ON THE LINEAR INDEPENDENCY
OF MONOIDAL NATURAL TRANSFORMATIONS

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Abstract. Let \( F, G : \mathcal{I} \to \mathcal{C} \) be monoidal functors from a monoidal category \( \mathcal{I} \) to a linear abelian rigid monoidal category \( \mathcal{C} \) over an algebraically closed field \( k \). Then the set \( \text{Nat}(F, G) \) of natural transformations \( F \to G \) is naturally a vector space over \( k \). Under certain assumptions, we show that the set of monoidal natural transformations \( F \to G \) is linearly independent as a subset of \( \text{Nat}(F, G) \).

As a corollary, we can show that the group of monoidal natural automorphisms on the identity functor on a finite tensor category is finite. We can also show that the set of pivotal structures on a finite tensor category is finite.

1. Introduction

Monoidal categories \([3]\) arise in many contexts in mathematics. In this paper, we prove a basic fact on monoidal natural transformations between monoidal functors.

Our terminology basically follows that of Mac Lane \([3]\). Recall that a monoidal functor \([3, XI.2]\) is a functor \( F : \mathcal{C} \to \mathcal{D} \) between monoidal categories equipped with a natural transformation \( \varphi_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y) \) and a morphism \( \varphi_0 : 1 \to F(1) \) in \( \mathcal{D} \) satisfying certain axioms and that it is said to be strong if \( \varphi \) and \( \varphi_0 \) are isomorphisms. In this paper, by a monoidal functor we always mean a strong monoidal functor.

Throughout, we work over an algebraically closed field \( k \). By a tensor category over \( k \) we mean a \( k \)-linear abelian monoidal category \( \mathcal{C} \) which is rigid \([1, \S 2.1]\) and satisfies the following conditions:

- The unit object \( 1 \in \mathcal{C} \) is simple.
- The tensor product \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) is \( k \)-linear in both variables.
- Every object of \( \mathcal{C} \) is of finite length.
- Every hom-set in \( \mathcal{C} \) is finite-dimensional over \( k \).

Let \( \mathcal{C} \) be a tensor category over \( k \) and let \( \mathcal{I} \) be a skeletally small monoidal category. If \( F, G : \mathcal{I} \to \mathcal{C} \) are functors, then the set \( \text{Nat}(F, G) \) of natural transformations \( F \to G \) is naturally a vector space over \( k \). Now we suppose that both \( F \) and \( G \) are monoidal functors. Then we can consider the set \( \text{Nat}_\otimes(F, G) \) of monoidal natural transformations. Our main result in this paper is the following:

Theorem 1.1. \( \text{Nat}_\otimes(F, G) \subset \text{Nat}(F, G) \) is linearly independent.
We will prove Theorem 1.1 in Section 2. In what follows, we give some applications of this theorem. The following criterion for finite-dimensionality of \( \text{Nat}(F,G) \) is important:

**Lemma 1.2.** Let \( F, G : A \to B \) be right exact \( k \)-linear functors between \( k \)-linear abelian categories. Suppose that \( A \) has a projective generator \( P \) and that every object of \( A \) is of finite length. Then the linear map

\[
(\cdot)|_P : \text{Nat}(F,G) \to \text{Hom}_B(F(P),G(P)), \quad f \mapsto f|_P
\]

is injective. In particular, if every hom-set in \( B \) is finite-dimensional, then

\[
\dim_k \text{Nat}(F,G) \leq \dim_k \text{Hom}_B(F(P),G(P)) < \infty.
\]

**Proof.** Let \( f \in \text{Nat}(F,G) \) and suppose that \( f|_P = 0 \). By the assumption, for every \( X \in A \), there exists an exact sequence \( P^{\oplus m} \to P^{\oplus n} \to X \to 0 \). By applying \( F \) and \( G \) to this sequence, we have a commutative diagram

\[
\begin{array}{ccc}
F(P)^{\oplus m} & \longrightarrow & F(P)^{\oplus n} \longrightarrow F(X) \longrightarrow 0 \\
(f|_P)^{\oplus m} \downarrow & & (f|_P)^{\oplus n} \downarrow f|_X \downarrow \\
G(P)^{\oplus m} & \longrightarrow & G(P)^{\oplus n} \longrightarrow G(X) \longrightarrow 0 \\
\end{array}
\]

in \( B \) with exact rows. Since \( f|_P = 0 \), \( f|_X = 0 \). This implies the injectivity. \( \square \)

