CHAOTIC SOLUTION FOR THE BLACK-SCHOLES EQUATION

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Abstract. The Black-Scholes semigroup is studied on spaces of continuous functions on \((0, \infty)\) which may grow at both 0 and at \(\infty\), which is important since the standard initial value is an unbounded function. We prove that in the Banach spaces \(Y_{s,\tau} := \{ u \in C((0, \infty)) : \lim_{x \to \infty} u(x) = 0, \lim_{x \to 0} u(x) = 0 \}\) with norm \(\|u\|_{Y_{s,\tau}} = \sup_{x > 0} \left| \frac{u(x)}{1 + x^s} \left( 1 + x^{-\tau} \right) \right| < \infty\), the Black-Scholes semigroup is strongly continuous and chaotic for \(s > 1, \tau \geq 0\) with \(s\nu > 1\), where \(\sqrt{2}\nu\) is the volatility. The proof relies on the Godefroy-Shapiro hypercyclicity criterion.

1. Introduction

In [B-S], F. Black and M. Scholes proved that under certain assumptions about the market, the value of a stock option, as a function of the current value of the underlying asset \(x \in \mathbb{R}^+ = [0, +\infty)\) and time, \(u(x, t)\), satisfies the final value problem

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx \frac{\partial u}{\partial x} + ru & \text{in } \mathbb{R}^+ \times [0, T]; \\
u(0, t) = 0 & \text{for } t \in [0, T]; \\
u(x, T) = (x - p)^+ & \text{for } x \in \mathbb{R}^+,
\end{cases}
\]

(\(\text{BS}\))

where \(p > 0\) represents a given strike price, \(\sigma > 0\) is the volatility and \(r > 0\) is the interest rate.

Let \(v(x, t) = u(x, T - t)\). Then \(v\) satisfies the forward Black-Scholes equation, which is a parabolic problem, defined for all time \(t \in \mathbb{R}^+\) by

\[
\begin{cases}
\frac{\partial v}{\partial t} = \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx \frac{\partial v}{\partial x} - rv & \text{in } \mathbb{R}^+ \times \mathbb{R}^+; \\
v(0, t) = 0 & \text{for } t \in \mathbb{R}^+; \\
v(x, 0) = f(x) & \text{for } x \in \mathbb{R}^+.
\end{cases}
\]

(\(\text{FBS}\))

Strictly speaking, the condition \(t \in \mathbb{R}^+\) should have been written as \(0 \leq t \leq T\). But once one notes that, there is no problem considering all nonnegative values of time.

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In (FBS) we have

\( f(x) = (x - p)^+ = \begin{cases} x - p & \text{if } x > p, \\ 0 & \text{if } x \leq p, \end{cases} \)

but for the time being we prefer to consider \( f \) merely as an arbitrary given function. Later we shall deal with (1.1). In order to put the (FBS) problem in an abstract form, let us denote by \( D_\nu = \nu x \frac{dx}{x} \), where \( \nu = \sigma/\sqrt{2} \), and let

\( B = D_\nu^2 + \gamma D_\nu - rI = \nu^2 C_1 + C_2, \)

with \( \gamma = r/\nu - \nu, C_1 := x^2 \frac{d^2}{dx^2} = D_1^2 - D_1 \) and \( C_2 := rD_1 - rI. \) Then (FBS) can be written as

\[
\begin{align*}
&dv/dt = Bv, \\
&v(0, t) = 0, \\
&v(x, 0) = f(x) \quad \text{for } x \in \mathbb{R}^+.
\end{align*}
\]

For European call options, Cruz-Báez and González-Rodríguez [CG], and Arendt and de Pagter [AdP] showed that (FBS) is governed by a \( C_0 \)-semigroup on a suitable Banach space. In [CG2] the authors have generalized [CG] to American call options, a topic of interest in mathematical finance. But by working in the context of a contraction semigroup, these authors could not consider the issue of chaos. Recently, [GMR] gave a simple explicit representation of the solution of (FBS), and this representation holds in the spaces \( Y^{s,\tau} \) considered here.

