CHAOTIC SOLUTION FOR THE BLACK-SCHOLES EQUATION

HASSAN EMAMIRAD, GISELE RUIZ GOLDSCHTEIN, AND JEROME A. GOLDSCHTEIN

(Communicated by Thomas Schlumprecht)

Abstract. The Black-Scholes semigroup is studied on spaces of continuous functions on \((0, \infty)\) which may grow at both 0 and at \(\infty\), which is important since the standard initial value is an unbounded function. We prove that in the Banach spaces

\[ Y_{s,\tau} := \{ u \in C((0, \infty)) : \lim_{x \to \infty} \frac{u(x)}{1+x^s} = 0, \lim_{x \to 0} \frac{u(x)}{1+x^{-\tau}} = 0 \} \]

with norm \( \|u\|_{Y_{s,\tau}} = \sup_{x>0} \frac{|u(x)(1+x^s)(1+x^{-\tau})|}{1+x^s} < \infty \), the Black-Scholes semigroup is strongly continuous and chaotic for \( s > 1, \tau \geq 0 \) with \( s\nu > 1 \), where \( \sqrt{2\nu} \) is the volatility. The proof relies on the Godefroy-Shapiro hypercyclicity criterion.

1. Introduction

In [B-S], F. Black and M. Scholes proved that under certain assumptions about the market, the value of a stock option, as a function of the current value of the underlying asset \( x \in \mathbb{R}^+ = [0, +\infty) \) and time, \( u(x,t) \), satisfies the final value problem

\[ \begin{align*}
\frac{\partial u}{\partial t} &= -\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx \frac{\partial u}{\partial x} + ru \\
\quad & \text{in } \mathbb{R}^+ \times [0, T]; \\
\quad & \text{for } t \in [0, T]; \\
u(0, t) &= 0 \\
\quad & \text{for } x \in \mathbb{R}^+, \\
u(x, T) &= (x - p)^+ \\
\quad & \text{for } x \in \mathbb{R}^+,
\end{align*} \]

where \( p > 0 \) represents a given strike price, \( \sigma > 0 \) is the volatility and \( r > 0 \) is the interest rate.

Let \( v(x,t) = u(x, T - t) \). Then \( v \) satisfies the forward Black-Scholes equation, which is a parabolic problem, defined for all time \( t \in \mathbb{R}^+ \) by

\[ \begin{align*}
\frac{\partial v}{\partial t} &= \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx \frac{\partial v}{\partial x} - rv \\
\quad & \text{in } \mathbb{R}^+ \times \mathbb{R}^+; \\
v(0, t) &= 0 \\
\quad & \text{for } t \in \mathbb{R}^+; \\
v(x, 0) &= f(x) \\
\quad & \text{for } x \in \mathbb{R}^+.
\end{align*} \]

Strictly speaking, the condition \( t \in \mathbb{R}^+ \) should have been written as \( 0 \leq t \leq T \). But once one notes that, there is no problem considering all nonnegative values of time.

Received by the editors August 18, 2009 and, in revised form, September 13, 2010; December 18, 2010; and February 9, 2011.

2010 Mathematics Subject Classification. Primary 47D06, 91G80, 35Q91.
Key words and phrases. Hypercyclic and chaotic semigroup, Black-Scholes equation.
In (FBS) we have

\[(1.1)\]
\[f(x) = (x - p)^+ = \begin{cases} x - p & \text{if } x > p, \\ 0 & \text{if } x \leq p, \end{cases}\]

but for the time being we prefer to consider \( f \) merely as an arbitrary given function. Later we shall deal with (1.1). In order to put the (FBS) problem in an abstract form, let us denote by \( D_v = \nu x \frac{dx}{dx} \), where \( \nu = \sigma/\sqrt{2} \), and let

\[(1.2)\]
\[B = D_v^n + \gamma D_v - rI = \nu^2 C_1 + C_2,\]

with \( \gamma = r/\nu - \nu, C_1 := x^2 \frac{d^2}{dx^2} = D_1^2 - D_1 \) and \( C_2 := rD_1 - rI \). Then (FBS) can be written as

\[(AFBS)\]
\[\begin{align*}
&dv/dt = Bv, \\
v(0, t) = 0, \\
v(x, 0) = f(x) \quad \text{for } x \in \mathbb{R}^+. 
\end{align*}\]

For European call options, Cruz-Báez and González-Rodríguez [C-G1] and Arendt and de Pagter [AdP] showed that (FBS) is governed by a \( C_0 \)-semigroup on a suitable Banach space. In [C-G2] the authors have generalized [C-G1] to American call options, a topic of interest in mathematical finance. But by working in the context of a contraction semigroup, these authors could not consider the issue of chaos. Recently, [GMR] gave a simple explicit representation of the solution of (FBS), and this representation holds in the spaces \( Y^{s, \tau} \) considered here.