A \( k \)-linear abelian category \( A \) is said to be finite if it is \( k \)-linearly equivalent to the category of finitely generated modules over a finite-dimensional \( k \)-algebra. We present some applications of Theorem 1.1 to finite tensor categories [2]. By a tensor functor we mean a \( k \)-linear monoidal functor between tensor categories. By Theorem 1.1 and Lemma 1.2 we have the following:

**Corollary 1.3.** Let \( F, G : C \to D \) be two right exact tensor functors between tensor categories \( C \) and \( D \). Suppose that \( C \) is finite. Then \( \text{Nat}_\otimes(F,G) \) is finite.

Let \( \text{Aut}_\otimes(F) \subset \text{Nat}_\otimes(F,F) \) denote the group of monoidal natural automorphisms on a monoidal functor \( F : C \to D \). As an immediate consequence of Corollary 1.3 we have the following:

**Corollary 1.4.** Let \( F : C \to C \) be a right exact tensor endofunctor on a finite tensor category \( C \). Then \( \text{Aut}_\otimes(F) \) is finite. In particular, \( \text{Aut}_\otimes(\text{id}_C) \) is finite.

We will give some remarks on the structure of \( \text{Aut}_\otimes(\text{id}_C) \) in Section 3. We note that the finiteness of \( \text{Aut}_\otimes(\text{id}_C) \) is well known in the case where \( C \) is the category \( \text{Rep}(H) \) of finite-dimensional representations of a finite-dimensional Hopf algebra \( H \). In fact, \( \text{Aut}_\otimes(\text{id}_C) \) is then isomorphic to the group of central grouplike elements of \( H \).

There is another important corollary. A pivotal structure on a rigid monoidal category \( C \) is an element of \( \text{Piv}(C) := \text{Nat}_\otimes(\text{id}_C, (-)^\ast) \), where \( (-)^\ast : C \to C \) is the left duality functor.

**Corollary 1.5.** If \( C \) is a finite tensor category, then \( \text{Piv}(C) \) is finite.

Also this corollary is well known in the case where \( C = \text{Rep}(H) \) for some finite-dimensional Hopf algebra \( H \). In fact, \( \text{Piv}(C) \) then is in one-to-one correspondence between grouplike elements \( g \in H \) such that \( S^2(x) = gxg^{-1} \) for all \( x \in H \), where \( S \) is the antipode of \( H \).
Lemma 2.2. The above map is injective.

Proof. Let \( V \in \mathcal{C} \) be an object. The tensor product of \( \mathcal{C} \) defines two \( \mathbf{k} \)-linear endofunctors \( V \otimes (-) \) and \( (-) \otimes V \) on \( \mathcal{C} \). Both these functors are exact since \( \mathcal{C} \) is rigid [1].

Let \( \mathcal{C} \) be a tensor category over \( \mathbf{k} \). Without loss of generality, we may suppose \( \mathcal{C} \) to be strict. Let \( V \in \mathcal{C} \) be an object. The tensor product of \( \mathcal{C} \) defines two \( \mathbf{k} \)-linear endofunctors \( V \otimes (-) \) and \( (-) \otimes V \) on \( \mathcal{C} \). Both these functors are exact since \( \mathcal{C} \) is rigid [1].

Lemma 2.1. If \( V \neq 0 \), then \( V \otimes (-) \) and \( (-) \otimes V \) are faithful.

Proof. Suppose that \( V \neq 0 \). Then the evaluation morphism \( d_V : V^* \otimes V \to 1 \) is an epimorphism. Indeed, otherwise, since \( 1 \) is simple, \( d_V = 0 \). It follows from the rigidity axiom that \( \text{id}_V = 0 \), and hence \( V = 0 \), a contradiction.