2. Multiplicative (\( C_0 \)) Semigroups on the Weighted Space

For representing the Black-Scholes semigroup, we begin by introducing the translation on the multiplication group of positive numbers, \( G = ((0, \infty), \cdot) \). We do this now and we postpone the Black-Scholes semigroup to Section 3.

Let \( \mu \) be the Haar measure on \( G \) and suppose \( \tau = \{ \tau_t : t \in \mathbb{R} \} \) is the group of translations on \( G \). Thus \( d\mu = \frac{dx}{x} \), and \( \tau_t(x) = e^{tx}, \) for \( x > 0, t \in \mathbb{R}. \)

Let \( s, \tau > 0 \) and let \( C(0, \infty) \) be the space of all complex continuous functions on \( (0, \infty) \). Define

\[
Y^{s,\tau} := \{ u \in C(0, \infty) : \lim_{x \to 0} \frac{u(x)}{1 + x^{-\tau}} = \lim_{x \to \infty} \frac{u(x)}{1 + x^s} = 0 \},
\]

with norm

\[
\|u\|_{s,\tau} = \sup_{x > 0} \frac{|u(x)|}{(1 + x^{-\tau})(1 + x^s)} < \infty.
\]

These are Banach spaces.

Fix \( \nu \in \mathbb{R} \setminus \{0\} \). Define the translation group with parameter \( \nu \), \( S_\nu := \{ S_\nu(t) : t \in \mathbb{R} \} \) on \( Y^{s,\tau} \), by

\[
(S_\nu(t)f)(x) = e^{tD_\nu} f(x) = f(\tau_{t\nu}(x))
\]

for \( f \in Y^{s,\tau}, x \in G \) and \( t \in \mathbb{R}. \) Since \( \tau_{t+s} = \tau_t \tau_s \) for all \( t, s \in \mathbb{R}, \) \( S_\nu \) forms a one-parameter group on each \( Y^{s,\tau}. \) Let \( D_\nu \) be its infinitesimal operator in the sense of Hille, that is,

\[
D_\nu f = \left. \frac{d}{dt} S_\nu(t)f \right|_{t=0}
\]

for all \( f \) for which this limit exists in \( Y^{s,\tau}; \) call this set \( \mathcal{D}(D_\nu). \) Then \( f \in \mathcal{D}(D_\nu) \) requires that \( x \to f(x) \) and \( x \to xf(x) \) are both in \( Y^{s,\tau}. \) Below we will establish
the strong continuity and characterize $D(D_s) = D(D_1)$ in $Y^{s,\tau}$ for $s \geq 1$, $\tau \geq 0$. The spaces $Y^{s,\tau}$ are Banach spaces, for all $s \geq 0$, $\tau \geq 0$.

Let $\mathcal{M}(0,\infty)$ be the set of all finite complex Borel measures on $(0,\infty)$. Any $\psi \in \mathcal{M}(0,\infty)$ can be written as

$$
\psi = \text{Re}(\psi) + i \text{Im}(\psi) = \sum_{j=1}^{4} c_j P_j
$$

where each $P_j$ is a probability measure on $(0,\infty)$ and the scalars $c_j$ satisfy $\text{Re}(\psi) = (\text{Re} \psi)_+ - (\text{Re} \psi)_- = \psi_1 - \psi_2$, with $\psi_1 = c_1 P_1$ and $\psi_2 = -c_2 P_2$, and $c_1, c_2 \geq 0$. In the same way $\text{Im}(\psi) = (\text{Im} \psi)_+ - (\text{Im} \psi)_- = \psi_3 - \psi_4$, with $i\psi_3 = c_3 P_3$ and $i\psi_4 = -c_4 P_4$, and $-ic_4, -ic_4 \geq 0$, and $P_j$ is uniquely determined for each $j$ for which $c_j \neq 0$. We also define $\psi \in \mathcal{M}_{loc}(0,\infty)$ to mean that for any $n \in \mathbb{N}$, the restriction of $\psi$ to Borel subsets of $[\frac{1}{n}, n]$ is a finite complex Borel measure $\psi_n$ satisfying

