square\)

As a particular case, letting $b = 0$ and $c + 2 \neq 0, -1, -2, \ldots$ we have $\beta^{(0,c,d)}_1 = \frac{1-q^{c+1}}{1-q} q^{-d+1}$ and
\[
\beta^{(0,c,d)}_{n+1} = \alpha^{(0,c,d)}_{n+1} = \frac{1-q^{c+n+1}}{1-q^{n+1}} q^{-d+1}, \quad n \geq 1.
\]
Moreover, $\mu^{(0,c,d)}_0 = 1$,
\[
\mu^{(0,c,d)}_j = 0 \quad \text{and} \quad \mu^{(0,c,d)}_{-j} = \frac{(q^{-c-1}; q)_j}{(q; q)_j} q^{j(c+2-d)}, \quad j = 1, 2, \ldots.
\]
Furthermore, the following corollary holds.

**Corollary 3.5.1.** If $\Re(c+2) > \Re(d) > 0$, then the sequence of polynomials
\[
\{Q^{(0,c,d)}_n(z)\}
\]
given by
\[
Q^{(0,c,d)}_n(z) = \frac{(q^{c+1}; q)_n}{(q; q)_n} q^{n(-d+1)} {}_2\Phi_1(q^{-n}, q; q^{-c-n}; q, q^{-c+d-1} z), \quad n \geq 1,
\]
are orthogonal with respect to the linear functional
\[
\frac{1}{2\pi i} \int_c z^{-s} Q^{(0,c,d)}_n(z) \frac{(q^{c-d+1}/z; q)\infty}{(q^{c+2-d}/z; q)\infty} \frac{1}{z} dz = \delta_{n,s}, \quad 0 \leq s \leq n.
\]
Moreover, uniformly on compact subsets of $|z| < \min\{q^{-\Re(c-d+2)}, q^{-\Re(-d+1)}\}$,
\[
\lim_{n \to \infty} Q^{*(0,c,d)}_n(z) = \frac{(q^{c-d+2}/z; q)\infty}{(q^{c-d+1}/z; q)\infty}.
\]
As another particular case, letting $c = b$ and $b + 1 \neq 0, -1, -2, \ldots$, we have
\[
\beta^{(b,b,d)}_n = \alpha^{(b,b,d)}_{n+1} = \frac{1-q^n}{1-q^{b+n}} q^{b-d+1}, \quad n \geq 1.
\]
Moreover,
\[
\mu^{(b,b,d)}_j = \frac{(q^{-b}; q)_j}{(q^2; q)_j} q^{jd}, \quad j = 0, \pm 1, \pm 2, \ldots.
\]
Furthermore, the following corollary can be stated.
Corollary 3.5.2. If \( b + 1 \neq 0 \) and \( \Re(b + 2) > \Re(d) > 0 \), then the sequence of polynomials \( \{ Q_n^{(b,b,d)} \} \) given by

\[
Q_n^{(b,b,d)}(z) = \frac{(q; q)_n}{(q^{b+1}; q)_n} q^{n(b-d+1)} \sum_{j=0}^{n} \frac{(q^{b+1}; q)_{j}}{(q; q)_{j}} q^{-j(b-d+1)z^j}, \quad n \geq 1,
\]

satisfies the orthogonality relations

\[
\frac{1}{2\pi i} \frac{(1-q)}{(1-q^{b+1})} \int_{c} z^{-s} Q_n^{(b,b,d)}(z) \frac{(q^{b-d}z; q)_{\infty}}{(q^d z; q)_{\infty}} \frac{z-q^{b-d+1}}{z^2} \, dz = \delta_{n,s}, \quad 0 \leq s \leq n.
\]

Moreover, uniformly on compact subsets of \( |z| < q^{-\Re(b-d+1)} \),

\[
\lim_{n \to \infty} Q_n^{(b,b,d)}(z) = \frac{1}{(1-q^{b-d+1})z}.
\]

4. \( q \)-Szegő Polynomials

From (3.3) the moment functional \( M^{(b,c,d)} \) is easily seen to be positive definite if \( b \neq -1, -2, \ldots, c - b + 1 \neq -1, -2, \ldots, \Re(c+2) > \Re(d) > 0, -b+d = b-d+1 \) and \( d = c+2-d \). That is, with the restrictions

\[
c = b + \bar{b} - 1, \quad d + \bar{d} = b + \bar{b} + 1 \quad \text{and} \quad \Re(b) > -1/2,
\]

the moment functional \( M^{(b,c,d)} \) is positive definite, and hence the polynomials \( Q_n^{(b,c,d)} \) are the associated Szegő polynomials.

