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F-BLOWUPS OF F-REGULAR SURFACE SINGULARITIES

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ABSTRACT. We prove that F-blowups of any F-regular surface singularity coincide with the minimal resolution.

In [Y1], Yasuda introduced the notion of the eth F-blowup, which is a canonical birational modification of varieties in characteristic $p > 0$ defined as a flattening of the direct image of the structure sheaf by the eth iterate of the Frobenius morphism. Yasuda also showed the monotonicity of the sequence of F-blowups for F-pure singularities [Y2]. Furthermore, it turns out that the eth F-blowup $\mathrm{FB}_e(X)$ of a surface singularity X coincides with the minimal resolution for $e \gg 0$ if X is a toric singularity [Y1], a tame quotient singularity [TY] or an F-rational double point [HS].

The above results seem to suggest a connection of F-blowups with F-singularities such as F-purity and F-regularity, which are the concepts from commutative algebra defined via splitting of the Frobenius ring homomorphism in characteristic $p > 0$; see e.g., [HH], [Hu]. Actually, toric and tame quotient singularities are F-regular as well as F-rational double points, while there are non-F-regular surface singularities whose F-blowups do not coincide with the minimal resolution [HS]. Thus it is natural to ask if the F-blowups of F-singularities have particularly nice properties such as the minimal resolution.

In this paper, we give an affirmative answer to the above question:

Theorem 3.1. *Let (X, x) be an F-regular surface singularity defined over an algebraically closed field of characteristic $p > 0$. Then the e-th F-blowup $\mathrm{FB}_e(X)$ of X coincides with the minimal resolution of X for $e \gg 0$.*

To prove the theorem we first give a characterization of F-regular surface singularities in Theorem 2.1: A surface singularity (X, x) in characteristic $p > 0$ is F-regular if and only if its complete local ring $R = \hat{\mathcal{O}}_{X,x}$ is a pure subring of a regular local ring that is module-finite over R. It then follows that if (X, x) is an F-regular surface singularity, then there are only finitely many isomorphism classes of indecomposable reflexive R-modules. Once we have established this property for R (which is referred as "finite representation type"), we can directly use the (strong) F-regularity of R to prove that the ring R^{1/p^e} of p^eth roots of R (or the eth Frobenius direct image $F^e_*\mathcal{O}_X$ of $R = \mathcal{O}_X$) contains all indecomposable reflexive $R = \mathcal{O}_X$-modules as its direct summands for $e \gg 0$ (Theorem 3.3). Thanks to this

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key theorem, we are able to prove Theorem 3.1 by the aid of Wunram's result [W] on the correspondence between "special" reflexive modules and exceptional curves of the minimal resolution.

There are some similarities between F-regular surface singularities in characteristic $p > 0$ and quotient surface singularities in characteristic zero. The local rings of both singularities are pure subrings of a module-finite regular local ring. Also, Corollary 3.6 is considered an analogue of Ishii's result [I] that the G-Hilbert scheme of a complex quotient surface singularity is the minimal resolution. These analogies seem to suggest that for surface singularities, F-regularity is the right notion in characteristic $p > 0$ that corresponds to quotient singularity in characteristic zero.

1. Preliminaries

F-regularity. The notion of F-regularity was first defined in terms of "tight closure" by Hochster and Huneke. To avoid technicality involving tight closure, we will adopt another version of F-regularity called strong F-regularity, which is known to coincide with the one defined via tight closure for $\mathbb{Q}$-Gorenstein rings (and in particular in dimension two). Although it is not known whether or not these two variants of F-regularity coincide in general, we often say just "F-regular" to mean "strongly F-regular," since we mostly work on surface singularities in this paper.

Definition 1.1 ([HH])**.** Let R be an integral domain of characteristic $p > 0$ which is F-finite (i.e., the inclusion map $R \hookrightarrow R^{1/p}$ is module-finite). We say that R is *strongly F-regular* if for every nonzero element $c \in R$, there exists a power $q = p^e$ such that the inclusion map $c^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-module homomorphism.

1.2. We use the following basic properties of (strongly) F-regular rings [HH]:

(1) Regular rings are F-regular.
(2) Pure subrings[1] of an F-regular ring are F-regular.
(3) In particular, if R is a subring of an F-regular ring S such that R is a direct summand of S as an R-module, then R is F-regular.

1.3. *F-regular vs. splinter.* One of the important ring-theoretic properties of F-regularity is that an F-regular domain R of characteristic $p > 0$ is a direct summand as an R-module of every module-finite extension ring; cf. [Hu, Theorem 1.7]. Rings having this property are called *splinters*. It is proved by Singh [Si] that this splinter property characterizes F-regularity for $\mathbb{Q}$-Gorenstein rings in characteristic $p > 0$.

For the reader's convenience, we give a brief proof to an a priori weaker statement which will be used later.

Proposition 1.4. *Let R be an F-finite domain of characteristic $p > 0$. If R is strongly F-regular, then R is a splinter.*

Proof. Let $R \subset S$ be any module-finite extension. Then there exists $\phi \in \mathrm{Hom}_R(S, R)$ such that $c := \phi(1)$ is a nonzero element of R. Then by the strong F-regularity of R, there exist a power $q = p^e$ of p and $\psi \in \mathrm{Hom}_R(R^{1/q}, R)$ such that $\psi(c^{1/q}) = 1$.

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[1]A ring extension $R \subseteq S$ is said to be *pure* if for all R-modules M, the induced R-module homomorphism $M = R \otimes_R M \to S \otimes_R M$ is injective. When S is module-finite over R, this is equivalent to the condition that the map $R \hookrightarrow S$ splits as an R-module homomorphism.