$$
\psi_n = \text{Re} \psi_n + i \text{Im} \psi_n
$$

Let $\xi_n$ denote any one of $(\text{Re} \psi_n)_\pm, (\text{Im} \psi_n)_\pm$. Then each such $\xi_n$ determines uniquely a $\sigma$-finite Borel measure on $(0,\infty)$ via

$$
\xi(A) = \lim_{n \to \infty} \xi_n(A \cap \left[\frac{1}{n}, n\right])
$$

for all Borel sets $A \subset [0, \infty]$. In this sense we can view $(\text{Re} \psi)_\pm, (\text{Im} \psi)_\pm$ as measures in a certain sense. Note that the set functions $\psi = (\text{Re} \psi)_+ - (\text{Re} \psi)_- + i[(\text{Im} \psi)_+ - (\text{Im} \psi)_-] \in \mathcal{M}_{loc}(0,\infty)$ are not in general complex measures, but nevertheless we can treat them locally (away from 0 and $\infty$) as if they were complex measures by using $\psi_n$ for $n \in \mathbb{N}$.

We begin our study of $Y^{s,\tau}$ with the case of $s = 0$, $\tau = 0$. Note that

$$
Y^{0,0} = C_0(0,\infty),
$$

the continuous complex functions on $(0,\infty)$, which vanish at both 0 and $\infty$, with the norm

$$
\|u\|_{0,0} = \frac{1}{4} \|u\|_{\infty}
$$

for $u \in Y^{0,0}$. Note that the constant function 1 is in $Y^{s,\tau}$ if and only if $s > 0$, $\tau > 0$. By the Riesz Representation Theorem, the dual space of $Y^{0,0} = ((C_0(0,\infty), \|\cdot\|_{0,0})$ can be identified with $\mathcal{M}(0,\infty)$ with the norm

$$
\|\psi\| = 4TV(\psi) = 4 \sum_{j=1}^{4} |c_j|
$$

when $c_j$ is as in (2.1), and $TV$ means total variation. The identification is made by mapping $u \in Y^{0,0}$ and $\psi \in \mathcal{M}(0,\infty)$ to

$$
\langle u, \psi \rangle = \int_{(0,\infty)} u(x)\psi(dx).
$$

We shall write $\int_{0}^{\infty}$ in place of $\int_{(0,\infty)}$.

One may view $Y^{0,0} \subset \{u \in C[0,\infty] : u(0) = u(\infty) = 0\}$, the continuous functions on the compact interval $[0,\infty]$, which vanish at both 0 and $\infty$. Similarly,
Lemma 2.1. Let \( U : X \to Y \) be an isometric isomorphism between Banach spaces. Then \( U^* : Y^* \to X^* \) is also an isometric isomorphism between their dual spaces.

Let
\[
\mathcal{C}_c := \mathcal{C}_c(0, \infty) = \{ u \in C(0, \infty) : u \text{ has compact support in } (0, \infty) \}.
\]
Then \( \mathcal{C}_c \) is dense in \( Y^{s, \tau} \) for all \( s, \tau \geq 0 \). Let \( u \in \mathcal{C}_c \). Then \( u \in C_0(\varepsilon, 1/\varepsilon) \) for some \( \varepsilon > 0 \). Let \( \varphi \in \mathcal{M}((\varepsilon, 1/\varepsilon)) \) be a finite complex measure on \([\varepsilon, 1/\varepsilon] \), which is the dual space of \( C[\varepsilon, 1/\varepsilon] \). Then if \( \psi \in (Y^{s, \tau})^* \), we have \( \langle u, \psi \rangle = \int_0^\infty u(x) \varphi(dx) \) for some \( \varphi \) as above. We extend \( \varphi \) by requiring that \( \varphi(A) = 0 \) for all Borel subsets \( A \) of \([0, \varepsilon] \cup [1/\varepsilon, \infty) \). For this \( \varphi \), define \( \chi \) by
\[
\chi(dx) = (1 + x^s)(1 + x^{-\tau})\varphi(dx).
\]
Then for \( u \in Y^{s, \tau} \), \( u = U_{s, \tau}v \) for a unique \( v \in Y^{0,0} \), and
\[
\langle u, \psi \rangle := \int_0^\infty u(x) \psi(dx) = \int_0^\infty \left( \frac{u(x)}{(1 + x^s)(1 + x^{-\tau})} \right) ((1 + x^s)(1 + x^{-\tau})\psi(dx)) = \langle 4U_{s, \tau}u, \frac{1}{4}\chi \rangle = \langle U_{s, \tau}u, \chi \rangle = \langle v, \chi \rangle = \langle u, U_{s, \tau}^*\chi \rangle,
\]
since \( u \in \mathcal{C}_c \), which is dense in \( Y^{a,b} \) for all \( a, b > 0 \). Here \( U_{s, \tau} \) is the \( U \) of Lemma 2.1 corresponding to \( X = Y^{s, \tau} \). Let
\[
Z_{s, \tau} := \{ \psi \in \mathcal{M}_{loc}(0, \infty) : \chi(dx) = (1 + x^s)(1 + x^{-\tau})\psi(dx) \text{ defines } \chi \in \mathcal{M}(0, \infty) \}
\]
for \( s, \tau \geq 0 \). Then (2.2) below holds for \( \psi \in Z_{s, \tau} \) and \( \psi = U_{s, \tau}^*\chi \) for a unique \( \chi \in \mathcal{M}(0, \infty) \), that is, \( \psi \in Y^{s, \tau} \), and conversely. Thus we have proved that \( Z_{s, \tau} \) can be identified with \( (Y^{s, \tau})^* \) for all \( s, \tau \geq 0 \). We restate this now proved result as follows.

