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EXTREMELY WEAK INTERPOLATION IN H^∞

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ABSTRACT. Given a sequence of points in the unit disk, a well known result due to Carleson states that if given any point of the sequence it is possible to interpolate the value one in that point and zero in all the other points of the sequence, with uniform control of the norm in the Hardy space of bounded analytic functions on the disk, then the sequence is an interpolating sequence (i.e. every bounded sequence of values can be interpolated by functions in the Hardy space). It turns out that such a result holds in other spaces. In this short paper we would like to show that for a given sequence it is sufficient to find just **one** function suitably interpolating zeros as well as ones to deduce interpolation in the Hardy space. The result has an interesting interpretation in the context of model spaces.

1. INTRODUCTION

The Hardy space H^∞ of bounded analytic functions on $\mathbb{D}$ is equipped with the usual norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. A sequence $\Lambda = \{\lambda_n\}_n \subset \mathbb{D}$ of points in the unit disk is called interpolating for H^∞, noted $\Lambda \in \operatorname{Int} H^\infty$, if every bounded sequence of values $v = (v_n)_n \in l^\infty$ can be interpolated by a function in H^∞. Clearly, for $f \in H^\infty$, the sequence $(f(\lambda_n))_n$ is bounded. Hence

$$\Lambda \in \operatorname{Int} H^\infty \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad H^\infty|\Lambda = l^\infty$$

(we identify the trace space with a sequence space). The sequence Λ is said to satisfy the Blaschke condition if $\sum_n (1-|\lambda_n|) < \infty$. In that case, the Blaschke product $B = \prod_n b_{\lambda_n}$, where $b_\lambda(z) = \frac{|\lambda|}{\lambda}\frac{z-\lambda}{1-\overline{\lambda}z}$ is the normalized Möbius transform ($\lambda \in \mathbb{D}$), converges uniformly on every compact set of $\mathbb{D}$ to a function in H^∞ with boundardy values $|B| = 1$ a.e. on $\mathbb{T}$. Carleson proved (see [Ca58]) that

$$\Lambda \in \operatorname{Int} H^\infty \quad \Longleftrightarrow \quad \inf_n |B_n(\lambda_n)| = \delta > 0,$$

where $B_n = \prod_{k\neq n} b_{\lambda_k}$. The latter condition will be termed the Carleson condition, and we shall write $\Lambda \in (C)$ when Λ satisfies this condition. Carleson's result can be reformulated using the notion of weak interpolation.

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Definition 1.1. A sequence Λ of points in $\mathbb{D}$ is called a weak interpolating sequence in H^∞, noted $\Lambda \in \operatorname{Int}_w H^\infty$, if for every $n \in \mathbb{N}$ there exists a function $\varphi_n \in H^\infty$ such that

- for every $n, k \in \mathbb{N}$, $\varphi_n(\lambda_k) = \delta_{nk}$,
- $\sup_n \|\varphi_n\|_\infty < \infty$.

Now, when $\Lambda \in (C)$, setting $\varphi_n = B_n/B_n(\lambda_n)$ we obtain a family of functions satisfying the conditions of the definition. Hence $\Lambda \in \operatorname{Int}_w H^\infty$. From Carleson's theorem and inner-outer factorization in H^∞ (see [Gar81]) we get

$$\Lambda \in \operatorname{Int} H^\infty \quad \Longleftrightarrow \quad \Lambda \in \operatorname{Int}_w H^\infty.$$

With suitable definitions of interpolating and weak interpolating sequences, such a result has been shown to be true in Hardy spaces H^p (see [ShHSh] for $1 \le p < \infty$ and [Ka63] for $0 < p < 1$) as well as in Bergman spaces (see [SchS98]) and in certain Paley-Wiener and Fock spaces (see [SchS00]).

One also encounters the notion of "dual boundedness" for such sequences (see [Am08]), and in a suitable context it is related to so-called uniform minimality of sequences of reproducing kernels (see e.g. [Nik02, Chapter C3] for some general facts).