Let \( X, Y \in \mathcal{C} \) be objects. Consider the linear map

\[
(-)^g : \text{Hom}_\mathcal{C}(V \otimes X, V \otimes Y) \to \text{Hom}_\mathcal{C}(V^* \otimes V \otimes X, Y)
\]

given by \( \phi^g = (d_V \otimes \text{id}_Y) \circ (\text{id}_V^* \otimes \phi) \). If \( f : X \to Y \) is a morphism in \( \mathcal{C} \), then

\[
(\text{id}_V \otimes f)^g = f \circ (d_V \otimes \text{id}_X).
\]

If \( \text{id}_V \otimes f = 0 \), then, by the above equation, we have \( f \circ (d_V \otimes \text{id}_X) = 0 \). As we observed above, \( d_V \) is an epimorphism. Since the tensor product of \( \mathcal{C} \) is exact, also \( d_V \otimes \text{id}_X \) is an epimorphism. Therefore, we conclude that \( f = 0 \). This means that the functor \( V \otimes (-) \) is faithful.

The faithfulness of \( (-) \otimes V \) can be proved in a similar way. \( \square \)

Let \( \mathcal{I} \) be a skeletally small category and let \( F, G : \mathcal{I} \to \mathcal{C} \) be functors. Then the set \( \text{Nat}(F, G) \) of natural transformations \( F \to G \) is a vector space over \( \mathbf{k} \). Let \( V, W \in \mathcal{C} \) be objects. The tensor product of \( \mathcal{C} \) defines a linear map

\[
\text{Hom}_\mathcal{C}(V, W) \otimes_{\mathbf{k}} \text{Nat}(F, G) \to \text{Nat}(V \otimes F(-), W \otimes G(-)).
\]

Lemma 2.2. The above map is injective.

In the case where \( F \) and \( G \) are constant functors sending all objects of \( \mathcal{I} \) respectively to \( X \in \mathcal{C} \) and \( Y \in \mathcal{C} \), Lemma 2.2 states that the map

\[
\text{Hom}_\mathcal{C}(V, W) \otimes_{\mathbf{k}} \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(V \otimes X, W \otimes Y)
\]

induced from the tensor product is injective. We need to consider a bit more general situation than this case.

Proof. Let \( f_1, \cdots, f_m \in \text{Nat}(F, G) \) be linearly independent elements. It suffices to show that if \( c_1, \cdots, c_m : V \to W \) are morphisms in \( \mathcal{C} \) such that

\[
(2.1) \quad c_1 \otimes f_1|_X + \cdots + c_m \otimes f_m|_X = 0
\]

for all \( X \in \mathcal{I} \), then \( c_i = 0 \) for \( i = 1, \cdots, m \).

We show the above claim by induction on the length \( \ell(V) \) of \( V \). If \( \ell(V) = 0 \), our claim is obvious. Suppose that \( \ell(V) \geq 1 \). Then there exists a simple subobject of \( V \), say \( L \). Let \( K \) be the image of the morphism

\[
(c_1|_L, c_2|_L, \cdots, c_m|_L) : L^\oplus m \to W.
\]
Since $L$ is simple, $K \cong L^{\oplus n}$ for some $n \geq 0$. Let $p_j : K \cong L^{\oplus n} \rightarrow L$ be the $j$-th projection. Then $p_j \circ c_i|L = \lambda_{ij} \id_L$ for some $\lambda_{ij} \in k$. By (2.1),

$$\id_L \otimes \left( \sum_{i=1}^{m} \lambda_{ij} f_i|X \right) = (p_j \otimes \id_{G(X)}) \circ \left( \sum_{i=1}^{m} c_i|L \otimes f_i|X \right) = 0.$$ 

Lemma 2.1 yields that $\sum_{i=1}^{m} \lambda_{ij} f_i|X = 0$ for all $X \in I$. By the linear independence of the $f_i$’s, we have that $\lambda_{ij} = 0$ for all $i$ and $j$. This means that $c_i|L = 0$ for all $i$.