2. Multiplicative \((C_0)\) semigroups on the weighted space

For representing the Black-Scholes semigroup, we begin by introducing the translation on the multiplication group of positive numbers, \( G = ((0, \infty), \cdot) \). We do this now and we postpone the Black-Scholes semigroup to Section 3.

Let \( \mu \) be the Haar measure on \( G \) and suppose \( \tau = \{\tau_t : t \in \mathbb{R}\} \) is the group of translations on \( G \). Thus \( d\mu = \frac{dx}{\tau} \), and \( \tau_t(x) = e^{tx} \), for \( x > 0, t \in \mathbb{R} \).

Let \( s, \tau \geq 0 \) and let \( C(0, \infty) \) be the space of all complex continuous functions on \((0, \infty)\). Define

\[Y^{s, \tau} := \{u \in C(0, \infty) : \lim_{x \to 0^+} \frac{u(x)}{1 + x^{-\tau}} = \lim_{x \to \infty} \frac{u(x)}{1 + x^s} = 0\},\]

with norm

\[\|u\|_{s, \tau} = \sup_{x > 0} \left| \frac{u(x)}{(1 + x^{-\tau})(1 + x^s)} \right| < \infty.\]

These are Banach spaces.

Fix \( \nu \in \mathbb{R} \setminus \{0\} \). Define the translation group with parameter \( \nu \), \( S_\nu := \{S_\nu(t) : t \in \mathbb{R}\} \) on \( Y^{s, \tau} \), by

\[(S_\nu(t)f)(x) = e^{tD_\nu} f(x) = f(\tau_{\nu t}(x)) \]

for \( f \in Y^{s, \tau}, x \in G \) and \( t \in \mathbb{R} \). Since \( \tau_{t+s} = \tau_t \tau_s \) for all \( t, s \in \mathbb{R} \), \( S_\nu \) forms a one-parameter group on each \( Y^{s, \tau} \). Let \( D_\nu \) be its infinitesimal operator in the sense of Hille, that is,

\[D_\nu f = \frac{d}{dt} S_\nu(t)f \bigg|_{t=0}\]

for all \( f \) for which this limit exists in \( Y^{s, \tau} \); call this set \( \mathcal{D}(D_\nu) \). Then \( f \in \mathcal{D}(D_\nu) \) requires that \( x \to f(x) \) and \( x \to xf(x) \) are both in \( Y^{s, \tau} \). Below we will establish
the strong continuity and characterize \( \mathcal{D}(D_n) = \mathcal{D}(D_1) \) in \( Y^{s,\tau} \) for \( s \geq 1, \tau \geq 0 \).
The spaces \( Y^{s,\tau} \) are Banach spaces, for all \( s \geq 0, \tau \geq 0 \).

Let \( \mathcal{M}(0,\infty) \) be the set of all finite complex Borel measures on \( (0,\infty) \). Any \( \psi \in \mathcal{M}(0,\infty) \) can be written as

\[
\psi = \text{Re}(\psi) + i \text{Im}(\psi) = \sum_{j=1}^{4} c_j P_j
\]

where each \( P_j \) is a probability measure on \( (0,\infty) \) and the scalars \( c_j \) satisfy \( \text{Re}(\psi) = (\text{Re} \, \psi)_+ - (\text{Re} \, \psi)_- = \psi_1 - \psi_2 \), with \( \psi_1 = c_1 P_1 \) and \( \psi_2 = -c_2 P_2 \), and \( c_1, c_2 \geq 0 \).
In the same way \( \text{Im}(\psi) = (\text{Im} \, \psi)_+ - (\text{Im} \, \psi)_- = \psi_3 - \psi_4 \), with \( i\psi_3 = c_3 P_3 \) and \( i\psi_4 = -c_4 P_4 \), and \( -ic_3, -ic_4 \geq 0 \), and \( P_j \) is uniquely determined for each \( j \) for which \( c_j \neq 0 \). We also define \( \psi \in \mathcal{M}_{\text{loc}}(0,\infty) \) to mean that for any \( n \in \mathbb{N} \), the restriction of \( \psi \) to Borel subsets of \( \left[ \frac{1}{n}, n \right] \) is a finite complex Borel measure \( \psi_n \) satisfying

\[
\psi_n = \text{Re} \psi_n + i \text{Im} \psi_n = (\text{Re} \, \psi_n)_+ - (\text{Re} \, \psi_n)_- + i(\text{Im} \, \psi_n)_+ - (\text{Im} \, \psi_n)_-.
\]