Hence, setting

\[
b = \lambda - i\eta, \quad c = 2\lambda - 1 \quad \text{and} \quad d = 1 - \lambda + i\phi,
\]

if \( \lambda > -1/2 \), our special case of Ramanujan identity (3.7) becomes

\[
\sum_{n=\infty}^{\infty} \frac{(q^{-\lambda+\eta}; q)_n}{(q^{\lambda+1+\eta}; q)_n} q^{n(c+1)} z^n = \tilde{\tau}^{(\lambda,\eta)} (q^{2\lambda+1}; q)_\infty (q^{2\lambda}; q)_\infty (q^{1\lambda - \eta}; q)_\infty (q^{1\lambda - \eta}; q)_\infty,
\]

which holds for \( q^{\lambda+1/2} < |z| < q^{-\lambda - 1/2} \), where

\[
\tilde{\tau}^{(\lambda,\eta)} = \frac{(q; q)_\infty (q^{2\lambda+1}; q)_\infty}{(q^{1\lambda + \eta}; q)_\infty (q^{1\lambda - \eta}; q)_\infty}.
\]

This means that we can write

\[
M^{(\lambda - i\eta, 2\lambda - 1, \lambda + i\phi + 1/2)}[z^{-j}] = \frac{(q^{-\lambda+i\eta}; q)_j}{(q^{\lambda+1+i\eta}; q)_j} q^{j(\frac{1}{2}+\lambda+i\phi)}
\]

\[
= \int_{c} z^{-j} d\mu^{(\lambda,\eta,\phi)}(z) = \int_{0}^{2\pi} e^{-ij\theta} \theta^{(\lambda,\eta,\phi)}(\theta) \, d\theta,
\]

for \( j = 0, \pm 1, \pm 2, \ldots \), where \( \omega^{(\lambda,\eta,\phi)}(\theta) d\theta = d\mu^{(\lambda,\eta,\phi)}(e^{i\theta}) \), with

\[
\frac{d\mu^{(\lambda,\eta,\phi)}(z)}{dz} = \frac{\tilde{\tau}^{(\lambda,\eta)}}{2\pi i} \frac{1}{z} (q^{\frac{1}{2}+i(\eta+\phi); q}_\infty (q^{\frac{1}{2}-i(\eta+\phi); q}_\infty (q^{\frac{1}{2}+\lambda+\eta+\phi; q}_\infty (q^{\frac{1}{2}+\lambda-\eta+\phi; q}_\infty (q^{\frac{1}{2}+\lambda-\eta; q}_\infty (q^{\frac{1}{2}+\lambda-\eta; q}_\infty
\]

and

\[
\omega^{(\lambda,\eta,\phi)}(\theta) = \frac{\tilde{\tau}^{(\lambda,\eta)}}{2\pi} (q^{\frac{1}{2}+i(\eta+\phi); q}_\infty (q^{\frac{1}{2}-i(\eta+\phi); q}_\infty (q^{\frac{1}{2}+\lambda+\eta+\phi; q}_\infty (q^{\frac{1}{2}+\lambda-\eta+\phi; q}_\infty (q^{\frac{1}{2}+\lambda-\eta; q}_\infty (q^{\frac{1}{2}+\lambda-\eta; q}_\infty
\]

As expected, \( \omega^{(\lambda,\eta,\phi)}(\theta) \) is a positive weight function in \([0, 2\pi]\).
Adopting the notation $S_n^{(\lambda, \eta, \phi)}(z) = Q_n^{(\lambda-i\eta, 2\lambda-1, \frac{1}{2}+\lambda+i\phi)}(z)$ we can write the following:

$$S_{n+1}^{(\lambda, \eta, \phi)}(z) = \left(z + \frac{1 - q^{\lambda+i\eta+n}}{1 - q^{\lambda+i\eta+n+1}} q^{\frac{1}{2}-(\eta+\phi)}\right) S_n^{(\lambda, \eta, \phi)}(z) - \frac{(1 - q^n)(1 - q^{2\lambda+n})}{(1 - q^{\lambda+i\eta+n})(1 - q^{\lambda+i\eta+n+1})} q^{\frac{1}{2}-(\eta+\phi)} z S_{n-1}^{(\lambda, \eta, \phi)}(z), \quad n \geq 1,$$

with $S_0^{(\lambda, \eta, \phi)}(z) = 1$ and $S_1^{(\lambda, \eta, \phi)}(z) = \left(z + \frac{1 - q^{\lambda+i\eta}}{1 - q^{\lambda+i\eta+\phi}} q^{\frac{1}{2}-(\eta+\phi)}\right)$. Moreover,

$$S_n^{(\lambda, \eta, \phi)}(0) = \frac{(q^{\lambda+i\eta}; q)_n}{(q^{\lambda+i\eta}; q^n)} q^{n\left(\frac{1}{2}-(\eta+\phi)\right)}, \quad n \geq 1.$$