Let $p : V \rightarrow V/L$ be the projection. By the above observation, for each $i$, there exists a morphism $\tau_i : V/L \rightarrow W$ such that $c_i = \tau_i \circ p$. By (2.1),

$$\sum_{i=1}^{m} \tau_i \otimes f_i|X = \left( \sum_{i=1}^{m} c_i \otimes f_i|X \right) \circ (p \otimes \id_{F(X)}) = 0$$

for all $X \in I$. Note that $\ell(V/L) = \ell(V) - 1$. By the induction hypothesis, $\tau_i = 0$ for all $i$. Therefore, $c_i = \tau_i \circ p = 0$. \hfill \Box

Now we generalize Lemma 2.2. Let $I_i (i = 1, 2)$ be skeletally small categories and let $E_i : I_i \rightarrow \mathcal{C} (i = 1, 2)$ be functors. We denote by $E_1 \otimes E_2$ the functor $E_1 \otimes E_2 : I_1 \times I_2 \rightarrow \mathcal{C} \times \mathcal{C}$.

**Lemma 2.3.** Let $F_i, G_i : I_i \rightarrow \mathcal{C} (i = 1, 2)$ be functors. Then the map

$$\text{Nat}(F_1, G_1) \otimes_k \text{Nat}(F_2, G_2) \rightarrow \text{Nat}(F_1 \otimes F_2, G_1 \otimes G_2)$$

induced from the tensor product is injective.

**Proof.** Let $f_1, \ldots, f_m \in \text{Nat}(F_2, G_2)$ be linearly independent elements. Suppose that $c_1, \ldots, c_m \in \text{Nat}(F_1, G_1)$ are elements such that

$$c_1|X \otimes f_1|Y + \cdots + c_m|X \otimes f_m|Y = 0$$

for all $(X, Y) \in I_1 \times I_2$. If we fix $X \in I_1$, we can apply Lemma 2.2 and obtain that $c_i|X = 0$ for $i = 1, \ldots, m$. By letting $X$ run through all objects of $I_1$, we have that $c_i = 0$ for $i = 1, \ldots, m$. Thus the map under consideration is injective. \hfill \Box

**Proof of Theorem 1.1.** Now we prove Theorem 1.1. Our proof is based on a proof of the linear independence of grouplike elements in a coalgebra over a field. Recall the assumptions: $I$ is a skeletally small monoidal category, $\mathcal{C}$ is a tensor category over $k$, and $F$ and $G$ are monoidal functors from $I$ to $\mathcal{C}$.

We first note that $0 \not\in \text{Nat}_\otimes(F, G)$. Indeed, if $g \in \text{Nat}_\otimes(F, G)$, $g|1 : F(1) \rightarrow G(1)$ must be an isomorphism. Since $F(1) \cong 1 \neq 0$, $g|1 \neq 0$, and hence $g \neq 0$.

Suppose to the contrary that $\text{Nat}_\otimes(F, G)$ is linearly dependent. Then there exist elements $g_1, \ldots, g_m \in \text{Nat}_\otimes(F, G)$ and $\lambda_1, \ldots, \lambda_m \in k$ such that

$$g := \lambda_1 g_1 + \cdots + \lambda_m g_m \in \text{Nat}_\otimes(F, G)$$

and $g \neq g_i$ for $i = 1, \ldots, m$. We may suppose that $g_1, \ldots, g_m$ are linearly independent. Since $g \neq 0$, we may also suppose that $\lambda_h \neq 0$ for some $h$.

By the definition of monoidal natural transformations, we have

$$\sum_{j=1}^{m} \lambda_j g_j|X \otimes g_j|Y = g|X \otimes g|Y = \sum_{i,j=1}^{m} \lambda_i \lambda_j g_i|X \otimes g_j|Y$$
for all $X, Y \in I$. By Lemma 2.3, $\sum_{i=1}^{n} \lambda_{i} y_{i} = \lambda_{j} g_{j}$ for each $j$. By the linear independence of the $g_{i}$’s, we have that $\lambda_{i} = 0$ for $i \neq h$. Therefore, $g = g_{h}$. This is a contradiction. \hfill $\square$

3. SOME REMARKS ON $\text{Aut}_\odot(\text{id}_C)$

3.1. Bound of the order. Let $C$ be a finite tensor category over $k$. In this section, we give some remarks on the structure of the group $G(C) = \text{Aut}_\odot(\text{id}_C)$. We first note that by the definition of natural transformations, $G(C)$ is abelian.