Let \( \xi_n \) denote any one of \( (\text{Re} \, \psi_n)_\pm, (\text{Im} \, \psi_n)_\pm \). Then each such \( \xi_n \) determines uniquely a \( \sigma \)-finite Borel measure on \( (0,\infty) \) via

\[
\xi(A) = \lim_{n \to \infty} \xi_n(A \cap \left[ \frac{1}{n}, n \right])
\]

for all Borel sets \( A \subset [0,\infty] \). In this sense we can view \( (\text{Re} \, \psi)_\pm, (\text{Im} \, \psi)_\pm \) as measures in a certain sense. Note that the set functions \( \psi = (\text{Re} \, \psi)_+ - (\text{Re} \, \psi)_- + i(\text{Im} \, \psi)_+ - (\text{Im} \, \psi)_- \) \( \in \mathcal{M}_{\text{loc}}(0,\infty) \) are not in general complex measures, but nevertheless we can treat them locally (away from 0 and \( \infty \)) as if they were complex measures by using \( \psi_n \) for \( n \in \mathbb{N} \).

We begin our study of \( Y^{s,\tau} \) with the case of \( s = 0, \tau = 0 \). Note that

\[
Y^{0,0} = C_0(0,\infty),
\]

the continuous complex functions on \( (0,\infty) \), which vanish at both 0 and \( \infty \), with the norm

\[
\|u\|_{0,0} = \frac{1}{4} \|u\|_{\infty}
\]

for \( u \in Y^{0,0} \). Note that the constant function \( 1 \) is in \( Y^{s,\tau} \) if and only if \( s > 0, \tau > 0 \).
By the Riesz Representation Theorem, the dual space of \( Y^{0,0} = (C_0(0,\infty), \|\cdot\|_{0,0}) \) can be identified with \( M(0,\infty) \) with the norm

\[
\|\psi\| = 4TV(\psi) = 4 \sum_{j=1}^{4} |c_j|
\]

when \( c_j \) is as in (2.1), and \( TV \) means total variation. The identification is made by mapping \( u \in Y^{0,0} \) and \( \psi \in M(0,\infty) \) to

\[
\langle u, \psi \rangle = \int_{(0,\infty)} u(x) \psi(dx).
\]

We shall write \( \int_{0}^{\infty} \) in place of \( \int_{(0,\infty)} \).
One may view \( Y^{0,0} \) as \( \{u \in C[0,\infty] : u(0) = u(\infty) = 0\} \), the continuous functions on the compact interval \( [0,\infty] \), which vanish at both 0 and \( \infty \). Similarly,
\( \mathcal{M}(0, \infty) \) may be viewed as the finite complex Borel measures \( \psi \) on \([0, \infty)\) satisfying \( \psi([0, \infty)) = 0 \).

We next recall a well-known fact.

**Lemma 2.1.** Let \( U : X \to Y \) be an isometric isomorphism between Banach spaces. Then \( U^* : Y^* \to X^* \) is also an isometric isomorphism between their dual spaces.

Let

$$C_c := C_c(0, \infty) = \{ u \in C(0, \infty) : u \text{ has compact support in } (0, \infty) \}.$$ 

Then \( C_c \) is dense in \( Y^* \) for all \( s, \tau \geq 0 \). Let \( u \in C_c \). Then \( u \in C_0(\varepsilon, 1/\varepsilon) \) for some \( \varepsilon > 0 \). Let \( \varphi \in \mathcal{M}([\varepsilon, 1/\varepsilon]) \) be a finite complex measure on \([\varepsilon, 1/\varepsilon] \), which is the dual space of \( C[\varepsilon, 1/\varepsilon] \). Then if \( \psi \in (Y^*)^* \), we have \( \langle u, \psi \rangle = \int_0^\infty u(x)\varphi(dx) \) for some \( \varphi \) as above. We extend \( \varphi \) by requiring that \( \varphi(A) = 0 \) for all Borel subsets \( A \) of \([0, \varepsilon] \cup [1/\varepsilon, \infty) \). For this \( \varphi \), define \( \chi \) by