Using a theorem by Hoffmann we want to show here that given a separated sequence Λ, there is a splitting of $\Lambda = \Lambda_0 \cup \Lambda_1$ such that if there exists **just one** function $f \in H^\infty$ vanishing on Λ_0 and being 1 on Λ_1, then the sequence is interpolating in H^∞.

The author does not claim that such a result is at all useful to test whether a sequence is interpolating or not but that it might be of some theoretical interest. For instance, in Subsection 2.1, we will discuss an interesting interpretation in terms of splitting of model spaces.

Finally it should be mentioned that it is possible to deduce our main result from [Iz93]. The proof presented here is elementary and based essentially on Hoffman's theorem and the minimum modulus principle.

2. The result

Let us begin by recalling Hoffman's result (which can e.g. be found in Garnett's book, [Gar81]):

Theorem 2.1 (Hoffman). *For $0 < \delta < 1$, there are constants $a = a(\delta)$ and $b = b(\delta)$ such that the Blaschke product $B(z)$ with zero set Λ has a nontrivial factorization $B = B_0 B_1$ and*

$$a|B_0(z)|^{1/b} \le |B_1(z)| \le \frac{1}{a}|B_0(z)|^b$$

for every $z \in \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} D(\lambda, \delta)$, where $D(\lambda, \delta) = \{z \in \mathbb{D} : |b_\lambda(z)| < \delta\}$ is the pseudohyperbolic disk. The factorization does not depend on δ.

In view of this theorem, given any Blaschke sequence of points Λ in the disk and a constant $\delta \in (0,1)$, we will set Λ_0 to be the zero set of B_0 and Λ_1 to be the zero set of B_1 where $B = B_0 B_1$ is a Hoffman factorization of B. We will refer to $\Lambda = \Lambda_0 \cup \Lambda_1$ as a *Hoffman decomposition* of Λ.

Recall that a sequence Λ is separated if there exists a constant $\delta_0 > 0$ such that for every $\lambda, \mu \in \Lambda$, $\lambda \ne \mu$, $|b_\lambda(\mu)| \ge \delta_0$. For such a sequence, we will suppose $\delta = \delta_0/2$.

Theorem 2.2. *A separated sequence Λ in the unit disk with corresponding Hoffman decomposition $\Lambda = \Lambda_0 \cup \Lambda_1$ is interpolating for H^∞ if and only if there exists a function $f \in H^\infty$ such that $f|\Lambda_0 = 0$ and $f|\Lambda_1 = 1$.*

The condition is clearly necessary.

Proof of Theorem. Here is a preliminary observation: By factorization in H^∞ (see e.g. [Gar81]), we have $f = B_0 F$ where F is a bounded analytic function (that could contain inner factors). Then for every $\mu \in \Lambda_1$,

$$1 = f(\mu) = |B_0(\mu)||F(\mu)| \leq c|B_0(\mu)|,$$

which shows that

$$|B_0(\mu)| \geq \eta := 1/c.$$

Replacing f by $g = 1 - f$ we obtain a function vanishing now on Λ_1 and being one on Λ_0. And the same argument as before shows that for $\mu \in \Lambda_0$,

$$|B_1(\mu)| \geq \eta$$

(let us agree to use the same η here).

Now pick $\mu \in \Lambda_1$. Then

$$|B_0(\mu)| \geq \eta.$$

We have to check whether such an estimate also holds for the second piece. Now, let $z \in \partial D(\mu, \delta)$ (note that $\delta = \delta_0/2$, where δ_0 is the separation constant of Λ, so that this disk is far from the other points of Λ). Then by Hoffman's theorem