Let $I$ be the set of isomorphism classes of simple objects of $C$. For each $i \in I$, we fix $S_{i} \in i$. Let $P_{i}$ be the projective cover of $S_{i}$. Then $P = \bigoplus_{i \in I} P_{i}$ is a projective generator. As every object of $C$ is of finite length, $C$ is $k$-linearly equivalent to the category of finite-dimensional right $\text{End}_C(P)$-modules. Hence the map

$$\text{Nat}(\text{id}_C, \text{id}_C) \to Z(\text{End}_C(P)), \quad \eta \mapsto \eta|_{P}$$

is an isomorphism of $k$-algebras (cf. Lemma 1.2).

Theorem 1.1 states that $G(C) \subset \text{Nat}(\text{id}_C, \text{id}_C)$ is linearly independent. Therefore, we have the following bound on the order of $G(C)$.

**Proposition 3.1.** $|G(C)| \leq \dim_k Z(\text{End}_C(P))$.

In the case where $C = \text{Rep}(H)$ for some finite-dimensional Hopf algebra $H$, this proposition is obvious since the right-hand side of the inequality is equal to the dimension of the center of $H$.

3.2. Values on simple objects. Let us consider the map

$$\text{Nat}(\text{id}_C, \text{id}_C) \to \prod_{i \in I} \text{End}_C(S_{i}), \quad \eta \mapsto (\eta|_{S_{i}})_{i \in I}.$$

This map is not injective anymore unless $C$ is semisimple. Since $k$ is algebraically closed, we can identify $\text{End}_C(S_{i})$ with $k$. The above map induces a group homomorphism

$$\varphi : G(C) \to \prod_{i \in I} \text{End}_C(S_{i})^\times \cong \text{Map}(I, k^\times), \quad g \mapsto (i \mapsto g|_{S_{i}}).$$

We show that $\varphi$ is injective if $k$ is of characteristic zero. To describe the kernel of $\varphi$, we introduce some subgroups of $G(C)$. If $p = \text{char}(k) > 0$, then we set

$$G(C)_{p} = \{g \in G(C) \mid g^{p^{k}} = 1 \text{ for some } k \geq 0\},$$

$$G(C)^{\prime} = \{g \in G(C) \mid \text{the order of } g \text{ is relatively prime to } p\}.$$  

Otherwise we set $G(C)_{p} = \{1\}$ and $G(C)^{\prime} = G(C)$. By the fundamental theorem of finite abelian groups, we have a decomposition $G(C) = G(C)_{p} \times G(C)^{\prime}_{p}$.

**Lemma 3.2.** Let $X \in C$ be an indecomposable object.

(a) If $g \in G(C)_{p}$, then $g|_{X}$ is unipotent.
(b) If $g \in G(C)^{\prime}_{p}$, then $g|_{X} = \lambda \cdot \text{id}_{X}$ for some $\lambda \in k^\times$.

**Proof.** Let $g \in G(C)$. As $X$ is indecomposable, $\text{End}_C(X)$ is a local algebra. Since $k$ is algebraically closed, $g|_{X}$ can be written uniquely in the form

$$g|_{X} = \lambda \cdot \text{id}_{X} + r \quad (\lambda \in k^\times, r \in m),$$

(3.1)
where \( m \) is the maximal ideal of \( \text{End}_C(X) \). If \( r \neq 0 \), then there exists \( k \geq 1 \) such that \( r \in m^{k-1} \) but \( r \not\in m^k \). By the binomial formula, we have

\[
(g|_X)^n \equiv \lambda^n \text{id}_X + nr \pmod{m^k}
\]

for every \( n \geq 0 \). This implies that \( (g|_X)^n = \text{id}_X \), then \( \lambda^n = 1 \) and \( nr = 0 \).