$$\chi(dx) = (1 + x^s)(1 + x^{-\tau})\varphi(dx).$$ 

Then for \( u \in Y^{s, \tau} \), \( u = U_{s, \tau}v \) for a unique \( v \in Y^{0,0} \), and

$$\langle u, \psi \rangle := \int_0^\infty u(x)\psi(dx) = \int_0^\infty \left( \frac{u(x)}{(1 + x^s)(1 + x^{-\tau})} \right) ((1 + x^s)(1 + x^{-\tau})\psi(dx))$$

$$= \langle 4U_{s, \tau}u, \frac{1}{4}\chi \rangle = \langle U_{s, \tau}u, \chi \rangle = \langle v, \chi \rangle = \langle u, U_{s, \tau}^*\chi \rangle,$$

since \( u \in C_c \), which is dense in \( Y^{a,b} \) for all \( a, b > 0 \). Here \( U_{s, \tau} \) is the \( U \) of Lemma 2.1 corresponding to \( X = Y^{s, \tau} \). Let

$$Z_{s, \tau} := \{ \psi \in \mathcal{M}_{loc}(0, \infty) : \chi(dx) := (1 + x^s)(1 + x^{-\tau})\psi(dx) \text{ defines } \chi \in \mathcal{M}(0, \infty) \}$$

for \( s, \tau \geq 0 \). Then \( (2.2) \) below holds for \( \psi \in Z_{s, \tau} \) and \( \psi = U_{s, \tau}^*\chi \) for a unique \( \chi \in \mathcal{M}(0, \infty) \), that is, \( \psi \in Y^{s, \tau} \), and conversely. Thus we have proved that \( Z_{s, \tau} \) can be identified with \( (Y^{s, \tau})^* \) for all \( s, \tau \geq 0 \). We restate this now proved result as follows.

**Lemma 2.2.** For \( s, \tau \geq 0 \), the dual space of \( Y^{s, \tau} \) is

\[
Y^{s, \tau} = \{ \varphi \in \mathcal{M}_{loc}((0, \infty)) : \eta(dx) := (1 + x^s)^{-1}(1 + x^{-\tau})^{-1}\varphi(dx) \in \mathcal{M}((0, \infty)) \}.
\]

Let us define the space \( \mathscr{S}_{s, \tau} := \{ f \in C^1(0, \infty) \cap Y^{s, \tau} : f' \in L^\infty(0, \infty) \} \). In order to prove that \( S_\nu \) is a \( (C_0) \) group on \( Y^{s, \tau} \), we need the following lemma.

**Lemma 2.3.** The space \( \mathscr{S}_{s, \tau} \) is dense in \( Y^{s, \tau} \) for all \( s, \tau \geq 0 \).

**Proof.** Note that the map

$$f(x) \quad \frac{4}{(1 + x^s)(1 + x^{-\tau})}$$

is an isometric isomorphism from \( Y^{0,0} \) onto \( Y^{s, \tau} \) which leaves invariant \( C_c^\infty(0, \infty) \), the smooth functions with compact support in \((0, \infty)\). Therefore \( C_c^\infty(0, \infty) \) and \( \mathscr{S}_{s, \tau} \) are both dense in \( Y^{s, \tau} \) since \( C_c^\infty(0, \infty) \) is dense in \( Y^{0,0} \). \( \square \)

**Theorem 2.4.** The family \( S_\nu \) forms a \( (C_0) \) group on \( Y^{s, \tau} \) for each \( s \geq 1 \) and \( \tau \geq 0 \).
Proof. First we note that the constant function 1 belongs to $Y^{s,\tau}$ if and only if $s$, $\tau$ are both positive. Next, we observe that for $f \in Y^{s,\tau}$ and $t \in \mathbb{R}$,

$$\|S_\nu(t)f\|_{s,\tau} = \sup_{x > 0} \frac{|f(e^{\nu t}x)|}{(1 + x^s)(1 + x^{-\tau})} = \sup_{y > 0} \frac{|f(y)|}{(1 + [e^{-\nu t}y]^s)(1 + [e^{-\nu t}y]^{-\tau})}.$$

Suppose $tv > 0$. Then

$$\|S_\nu(t)f\|_{s,\tau} \leq e^{\nu ts} \sup_{y > 0} \frac{|f(y)|}{(1 + y^s)(1 + y^{-\tau})} = e^{\nu ts}\|f\|_{s,\tau}.$$

For $tv \leq 0$, we have

$$\|S_\nu(t)f\|_{s,\tau} = \sup_{y > 0} \frac{|f(y)|}{(1 + [e^{-\nu t}y]^s)(1 + [e^{-\nu t}y]^{-\tau})} \leq e^{|\nu t|\tau} \sup_{y > 0} \frac{|f(y)|}{(1 + y^s)(1 + y^{-\tau})} = e^{|\nu t|\tau}\|f\|_{s,\tau}.$$

Thus $S_\nu(t) : Y^{s,\tau} \mapsto Y^{s,\tau}$ and $\|S_\nu(t)\| \leq e^{|\nu|t}$, $\omega = |\nu| \max\{s, \tau\}$.