Lemma 2.2. For \( s, \tau \geq 0 \), the dual space of \( Y^{s, \tau} \) is
\[
(Y^{s, \tau})^* = \{ \varphi \in \mathcal{M}_{loc}((0, \infty)) : \eta(dx) := (1 + x^s)^{-1}(1 + x^{-\tau})^{-1}\varphi(dx) \in \mathcal{M}((0, \infty)) \}.
\]

Let us define the space \( \mathcal{J}_{s, \tau} := \{ f \in C^1(0, \infty) \cap Y^{s, \tau} : f' \in L^\infty(0, \infty) \} \). In order to prove that \( S_\nu \) is a \( (C_0) \) group on \( Y^{s, \tau} \), we need the following lemma.

Lemma 2.3. The space \( \mathcal{J}_{s, \tau} \) is dense in \( Y^{s, \tau} \) for all \( s, \tau \geq 0 \).

Proof. Note that the map
\[
\frac{f(x)}{4} \to \frac{f(x)}{(1 + x^s)(1 + x^{-\tau})}
\]
is an isometric isomorphism from \( Y^{0,0} \) onto \( Y^{s, \tau} \) which leaves invariant \( C_\infty^\infty(0, \infty) \), the smooth functions with compact support in \( (0, \infty) \). Therefore \( C_\infty^\infty(0, \infty) \) and \( \mathcal{J}_{s, \tau} \) are both dense in \( Y^{s, \tau} \) since \( C_\infty^\infty(0, \infty) \) is dense in \( Y^{0,0} \).

Theorem 2.4. The family \( S_\nu \) forms a \( (C_0) \) group on \( Y^{s, \tau} \) for each \( s \geq 1 \) and \( \tau \geq 0 \).
Proof. First we note that the constant function 1 belongs to $Y^{s,\tau}$ if and only if $s,\tau$ are both positive. Next, we observe that for $f \in Y^{s,\tau}$ and $t \in \mathbb{R}$,
\[
\| S_t(t)f \|_{s,\tau} = \sup_{x > 0} \frac{|f(e^{ix}t)|}{(1 + x^s)(1 + x^{-\tau})} = \sup_{y > 0} \frac{|f(y)|}{(1 + [e^{-\nu y}]^s)(1 + [e^{-\nu y}]^{-\tau})}.
\]
Suppose $tv > 0$. Then
\[
\| S_t(t)f \|_{s,\tau} \leq e^{\nu s} \sup_{y > 0} \frac{|f(y)|}{(1 + y^s)(1 + y^{-\tau})} = e^{\nu s} \| f \|_{s,\tau}.
\]
For $tv \leq 0$, we have
\[
\| S_t(t)f \|_{s,\tau} = \sup_{y > 0} \frac{|f(y)|}{(1 + [e^{-\nu y}]^s)(1 + [e^{-\nu y}]^{-\tau})} \leq e^{\nu |\tau|} \sup_{y > 0} \frac{|f(y)|}{(1 + y^s)(1 + y^{-\tau})} = e^{\nu |\tau|} \| f \|_{s,\tau}.
\]
Thus $S_t(t) : Y^{s,\tau} \mapsto Y^{s,\tau}$ and $\| S_t(t) \| \leq e^{\omega |t|}$, $\omega = |\nu| \max\{s,\tau\}$.