$$|B_{\Lambda_1}(z)| \geq a|B_0(z)|^{1/b}.$$

Hence

$$|B_{\Lambda_1\setminus\{\mu\}}(z)||b_\mu(z)| \geq a|B_0(z)|^{1/b}$$

and

$$|B_{\Lambda_1\setminus\{\mu\}}(z)| \geq \frac{a}{\delta}|B_0(z)|^{1/b}.$$

Now $B_{\Lambda_1\setminus\{\mu\}}$ and B_0 do not vanish in $D(\mu, \delta)$. We thus can take powers of B_0 and divide through getting a function $B_{\Lambda_1\setminus\{\mu\}}/B_0^{1/b}$ not vanishing in $D(\mu, \delta)$. By the minimum modulus principle we obtain

$$\left|\frac{B_{\Lambda_1\setminus\{\mu\}}(z)}{B_0^{1/b}(z)}\right| \geq \frac{a}{\delta}$$

for every $z \in D(\mu, \delta)$ and especially in $z = \mu$ so that

$$|B_{\Lambda_1\setminus\{\mu\}}(\mu)| \geq \frac{a}{\delta}\eta^{1/b}.$$

Hence

$$|B_{\Lambda\setminus\{\mu\}}(\mu)| = |B_0(\mu)||B_{\Lambda_1\setminus\{\mu\}}(\mu)| \geq \frac{a}{\delta}\eta^{1+1/b}.$$

By the preliminary observation above, the same argument can be carried through when $\mu \in \Lambda_0$, so that for every $\mu \in \Lambda$ we get

$$|B_{\Lambda\setminus\{\mu\}}(\mu)| \geq c$$

for some suitable $c > 0$. Hence $\Lambda \in (C)$ and we are done. ∎

Remark 2.3. 1) It is clear from the proof that it is sufficient that there is an $\eta > 0$ with

$$\inf_{\mu\in\Lambda_1} |B_0(\mu)| \geq \eta \quad \text{and} \quad \inf_{\mu\in\Lambda_0} |B_1(\mu)| \geq \eta. \tag{2.1}$$

This means that in terms of Blaschke products, we need **two** functions instead of the sole function f (which is of course not unique) as stated in Theorem 2.2. One could raise the question whether it would be sufficient to have only one of the conditions in (2.1) (the condition is clearly necessary). Suppose we had the first condition,

$$\inf_{\mu\in\Lambda_1} |B_0(\mu)| \geq \eta.$$

Then in order to obtain the condition of Theorem 2.2, we would need to multiply B_0 by a function $F \in H^\infty$ such that $(B_0F)(\mu) = 1$ for every $\mu \in \Lambda_1$. In other words we need that $B_0 + B_1H^\infty$ is invertible in the quotient algebra H^∞/B_1H^∞ under the condition that $0 < \eta \leq |B_0(\mu)| \leq 1$. This is possible when Λ_1 is a finite union of interpolating sequences in H^∞ (which in our case boils down to interpolating sequences since we have somewhere assumed that Λ, and hence Λ_1, is separated). See for example [Har96] for this, but it can also be deduced from Vasyunin's earlier characterization of the trace of H^∞ on the finite union of interpolating sequences (see [Vas84]).

We do not know the general answer to this invertibility problem when Λ_1 is not assumed to be a finite union of interpolating sequences.

2) Though separation is necessary for interpolation in H^∞, the question could be raised whether in Theorem 2.2 the assumption of being separated can be abandoned. At least Hoffman's theorem does not allow us to deduce that the sequence is separated. As an example, one could have a union of two interpolating sequences, the elements of which come arbitrarily close to each other. Write $\Lambda = \bigcup_n \sigma_n$ where σ_n contains two close points of Λ, one of which is of the first interpolating sequence and the other one from the second interpolating sequence. Let Λ_0 be the union of the even indexed σ_n's and let Λ_1 be the odd indexed σ_n's. We obtain a Hoffman decomposition for which we can find f as in the theorem, but Λ is not interpolating.

2.1. **Model spaces.** Here we will apply our result to show that with the Hoffman decomposition it is sufficient to have boundedness of **one** specified spectral projection in the model space (see definitions below) in order to have uniform boundedness of **all** possible spectral projections.