Thanks to Lemma 2.3 it is enough to show the strong continuity on $\mathcal{S}_{s,\tau}$. In fact, for any $f \in \mathcal{S}_{s,\tau}$, choose $\chi \in C^\infty(0, \infty)$ such that $\chi(x) = 0$ for $0 \leq x \leq 1$, $\chi$ is increasing on $(1, 2)$, and $\chi(x) = 1$ for $x \geq 2$.

Let $f_1 = f\chi$, $f_2 = f(1 - \chi)$. Then $f_1, f_2 \in \mathcal{S}_{s,\tau}$, $\text{supp}f_1 \subset (1, \infty)$, $\text{supp}f_2 \subset (0, 2)$, and $f_1 + f_2 = f$. Now for $f_1$,

$$\|S_\nu(t)f_1 - f_1\|_{s,\tau} = \sup_{x \geq 1} \frac{|f_1(e^{\nu t}x) - f_1(x)|}{(1 + x^s)(1 + x^{-\tau})} \leq \|f'_1\|_\infty \sup_{x \geq 1} \frac{|e^{\nu t}x - x|}{1 + x^s} \leq \|f'_1\|_\infty |e^{\nu t} - 1| \to 0, \quad \text{as} \quad t \to 0^+ \text{ since } s \geq 1.$$

For $f_2$, we have

$$\|S_\nu(t)f_2 - f_2\|_{s,\tau} \leq \|f'_2\|_\infty \sup_{0 < x < 2} \frac{|e^{\nu t}x - x|}{1 + x^{-\tau}} \leq \frac{2^{\tau+1}}{1 + 2\tau}\|f'_2\|_\infty |e^{\nu t} - 1| \to 0,$$

as $t \to 0^+$, and this proves the theorem. \hfill \square

In the sequel we will need the following result, which is proved in [GT] and [deL] Theorem 11.

**Lemma 2.5.** Suppose $iA$ generates a strongly continuous group. Let $p(t) = t^{2n} + q(t)$, where $q$ is a polynomial of degree less than $2n$. Then $-p(A)$ generates a holomorphic ($C_0$) semigroup of angle $\pi/2$.

Take $A = -iD_\nu$, so that $iA$ generates a strongly continuous group on $X = Y^{s,\tau}$ and take $p(t) = t^2 - i\gamma t + r$. Hence we have the following result.

**Theorem 2.6.** The operator $B$ defined in (122) generates a holomorphic ($C_0$) semigroup of angle $\pi/2$ on any $Y^{s,\tau}$, where $s \geq 1, \tau \geq 0$. 
3. The Chaotic Character of the Black-Scholes Semigroup

Let $X$ be a separable complex Banach space.

**Definition 3.1.** A strongly continuous semigroup (or ($C_0$) semigroup) $T = \{T(t) : t \geq 0\}$ of bounded linear operators on $X$ is called **hypercyclic** if there exists a vector $x \in X$ such that its orbit $\{T(t)x : t \geq 0\}$ is dense in $X$, and $T$ is called **chaotic** if in addition the set of periodic points of $T$, $\mathcal{P}_{\text{per}} := \{x \in X : \text{there exists } t_0 > 0 \text{ such that } T(t_0)x = x\}$, is dense in $X$.

The notion of chaotic ($C_0$) semigroups was introduced independently by MacCluer $[\text{McC}]$ and Protopopescu and Azmy $[\text{P-A}]$; the first systematic study of this concept is due to Desch, Schappacher and Webb $[\text{DSW}]$. So far, several specific examples of hypercyclic ($C_0$) semigroups have come up in the literature (see $[\text{GE1}, \text{GE2}]$ for complete citations).

The following lemma is proved by G. Godefroy and J. Shapiro in $[\text{G-S, Corollary 1.5}]$.