Thanks to Lemma 2.3, it is enough to show the strong continuity on $J_{s,\tau}$. In fact, for any $f \in J_{s,\tau}$, choose $\chi \in C^\infty(0,\infty)$ such that $\chi(x) = 0$ for $0 \leq x \leq 1$, $\chi$ is increasing on $(1, 2)$, and $\chi(x) = 1$ for $x \geq 2$.

Let $f_1 = f\chi$, $f_2 = f(1 - \chi)$. Then $f_1, f_2 \in J_{s,\tau}$, $\text{supp}f_1 \subset (1, \infty)$, $\text{supp}f_2 \subset (0, 2)$, and $f_1 + f_2 = f$. Now for $f_1$,
\[
\| S_t(t)f_1 - f_1 \|_{s,\tau} = \sup_{x \geq 1} \frac{|f_1(e^{ix}t) - f_1(x)|}{(1 + x^s)(1 + x^{-\tau})} \leq \| f_1 \|_\infty \sup_{x \geq 1} \frac{|e^{ix}t - x|}{1 + x^s} \leq \| f_1 \|_\infty |e^{ix}t - 1| \rightarrow 0, \quad \text{as} \quad t \rightarrow 0^+ \quad \text{since} \quad s \geq 1.
\]
For $f_2$, we have
\[
\| S_t(t)f_2 - f_2 \|_{s,\tau} \leq \| f_2 \|_\infty \sup_{0 < x < 2} \frac{|e^{ix}t - x|}{1 + x^{-\tau}} \leq \frac{2^{\tau+1}}{1 + 2\tau} \| f_2 \|_\infty |e^{ix}t - 1| \rightarrow 0,
\]
as $t \rightarrow 0^+$, and this proves the theorem. \]

In the sequel we will need the following result, which is proved in [1] and [2], Theorem 11.

Lemma 2.5. Suppose $iA$ generates a strongly continuous group. Let $p(t) = t^{2n} + q(t)$, where $q$ is a polynomial of degree less than $2n$. Then $-p(A)$ generates a holomorphic $(C_0)$ semigroup of angle $\pi/2$.

Take $A = -iD_y$, so that $iA$ generates a strongly continuous group on $X = Y^{s,\tau}$ and take $p(t) = t^2 - i\gamma t + r$. Hence we have the following result.

Theorem 2.6. The operator $B$ defined in (12) generates a holomorphic $(C_0)$ semigroup of angle $\pi/2$ on any $Y^{s,\tau}$, where $s \geq 1, \tau \geq 0$. 
3. The chaotic character of the Black-Scholes semigroup

Let $X$ be a separable complex Banach space.

**Definition 3.1.** A strongly continuous semigroup (or $(C_0)$ semigroup) $T = \{T(t) : t \geq 0\}$ of bounded linear operators on $X$ is called hypercyclic if there exists a vector $x \in X$ such that its orbit $\{T(t)x : t \geq 0\}$ is dense in $X$, and $T$ is called chaotic if in addition the set of periodic points of $T$,

$$\mathcal{P}_{per} := \{x \in X : \text{there exists } t_0 > 0 \text{ such that } T(t_0)x = x\},$$

is dense in $X$.

The notion of chaotic $(C_0)$ semigroups was introduced independently by MacCluer [McC] and Protopopescu and Azmy [P-A]; the first systematic study of this concept is due to Desch, Schappacher and Webb [DSW]. So far, several specific examples of hypercyclic $(C_0)$ semigroups have come up in the literature (see [GE1, GE2] for complete citations).