(a) Suppose that \( g \in G(C)_p \). Since the claim is obvious for \( p = 0 \), we assume that \( p > 0 \). Then the order of \( g|_X \) is a power of \( p \). Thus, by the above observation, \( \lambda = 1 \). This implies that \( g|_X \) is unipotent.

(b) Suppose that \( g \in G(C)'_p \). Then the order of \( g|_X \) is nonzero in \( k \). Thus, by the above observation, \( r = 0 \). This implies that \( g|_X = \lambda \text{id}_X \).

In what follows, we denote \( \lambda \) in equation (3.1) by \( \lambda_g(X) \). We note the following easy but important property of \( \lambda_g(X) \).

**Lemma 3.3.** Let \( X \) and \( Y \) be indecomposable objects of \( C \). If \( X \) and \( Y \) belong to the same block, then \( \lambda_g(X) = \lambda_g(Y) \) for every \( g \in G(C)'_p \).

Here, a block is an equivalence class of indecomposable objects of \( C \) under the weakest equivalence relation such that two indecomposable objects \( X \) and \( Y \) of \( C \) are equivalent whenever \( \text{Hom}_C(X,Y) \neq 0 \).

**Proof.** Let \( g \in G(C)'_p \). By the definition of blocks, it is sufficient to prove this in the case when there exists a nonzero morphism \( f : X \to Y \). By the naturality of \( g \),

\[
\lambda_g(X)f = (g|_X) \circ f = f \circ (g|_Y) = \lambda_g(Y)f.
\]

Since \( f \neq 0 \), we have \( \lambda_g(X) = \lambda_g(Y) \). \( \square \)

Now we have a description of the kernel of \( \varphi \) as follows:

**Proposition 3.4.** \( \text{Ker}(\varphi) = G(C)_p \). In particular, \( \varphi \) is injective if \( p = 0 \).

**Proof.** Let \( g \in G(C)_p \). Then, by Lemma 3.2 \( g|_S = \text{id}_S \) for every simple object \( S \) of \( C \) and hence \( g \in \text{Ker}(\varphi) \). This implies that \( G(C)_p \subset \text{Ker}(\varphi) \).

Next let \( g \in G(C)'_p \cap \text{Ker}(\varphi) \). Let \( X \) be an indecomposable object of \( C \) and fix a simple subobject \( S \) of \( X \). Then, by Lemma 3.3 \( \lambda_g(X) = \lambda_g(S) \). On the other hand, \( \lambda_g(S) = 1 \) since \( g \in \text{Ker}(\varphi) \). This implies that \( g|_X = \text{id}_X \) for all indecomposable objects \( X \) and hence \( g = 1 \). Therefore \( G(C)'_p \cap \text{Ker}(\varphi) = \{1\} \).

Now we recall that \( G(C) = G(C)_p \times G(C)'_p \). Our claim follows immediately from the above observations. \( \square \)

It is interesting to characterize the image of \( \varphi \). Let \( N^k_{ij} \) \((i, j, k \in I)\) be the multiplicity of \( S_k \) as a composition factor of \( S_i \otimes S_j \). In the case where \( C \) is semisimple, it is known that the image of \( \varphi \) is the set of functions \( \lambda : I \to k^\times \) such that

\[
\lambda(i)\lambda(j) = \lambda(k) \quad \text{whenever} \quad N^k_{ij} \neq 0 \quad (i, j, k \in I).
\]

In general, the image of \( \varphi \) is smaller than the set of such functions.

**Proposition 3.5.** The image of \( \varphi \) is the set of all functions satisfying (3.2) and the following condition:

\[
\lambda(i) = \lambda(j) \quad \text{whenever} \quad S_i \text{ and } S_j \text{ belong to the same block} \quad (i, j \in I).
\]
Theorem 3.6. If \( \lambda \) of functions \( P \) morphism \( K \) in \( \mathbb{C} \). We remark that if an indecomposable object \( X \), then \( X \) and \( Y \) belong to the same block. Indeed, then there exists a nonzero morphism \( P_i \to X \) and hence \( P_i \) and \( X \) belong to the same block. On the other hand, since \( S_i \) is a quotient of \( P_i, S_i \) and \( P_i \) belong to the same block. Therefore the claim follows.