**Lemma 3.2.** Suppose $A$ is a linear bounded operator on a Banach space $X$, $Q_1, Q_2$ are dense subsets of $X$ and $Z : Q_1 \to Q_1$ such that

1. $AZy = y$, for all $y \in Q_1$,
2. $\lim_{n \to \infty} Z^n y = 0$, for all $y \in Q_1$ and
3. $\lim_{n \to \infty} A^n w = 0$, for all $w \in Q_2$.

Then $A$ is hypercyclic.

Let $s > 1/\nu$, where $\nu > 0$ is given. Denote by

$$S_s = \{\lambda \in \mathbb{C} : 0 < \text{Re } \lambda < \nu s\}$$

the open strip in $\mathbb{C}$ and let $h_\lambda(x) = x^\lambda$. This function is well-defined in $\mathbb{R}^+$ for any $\lambda \in S_s$.

**Lemma 3.3.** The function $\lambda \mapsto h_\lambda(x)$ is analytic from $S_s$ into $Y^{s, \tau}$ for each $sv > 1$ and $\tau \geq 0$.

**Proof.** Note that when $\tau = 0$, any $\psi \in \mathcal{M}(0, \infty)$ cannot have an atom at 0 and $1 \notin Y^{s,0}$. Now, since weak analyticity is equivalent to analyticity, we have only to prove that

$$\lambda \mapsto \int_{(0, \infty)} h(x, \lambda) \psi(dx)$$

is analytic for any $\psi \in F$, where $F$ is a norm-determining subset of $(Y^{s, \tau})^\ast$. The norm-determining set we use is

$$F := \{c\delta_x : c \in \mathbb{C}, x \in (0, \infty)\},$$

where $\delta_x$ denotes the Dirac point mass measure at $x$. Note that

$$\|f\|_{s, \tau} = \sup \{|c f(x)| = |(f, \psi)| : \psi = c\delta_x, c \in \mathbb{C}, x \in (0, \infty), \|\psi\|_{(Y^{s,\tau})^\ast} = 1\},$$
and a choice of \( c \) that works above is \( c = \frac{1}{(1+x^s)(1+x^{-r})} \), when the supremum defining the norm of \( f \) is a maximum attained at \( x \). Furthermore
\[
\lambda \mapsto x^\lambda = e^{(\ln x)^\lambda} = \langle h_\lambda(x), \delta_x \rangle
\]
is an entire function of \( \lambda \in \mathbb{C} \) for all \( x > 0 \); and for \( \lambda \in S_s \) and \( x > 0 \), \( \lim_{x \to \infty} x^\lambda / (1 + x^s) = 0 \). Hence \( x^\lambda \in Y^{s, \tau} \) for all \( \tau > 0 \). □

If a linear operator \( L \) generates a \((C_0)\) group on a Banach space \( X \), then some polynomials in \( L \) (such as \( L^2 + \alpha L + \beta I \) for arbitrary scalars \( \alpha, \beta \)) generate \((C_0)\) semigroups on \( X \). For the operator \( B \), defined in (1.2), the Black-Scholes semigroup can be represented by \( T(t) := f(D_\nu) \), where
\[
(3.2) \quad f(z) = e^{ig(z)} \quad \text{with} \quad g(z) = z^2 + \gamma z - r.
\]
According to Theorem 2.6 this \((C_0)\) semigroup is well defined for each \( t \in \mathbb{C} \), with Re(\( t \)) > 0. These operators will be shown to be chaotic on \( X = Y^{s, \tau} \) for \( s > 1 \), \( \tau \geq 0 \) when \( s\nu > 1 \). We begin by recalling the following lemma, which was proved in [DSW] and [dL-E], and we reproduce this proof in our case.

**Lemma 3.4.** Suppose that there exists a set \( \Omega \subset S_s \) which has an accumulation point in \( S_s \). Then
\[
Q := \text{Span}\{h_\lambda : \lambda \in \Omega\}
\]
is dense in \( Y^{s, \tau} \) for \( s > 1 \), \( \tau \geq 0 \).

**Proof.** Suppose \( \psi \in Q^\perp \). Since \( \psi \) belongs to the dual of \( Y^{s, \tau} \) and \( h_\lambda = x^\lambda \in Y^{s, \tau} \), Lemma 3.3 asserts that \( p(\lambda) = \langle \psi, x^\lambda \rangle \) is well defined and \( p(\lambda) \) is analytic in \( S_s \). Since \( p(\lambda) = 0 \) for all \( \lambda \in \Omega \), which is a set with an accumulation point, then \( p = 0 \) in all of \( S_s \) and so \( \psi = 0 \), as desired. □

We continue to work in the spaces \( Y^{s, \tau}, s > 1, \tau \geq 0 \) with \( s\nu > 1 \).