The following lemma is proved by G. Godefroy and J. Shapiro in [G-S, Corollary 1.5].

**Lemma 3.2.** Suppose $A$ is a linear bounded operator on a Banach space $X$, $Q_1, Q_2$ are dense subsets of $X$ and $Z : Q_1 \mapsto Q_1$ such that

1. $AZy = y$, for all $y \in Q_1$,
2. $\lim_{n \to \infty} Z^n y = 0$, for all $y \in Q_1$ and
3. $\lim_{n \to \infty} A^n w = 0$, for all $w \in Q_2$.

Then $A$ is hypercyclic.

Let $s > 1/\nu$, where $\nu > 0$ is given. Denote by

$$S_s = \{\lambda \in \mathbb{C} : 0 < \Re \lambda < \nu s\}$$

the open strip in $\mathbb{C}$ and let $h_\lambda(x) = x^\lambda$. This function is well-defined in $\mathbb{R}_+$ for any $\lambda \in S_s$.

**Lemma 3.3.** The function $\lambda \mapsto h_\lambda(x)$ is analytic from $S_s$ into $Y^{s,\tau}$ for each $sv > 1$ and $\tau \geq 0$.

**Proof.** Note that when $\tau = 0$, any $\psi \in M(0, \infty)$ cannot have an atom at 0 and $1 \notin Y^{s,0}$. Now, since weak analyticity is equivalent to analyticity, we have only to prove that

$$\lambda \mapsto \int_{(0, \infty)} h(x, \lambda) \psi(dx)$$

is analytic for any $\psi \in F$, where $F$ is a norm-determining subset of $(Y^{s,\tau})^*$. The norm-determining set we use is

$$F := \{c\delta_x : c \in \mathbb{C}, x \in (0, \infty)\},$$

where $\delta_x$ denotes the Dirac point mass measure at $x$. Note that

$$\|f\|_{s,\tau} = \sup \{|cf(x)| = |(f, \psi)| : \psi = c\delta_x, c \in \mathbb{C}, x \in (0, \infty), \|\psi\|_{(Y^{s,\tau})^*} = 1\},$$
and a choice of $c$ that works above is $c = \frac{1}{(1 + x^s)(1 + x^{-\tau})}$, when the supremum defining the norm of $f$ is a maximum attained at $x$. Furthermore,

$$\lambda \mapsto x^\lambda = e^{(\ln x)^\lambda} = \langle h_\lambda(x), \delta_x \rangle$$

is an entire function of $\lambda \in \mathbb{C}$ for all $x > 0$; and for $\lambda \in S_s$ and $x > 0$, $\lim_{x \to \infty} x^\lambda / (1 + x^s) = 0$. Hence $x^\lambda \in Y^{s,\tau}$ for all $\tau > 0$. $\square$

If a linear operator $L$ generates a $(C_0)$ group on a Banach space $X$, then some polynomials in $L$ (such as $L^2 + \alpha L + \beta I$ for arbitrary scalars $\alpha, \beta$) generate $(C_0)$ semigroups on $X$. For the operator $B$, defined in (1.2), the Black-Scholes semigroup can be represented by $T(t) := f(D_\nu)$, where

$$f(z) = e^{tg(z)} \quad \text{with} \quad g(z) = z^2 + \gamma z - r.$$  

According to Theorem 2.6 this $(C_0)$ semigroup is well defined for each $t \in \mathbb{C}$, with $\text{Re}(t) > 0$. These operators will be shown to be chaotic on $X = Y^{s,\tau}$ for $s > 1, \tau \geq 0$ when $s\nu > 1$. We begin by recalling the following lemma, which was proved in [DSW] and [dL-E], and we reproduce this proof in our case.

**Lemma 3.4.** Suppose that there exists a set $\Omega \subset S_s$ which has an accumulation point in $S_s$. Then $Q := \text{Span}\{h_\lambda : \lambda \in \Omega\}$ is dense in $Y^{s,\tau}$ for $s > 1, \tau \geq 0$.