Proof. We remark that if an indecomposable object \( X \in \mathcal{C} \) has \( S_i \) as a composition factor, then \( S_i \) and \( X \) belong to the same block. Indeed, then there exists a nonzero morphism \( P_i \to X \) and hence \( P_i \) and \( X \) belong to the same block. On the other hand, since \( S_i \) is a quotient of \( P_i, S_i \) and \( P_i \) belong to the same block. Therefore the claim follows.

Let \( \lambda = \varphi(g) \). By Proposition \([3.4]\), we may assume \( g \in G(\mathcal{C})_p \). \([3.3]\) follows from Lemma \([3.3]\). Thus we check that \( \lambda \) satisfies \([3.2]\). Let \( i, j \in I \). By the definition of monoidal natural transformations,

\[
g|_{s_i \otimes s_j} = g|_{s_i} \otimes g|_{s_j} = \lambda(i)\lambda(j) \text{id}_{s_i \otimes s_j}.
\]

Suppose that \( N_{ij}^k \neq 0 \). This means that \( S_i \otimes S_j \) has \( S_k \) as a composition factor. Let \( X \) be an indecomposable direct summand of \( S_i \otimes S_j \) having \( S_k \) as a composition factor. By the above equation, \( g|_X = \lambda(i)\lambda(j) \text{id}_X \). On the other hand, since \( X \) and \( S_k \) belong to the same block, \( g|_X = \lambda(k) \text{id}_X \). Therefore \( \lambda(k) = \lambda(i)\lambda(j) \).

Conversely, given a function \( \lambda : I \to \mathbb{k}^\times \) satisfying \([3.2]\) and \([3.3]\), we define a natural automorphism \( g : \text{id}_C \to \text{id}_C \) as follows: If \( X \) is an indecomposable object of \( \mathcal{C} \), then \( g|_X = \lambda(i) \text{id}_X \), where \( i \in I \) is such that \( X \) has \( S_i \) as a composition factor. As \( \lambda \) satisfies \([3.3]\), this does not depends on the choice of \( i \). We can extend \( g \) to all objects of \( \mathcal{C} \), since they are direct sums of indecomposable objects.

Now we need to show that \( g \in G(\mathcal{C}) \), that is, \( g|_{X \otimes Y} = g|_X \otimes g|_Y \) for all objects \( X, Y \in \mathcal{C} \). We may assume that \( X \) and \( Y \) are indecomposable. Suppose that

\[
X = \sum_{i \in I} m_i S_i \quad \text{(} m_i \in \mathbb{Z}_{\geq 0} \text{)} \quad \text{and} \quad Y = \sum_{j \in I} n_j S_j \quad \text{(} n_i \in \mathbb{Z}_{\geq 0} \text{)}
\]

in the Grothendieck ring \( K(\mathcal{C}) \). Then

\[
X \cdot Y = \sum_{k \in I} \left( \sum_{i,j \in I} m_i n_j N_{ij}^k \right) S_k
\]

in \( K(\mathcal{C}) \). This equation means that if \( X \otimes Y \) has \( S_k \) as a composition factor, then there exist \( i, j \in I \) such that \( X \) has \( S_i \) as a composition factor, \( Y \) has \( S_j \) as a composition factor, and \( N_{ij}^k \neq 0 \). By the definition of \( g \) and \([3.2]\), we have that \( g|_{X \otimes Y} = g|_X \otimes g|_Y \).

It is obvious that \( \lambda = \varphi(g) \). The proof is completed.

The following theorem is a direct consequence of Propositions \([3.4]\) and \([3.5]\).

**Theorem 3.6.** If \( p = 0 \), then \( \varphi \) gives an isomorphism between \( G(\mathcal{C}) \) and the group of functions \( \lambda : I \to \mathbb{k}^\times \) satisfying \([3.2]\) and \([3.3]\).

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**References**


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