Hence, in particular, using Theorems 5.3 and 5.4 we have

**Theorem 4.1.** If $\lambda, \eta, \phi \in \mathbb{R}$ and $\lambda > -1/2$, then the polynomials

$$S_n^{(\lambda, \eta, \phi)}(z) = \frac{(q^{\lambda+i\eta}; q)_n q^{n\left(\frac{1}{2}-(\eta+\phi)\right)}}{(q^{\lambda+1-i\eta}; q)_n} 2\Phi_1(q^{-n}, q^{\lambda+1-i\eta}; q^{-\lambda-n+1-i\eta}; q, q^{\frac{1}{2}+\lambda+\phi} z)$$

are the monic Szegő polynomials satisfying

$$\int_0^{2\pi} \frac{S_n^{(\lambda, \eta, \phi)}(e^{i\theta}) S_m^{(\lambda, \eta, \phi)}(e^{i\phi}) \omega(\phi, \eta, \lambda)(\theta)}{(ei\phi - ei\phi)} d\theta = [\kappa_n^{(\lambda, \eta)}]^{-2} \delta_{nm}, \quad n, m = 0, 1, 2, \ldots,$$

with respect to the weight function $\omega(\lambda, \eta, \phi)(\theta)$ given by (4.1). Here,

$$[\kappa_n^{(\lambda, \eta)}]^{-2} = \kappa_n^{(\lambda-i\eta, 2\lambda-1)} = \frac{(q; q)_n (q^{2\lambda+1}; q)_n}{(q^{\lambda+1+i\eta}; q)_n (q^{\lambda+i\eta}; q)_n}.$$

Moreover, these polynomials satisfy the Szegő recurrence relation

$$S_n^{(\lambda, \eta, \phi)}(z) = a_n^{(\lambda, \eta, \phi)} z S_{n-1}^{(\lambda, \eta, \phi)}(z) + S_n^{(\lambda, \eta, \phi)}(z), \quad n \geq 1,$$

where the reflection (or Verblunsky) coefficients $a_n^{(\lambda, \eta, \phi)} = S_n^{(\lambda, \eta, \phi)}(0)$ are given by (4.2).

Now using Theorem 5.3 we can state the following. Let $\lambda, \eta, \phi \in \mathbb{R}$, $\lambda > -1/2$ and $\sigma = \min\{q^{-1/2}, q^{-\lambda-1/2}\}$. Then uniformly on compact subsets of $|z| < \sigma$,

$$\lim_{n \to \infty} S_n^{(\lambda, \eta, \phi)}(z) = \frac{(q^{\lambda+\frac{1}{2}+\phi} z; q)_\infty}{(q^{\frac{1}{2}+(\eta+\phi) z}; q)_\infty}.$$

Moreover,

$$\sum_{n=1}^{\infty} |a_n^{(\lambda, \eta, \phi)}|^2 = |1 - q^{\lambda+i\eta}|^2 \sum_{n=1}^{\infty} \frac{q^n}{|1 - q^{n+\lambda+i\eta}|^2} \leq |1 - q^{\lambda+i\eta}|^2 \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n+\lambda)^2} < \infty.$$

This last result means that the Szegő condition

$$\frac{1}{2\pi} \int_0^{2\pi} \log(\omega^{(\lambda, \eta, \phi)}(\theta)) d\theta > -\infty$$

holds and we can now give an expression for the associated Szegő function

$$D^{(\lambda, \eta, \phi)}(z) = \exp \left(\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\omega^{(\lambda, \eta, \phi)}(\theta)) d\theta\right).$$
Theorem 4.2. If \( \lambda, \eta, \phi \in \mathbb{R} \) and \( \lambda > -1/2 \), then for \( |z| < 1 \),

\[
D(\lambda, \eta, \phi)(z) = \sqrt{\frac{\Gamma_q(2\lambda + 1)}{\Gamma_q(\lambda + 1 - i\eta) \Gamma_q(\lambda + 1 + i\eta)}} \frac{(q^{1/2 + i(\eta + \phi)} z; q)_\infty}{(q^{\lambda + 1/2 + i\phi} z; q)_\infty}.
\]

Proof. Since, \( \kappa_n(\lambda, \eta) S_n(\lambda, \eta, \phi) \to [D(\lambda, \eta, \phi)(z)]^{-1} \) for \( |z| < 1 \) (see [23, p. 144]), the result follows from part c) of Theorem 3.2 and from (4.3).

\( \square \)

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