**Proof.** Suppose $\psi \in Q^\perp$. Since $\psi$ belongs to the dual of $Y^{s,\tau}$ and $h_\lambda = x^\lambda \in Y^{s,\tau}$, Lemma 3.3 asserts that $p(\lambda) = \langle \psi, x^\lambda \rangle$ is well defined and $p(\lambda)$ is analytic in $S_s$. Since $p(\lambda) = 0$ for all $\lambda \in \Omega$, which is a set with an accumulation point, then $p = 0$ in all of $S_s$ and so $\psi = 0$, as desired. $\square$

We continue to work in the spaces $Y^{s,\tau}$, $s > 1, \tau \geq 0$ with $s\nu > 1$.

**Lemma 3.5.** Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and $\mathbb{T}$, the unit circle, be its boundary. The set $f(S_s) \cap \mathbb{T}$ is nonempty and possesses infinitely many accumulation points in the strip $S_s$, where $f$ is as in (3.2).

**Proof.** For $f(z) = e^{tg(z)}$ with $t > 0$, in order to have $f(S_s) \cap \mathbb{T} \neq \emptyset$ we must find $z \in S_s$ such that

$$\text{Re } g(z) = \text{Re } (\nu^2z^2 + (r - \nu^2)z - r) = \nu^2(x^2 - y_0^2 - x) + rx - r = 0$$

with $z = x + iy_0$. Equivalently, we must find $(x, y_0)$ with $0 < x < \nu s, y_0 \in \mathbb{R}$ such that

$$x^2 + \left(\frac{r}{\nu^2} - 1\right)x - \frac{r}{\nu^2} = y_0^2.$$  

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Call $\mathcal{C}$ the curve represented by the graph of the quadratic function $y = x^2 + \left(\frac{1}{\nu^2} - 1\right)x - \frac{2}{\nu^2}$. As Figure 1 shows, for $1 < x < \nu s$, there are uncountably many points $(x, y)$ on the dashed portion of $\mathcal{C}$ with $y > 0$. For each such point let $y_0 = \sqrt{y}$. Then this gives uncountably many solutions of (3.3).

Now we can prove our main theorem.

**Theorem 3.6.** The Black-Scholes $(C_0)$ semigroup $T$ is chaotic in $Y_{s,\tau}$ for each $s > 1$, $\tau \geq 0$ with $sv > 1$.

**Proof.** First let us prove that the $(C_0)$ semigroup $T = \{T(t) = f(D_\nu) = e^{tB} : t \geq 0\}$ is hypercyclic. For this we will use Lemma 3.2, taking

$\Omega_1 = \{\lambda \in \frac{1}{\nu}S_s : |f(\nu\lambda)| > 1\}$, $\Omega_2 = \{\lambda \in \frac{1}{\nu}S_s : |f(\nu\lambda)| < 1\}$

and

$Q_j := \text{Span}\{h_{\lambda} : \lambda \in \Omega_j\}$ for $j = 1, 2$.

Now, let $z_0 \in f(S_s) \cap \mathbb{T}$, since $f$ is holomorphic and nonconstant, $f(S_s)$ is an open set, and $\Omega_1 = f(S_s) \cap \{z \in \mathbb{C} : |z| > 1\}$ and $\Omega_2 = f(S_s) \cap \{z \in \mathbb{C} : |z| < 1\}$ are also open, and any point in $\Omega_j$ is an accumulation point. So according to Lemma 3.3, $Q_j$ is dense in $Y_{s,\tau}$ for $j = 1, 2$.

Let $A = f(D_\nu)$ and define $Z = (f(D_\nu))^{-1}$ on $Q_1$ so that

$Z \left(\sum_{k=1}^{N} \alpha_k h_{\lambda_k}\right) = \sum_{k=1}^{N} \alpha_k (f(\nu\lambda_k))^{-1} h_{\lambda_k}$

for $\lambda_k \in \Omega_1$, $\alpha_k \in \mathbb{C}$ and $N \in \mathbb{N}$. It is clear that for any $y = \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \in Q_1$, we have $AZy = y$. Furthermore for $\lambda_k \in \Omega_1$, $|f(\nu\lambda_k)| > 1$, and consequently

$\lim_{n \to \infty} Z^n y = \lim_{n \to \infty} \sum_{k=1}^{N} \alpha_k (f(\nu\lambda_k))^{-n} h_{\lambda_k} = 0.$
Finally, for \( w = \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \in Q_3 \) with \( |f(\nu \lambda_k)| < 1 \) for each \( k \),
\[
\lim_{n \to \infty} A^n y = \lim_{n \to \infty} \sum_{k=1}^{N} \alpha_k f(\nu \lambda_k)^n h_{\lambda_k}(x) = 0.
\]

These imply that the hypotheses of the Godefroy-Shapiro Lemma 3.2 are satisfied and \( A \) is hypercyclic.