**Lemma 3.5.** Let \( \mathbb{D} \) be the unit disk in \( \mathbb{C} \) and \( \mathbb{T} \), the unit circle, be its boundary. The set \( f(S_s) \cap \mathbb{T} \) is nonempty and possesses infinitely many accumulation points in the strip \( S_s \), where \( f \) is as in (3.2).

**Proof.** For \( f(z) = e^{ig(z)} \) with \( t > 0 \), in order to have \( f(S_s) \cap \mathbb{T} \neq \emptyset \) we must find \( z \in S_s \) such that
\[
\text{Re } g(z) = \text{Re}(\nu^2 z^2 + (r - \nu^2)z - r) = \nu^2(x^2 - y_0^2 - x) + rx - r = 0
\]
with \( z = x + iy_0 \). Equivalently, we must find \( (x, y_0) \) with \( 0 < x < \nu s, y_0 \in \mathbb{R} \) such that
\[
(3.3) \quad x^2 + \left( \frac{r}{\nu^2} - 1 \right) x - \frac{r}{\nu^2} = y_0^2.
\]
Call \( C \) the curve represented by the graph of the quadratic function \( y = x^2 + \left( \frac{r}{\nu^2} - 1 \right)x - \frac{r}{\nu^2} \). As Figure 1 shows, for \( 1 < x < \nu s \), there are uncountably many points \((x, y)\) on the dashed portion of \( C \) with \( y > 0 \). For each such point let \( y_0 = \sqrt{y} \). Then this gives uncountably many solutions of (3.3).

\[ \square \]

Now we can prove our main theorem.

**Theorem 3.6.** The Black-Scholes \((C_0)\) semigroup \( T \) is chaotic in \( Y^{s, \tau} \) for each \( s > 1, \tau \geq 0 \) with \( s\nu > 1 \).

**Proof.** First let us prove that the \((C_0)\) semigroup \( T = \{ T(t) = f(D_\nu) = e^{tB} : t \geq 0 \} \) is hypercyclic. For this we will use Lemma 3.2, taking

\[ \Omega_1 = \{ \lambda \in 1/\nu S_{\nu} : |f(\nu\lambda)| > 1 \}, \quad \Omega_2 = \{ \lambda \in 1/\nu S_{\nu} : |f(\nu\lambda)| < 1 \} \]

and

\[ Q_j := \text{Span} \{ h_\lambda : \lambda \in \Omega_j \} \text{ for } j = 1, 2. \]

Now, let \( z_0 \in f(S_{\nu}) \cap \mathbb{T} \), since \( f \) is holomorphic and nonconstant, \( f(S_{\nu}) \) is an open set, and \( \Omega_1 = f(S_{\nu}) \cap \{ z \in \mathbb{C} : |z| > 1 \} \) and \( \Omega_2 = f(S_{\nu}) \cap \{ z \in \mathbb{C} : |z| < 1 \} \) are also open, and any point in \( \Omega_j \) is an accumulation point. So according to Lemma 3.4 \( Q_j \) is dense in \( Y^{s, \tau} \) for \( j = 1, 2 \).

Let \( A = f(D_\nu) \) and define \( Z = (f(D_\nu))^{-1} \) on \( Q_1 \) so that

\[ Z \left( \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \right) = \sum_{k=1}^{N} \alpha_k (f(\nu\lambda_k))^{-1} h_{\lambda_k} \]

for \( \lambda_k \in \Omega_1, \alpha_k \in \mathbb{C} \) and \( N \in \mathbb{N} \). It is clear that for any \( y = \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \in Q_1 \), we have \( AZy = y \). Furthermore for \( \lambda_k \in \Omega_1, |f(\nu\lambda_k)| > 1 \), and consequently

\[ \lim_{n \to \infty} Z^n y = \lim_{n \to \infty} \sum_{k=1}^{N} \alpha_k (f(\nu\lambda_k))^{-n} h_{\lambda_k} = 0. \]
Finally, for \( w = \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \in Q_3 \) with \( |f(\nu \lambda_k)| < 1 \) for each \( k \),
\[
\lim_{n \to \infty} A^n y = \lim_{n \to \infty} \sum_{k=1}^{N} \alpha_k f(\nu \lambda_k)^n h_{\lambda_k}(x) = 0.
\]

These imply that the hypotheses of the Godefroy-Shapiro Lemma 3.2 are satisfied and \( A \) is hypercyclic.