We begin by introducing the necessary notation. Let H^2 denote the Hardy space on $\mathbb{D}$ and let L^2 denote the classical Lebesgue space on $\mathbb{T}$. H^2 is regarded as a closed subspace of L^2 in the usual way via non-tangential boundary values. For an inner function I, i.e. $I \in H^\infty$ with $|I| = 1$ a.e. on $\mathbb{T}$, we let $K_I = H^2 \ominus IH^2$ be the well-studied model space [Nik86]. We consider the special case when I is a Blaschke product B with simple zeros $\Lambda = \{\lambda_k\}_k$. For a subset $\sigma \subset \Lambda$, the *spectral projection* $\mathcal{E}_\sigma : K_B \longrightarrow K_B$ is the skew projection, if it exists, onto K_{B_σ} parallel to $K_{B_{\Lambda\setminus\sigma}}$. Here, as usual, $B_E = \prod_{\lambda\in E} b_\lambda$ for a set E satisfying the Blaschke condition. It is known (as a special case of [Nik86, Theorem on Unconditional Bases, p. 138]) that if $B = B_\Lambda = \prod_n b_{\lambda_n}$, then $(K_{b_{\lambda_n}})_n$ is an unconditional basis in K_B if and only if $\sup_{\sigma\subset\Lambda} \|\mathcal{E}_\sigma\| < \infty$. And it is well known that $(K_{b_{\lambda_n}})_n$ is an unconditional basis in K_B if and only if $\Lambda \in (C)$ (this is essentially the Shapiro-Shields result; see also [Nik86, p. 188]).

With these remarks in mind, Theorem 2.2 may be read as follows.

Corollary 2.4. *Let Λ be a separated sequence with Hoffman decomposition $\Lambda = \Lambda_1 \cup \Lambda_2$, and let $B = B_0 B_1$ be the corresponding Blaschke product factorization. The spectral projection $\mathcal{E}_{\Lambda_1}$ (or $\mathcal{E}_{\Lambda_2}$) is bounded on K_B if and only if $\sup_{\sigma \subset \Lambda} \|\mathcal{E}_\sigma\| < \infty$.*

Proof. In view of our preliminary observations, the central step in the proof of this corollary is the observation that for $\sigma \subset \Lambda$ the spectral projection $\mathcal{E}_\sigma$ is given by $\mathcal{E}_\sigma = P_I \overline{\varphi_\sigma} : K_I \longrightarrow K_I$, $\mathcal{E}_\sigma f = P_I \overline{\varphi_\sigma} f$, where $\varphi_\sigma \in H^\infty$ satisfies

$$\begin{aligned} &\varphi_\sigma | \sigma = 1, \\ &\varphi_\sigma | (\Lambda \setminus \sigma) = 0. \end{aligned} \tag{2.2}$$

Note that by direct inspection we can check that if $\varphi_\sigma \in H^\infty$ satisfies the latter two conditions, then $\mathcal{E}_\sigma = P_I \overline{\varphi_\sigma}$. If we merely suppose the boundedness of $\mathcal{E}_\sigma$, then the commutant lifting theorem (see [Nik86, p. 191]) applied to $\mathcal{E}_\sigma$ gives the existence of a function $\varphi_\sigma \in H^\infty$ with (2.2) such that $\mathcal{E}_\sigma = P_I \overline{\varphi_\sigma}$. This allows us to deduce the existence of the function f in Theorem 2.2 from the boundedness of $\mathcal{E}_{\Lambda_1}$ (or from that of $\mathcal{E}_{\Lambda_2}$). ∎

Yet another way of stating this result is in terms of angles. If the angle between the two specified subspaces K_{B_0} and K_{B_1} is strictly positive, then all the angles between subspaces K_{B_σ} and $K_{B_{\Lambda \setminus \sigma}}$ are uniformly bounded below by a strictly positive constant.

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