To see that \( T(t) = f(D_v) \) is chaotic, we define \( \Omega_3 = \{ \lambda \in \mathbb{R} : f(\nu \lambda) \in e^{2\pi i \mathbb{Q}} \} \) and \( Q_3 := \text{Span}\{h_{\lambda} : \lambda \in \Omega_3\} \). \( Q_3 \) is contained in the set of all periodic points of \( A = f(D_v) \). Suppose \( f(\nu \lambda_k) = e^{2\pi i n_k/m_k} \). Then for \( y = \sum_{k=1}^{N} r_k h_{\lambda_k} \) and \( m = \prod_{k=1}^{N} m_k \), one has \( f(D_v)^m y = y \). So the set of all periodic points \( P_{per} \) of \( f(D_v) \) is dense, and consequently \( T(t) \) is chaotic.

The real-world applications of (FBS) require nonnegative initial data and nonnegative solutions. The above proof that the Black-Scholes semigroup \( T \) is chaotic uses holomorphic functions and thus requires the use of spaces of complex-valued functions. Theorem 3.6 would be more satisfying from an applied standpoint if it were valid for real functions. This is precisely the content of the next result.

Let \( Y_{s,\tau}^s \) be the real functions in \( Y_{s,\tau}^s \). This is a real Banach space. If \( f \in Y_{s,\tau}^s \), then by [GMR, eq. (17)], the solution of (FBS) is given by
\[
v(x,t) = (T(t)f)(x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-y^2/(4t)} f \left( x e^{(r-\sigma^2/2)t-(\sigma/\sqrt{2})y} \right) dy.
\]

Thus \( T(t)f \) is real (resp., nonnegative) for each \( t \geq 0 \) if and only if \( f \) is real (resp., nonnegative). Let \( S_T \) be the restriction of \( T \) to \( Y_{s,\tau}^s \). Then \( S_T = \{ S_T(t) : t \geq 0 \} \) is a \( (C_0) \) semigroup on \( Y_{s,\tau}^s \) for \( s \geq 1, \tau \geq 0 \).

**Theorem 3.7.** The semigroup \( S_T \) on \( Y_{s,\tau}^s \) is chaotic if \( s > 1 \) and \( \tau \geq 0 \) when \( su > 1 \).

**Proof.** Let \( f \in Y_{s,\tau}^s \) be given, where \( s > 1 \) with \( su > 1 \), and \( \tau \geq 0 \). Let \( g \in Y_{s,\tau}^s \) have a dense \( T \)-orbit. Then there is a sequence of times \( t_n \to \infty \) such that \( \|T(t_n)g - f\|_{s,\tau} \to 0 \) as \( n \to \infty \). Consequently, since \( \text{Re}(T(t)h) = T(t)(\text{Re}(h)) \) for all \( h \in Y_{s,\tau}^s \),
\[
\|S_T(t_n)(\text{Re}(g)) - f\|_{s,\tau} \leq \left\| \sqrt{S_T(t_n)(\text{Re}(g)) - f} + S_T(t_n)(\text{Im}(g)) \right\|_{s,\tau}
\]
\[
= \|\text{Re}(T(t_n)g - f) + i[\text{Im}(T(t_n)g)]\|_{s,\tau}
\]
\[
= \|T(t_n)g - f\|_{s,\tau} \to 0
\]
as \( n \to \infty \). It follows that \( S_T \) is hypercyclic.

Next, if \( f \) is periodic of period \( p \), then so are \( \text{Re}(f) \) and \( \text{Im}(f) \). Thus \( S_T \) has a dense set of periodic points since \( T \) does. The theorem follows.

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