To see that \( T(t) = f(D_v) \) is chaotic, we define \( \Omega_3 = \{ \lambda \in \mathbb{Z} : f(\nu \lambda) \in e^{2\pi i Q} \} \) and \( Q_3 := \text{Span}\{h_{\lambda} : \lambda \in \Omega_3\} \). \( Q_3 \) is contained in the set of all periodic points of \( A = f(D_v) \). Suppose \( f(\nu \lambda_k) = e^{2\pi i n_k} \). Then for \( y = \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \) and \( m = \prod_{k=1}^{N} m_k \), one has \( f(D_v)^m y = y \). So the set of all periodic points \( P_{per} \) of \( f(D_v) \) is dense, and consequently \( T(t) \) is chaotic.

The real-world applications of (FBS) require nonnegative initial data and non-negative solutions. The above proof that the Black-Scholes semigroup \( T \) is chaotic uses holomorphic functions and thus requires the use of spaces of complex-valued functions. Theorem 3.4 would be more satisfying from an applied standpoint if it were valid for real functions. This is precisely the content of the next result.

Let \( Y^{s,\tau}_R \) be the real functions in \( Y^{s,\tau} \). This is a real Banach space. If \( f \in Y^{s,\tau} \), then by [GMR, eq. (17)], the solution of (FBS) is given by
\[
v(x, t) = (T(t)f)(x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-y^2/(4t)} f \left( xe^{(r-\sigma^2/2)t-(\sigma/\sqrt{2})y} \right) dy.
\]

Thus \( T(t)f \) is real (resp., nonnegative) for each \( t \geq 0 \) if and only if \( f \) is real (resp. nonnegative). Let \( S_T \) be the restriction of \( T \) to \( Y^{s,\tau}_R \). Then \( S_T = \{ S_T(t) : t \geq 0 \} \) is a \( (C_0) \) semigroup on \( Y^{s,\tau}_R \) for \( s \geq 1, \tau \geq 0 \).

**Theorem 3.7.** The semigroup \( S_T \) on \( Y^{s,\tau}_R \) is chaotic if \( s > 1 \) and \( \tau \geq 0 \) when \( sv > 1 \).

**Proof.** Let \( f \in Y^{s,\tau} \) be given, where \( s > 1 \) with \( sv > 1 \), and \( \tau \geq 0 \). Let \( g \in Y^{s,\tau} \) have a dense \( T \)-orbit. Then there is a sequence of times \( t_n \to \infty \) such that \( \| T(t_n)g - f \|_{s,\tau} \to 0 \) as \( n \to \infty \). Consequently, since \( \text{Re}(T(t)h) = T(t)(\text{Re}(h)) \) for all \( h \in Y^{s,\tau} \),
\[
\| S_T(t_n)(\text{Re}(g)) - f \|_{s,\tau} \leq \left\| \sqrt{S_T(t_n)(\text{Re}(g)) - f} \right\|^2 + \| S_T(t_n)(\text{Im}(g)) \|_{s,\tau}^2 \leq \| \text{Re}(T(t_n)g - f) + i\| \| \text{Im}(T(t_n)g) \|_{s,\tau} \]
\[
= \| T(t_n)g - f \|_{s,\tau} \to 0
\]
as \( n \to \infty \). It follows that \( S_T \) is hypercyclic.

Next, if \( f \) is periodic of period \( p \), then so are \( \text{Re}(f) \) and \( \text{Im}(f) \). Thus \( S_T \) has a dense set of periodic points since \( T \) does. The theorem follows.

**Acknowledgements**

The three authors thank the referee for very careful and thorough reports, containing many helpful suggestions. This work was started while the second and third authors visited Poitiers, where they received marvelous hospitality. They wish to thank Alain Miranville, Morgan Pierre, Arnaud Rougirel, and the other PDE people at Poitiers for their many kindnesses and support.
References


Laboratoire de Mathématiques, Université de Poitiers, Teleport 2, BP 179, 86960 Chasseneuil du Poitou, Cédex, France
E-mail address: emamirad@math.univ-poitiers.fr

Department of Mathematical Sciences, The University of Memphis, Tennessee 38152
E-mail address: ggoldste@memphis.edu

Department of Mathematical Sciences, The University of Memphis, Tennessee 38152
E-mail address: jgoldste@memphis.edu