Let $\phi^{1/q} \in \mathrm{Hom}_{R^{1/q}}(S^{1/q}, R^{1/q})$ be the map corresponding to ϕ. Then $\psi \circ \phi^{1/q} \in \mathrm{Hom}_R(S^{1/q}, R)$ gives a splitting of $R \hookrightarrow R^{1/q} \hookrightarrow S^{1/q}$. Since this ring extension factors through S, the extension $R \hookrightarrow S$ also splits, via which R is a direct summand of S as an R-module. □

1.5. *F-regular vs. log terminal* ([H], [HW]). F-regularity is closely related to log terminal singularity. Namely, we have the implication

$$\text{F-regular and } \mathbb{Q}\text{-Gorenstein} \Rightarrow \text{log terminal singularity}$$

in arbitrary characteristic. The converse of this implication also holds in characteristic $p \gg 0$. In dimension two, F-regular rings are always $\mathbb{Q}$-Gorenstein, so that the implication "F-regular $\Rightarrow$ log terminal singularity" holds without assuming the $\mathbb{Q}$-Gorensteinness. Let us discuss the case of surface singularities more in detail.

Let (X, x) be a normal surface singularity defined over an algebraically closed field k and let $\mu\colon \widetilde{X} \to X$ be the minimal resolution with irreducible exceptional curves $E_1, \dots, E_s$. The numerical anti-discrepancy divisor Δ of μ is defined to be the μ-exceptional $\mathbb{Q}$-divisor on $\widetilde{X}$ such that $\Delta E_i = -K_{\tilde{X}} E_i$ for $1 \le i \le s$. Note that Δ is an effective divisor by the minimality of μ. The singularity (X, x) is said to be *log terminal* if the integral part of Δ is zero, i.e., $\lfloor \Delta \rfloor = 0$.[2] Since this is a numerical condition depending only on the intersection matrix $(E_i E_j)_{1 \le i,j \le s}$ of the exceptional curves and their genera, log terminal surface singularities are classified in terms of the weighted dual graph Γ associated to the exceptional set $E = \bigcup_{i=1}^{s} E_i$. In particular, if (X, x) is log terminal, then Γ is a chain or a star-shaped graph with three branches, and each $E_i \cong \mathbb{P}^1$. In the latter case, we associate to Γ a triple (d_1, d_2, d_3) consisting of the absolute values of the determinants of the intersection matrices of the three branches. In this notation, (X, x) is a log terminal singularity if and only if $E_i \cong \mathbb{P}^1$ for $1 \le i \le s$ and either one of the following holds:

- (A) Γ is a chain;
- (D) Γ is star-shaped with $(d_1, d_2, d_3) = (2, 2, n)$, where $n \ge 2$;
- (E_6) Γ is star-shaped with $(d_1, d_2, d_3) = (2, 3, 3)$;
- (E_7) Γ is star-shaped with $(d_1, d_2, d_3) = (2, 3, 4)$;
- (E_8) Γ is star-shaped with $(d_1, d_2, d_3) = (2, 3, 5)$.

The graphs appearing in this classification are exactly the same as the graphs of quotient surface singularities in characteristic zero, but a log terminal surface singularity in characteristic $p > 0$ is not a quotient singularity in general.

The classification of F-regular surface singularities obtained in [H] is as follows: (X, x) is F-regular if and only if the type of the singularity and characteristic $p > 0$ is either one of the following:

(1) type A (possibly smooth), p is arbitrary;
(2) type D, $p \neq 2$;
(3) type E_6 or E_7, $p > 3$;
(4) type E_8, $p > 5$.

In particular, log terminal surface singularities in characteristic $p > 5$ are F-regular.

F-blowups. In what follows, we work over an algebraically closed field k of characteristic $p > 0$. For a variety X over k and an integer $e \ge 0$, let $F^e_{\mathrm{rel}}\colon X^{(e)} \to X$ be

[2] This condition implies that (X, x) is a rational surface singularity from which the $\mathbb{Q}$-Gorensteinness automatically follows.

the e-times iterated k-linear Frobenius morphism. We identify this relative Frobenius with the morphism $X^{1/p^e} := \mathrm{Spec}_X \mathcal{O}_X^{1/p^e} \to X$ induced by the inclusion map $\mathcal{O}_X \hookrightarrow \mathcal{O}_X^{1/p^e}$ into the ring of p^eth roots. (We also abuse the absolute and relative Frobenius, since it is harmless under our assumption that k is algebraically closed.)

Now, the fiber $(F^e_{\mathrm{rel}})^{-1}(x)$ of the e-times iterated Frobenius over a smooth (closed) point $x \in X$ is a fat point of $X^{(e)}$ of length p^{ne}, where $n = \dim X$. Thus, it is regarded as a reduced point of the Hilbert scheme $\mathrm{Hilb}_{p^{ne}}(X^{(e)})$ of zero-dimensional subschemes of $X^{(e)}$ of length p^{ne}.

Definition 1.6 (Yasuda [Y1])**.** Let X be a variety of $\dim X = n$ over k. The eth *F-blowup* $\mathrm{FB}_e(X)$ of X is defined to be the closure of the subset

$$\{(F^e_{\mathrm{rel}})^{-1}(x) \mid x \in X(k) \text{ smooth}\} \subseteq \mathrm{Hilb}_{p^{ne}}(X^{(e)}).$$

It is proved in [Y1] that it is birational and projective over X. Indeed, $\mathrm{FB}_e(X)$ is isomorphic to the irreducible component $\mathrm{Hilb}_{p^{ne}}(X^{(e)}/X)^\circ$ of the relative Hilbert scheme $\mathrm{Hilb}_{p^{ne}}(X^{(e)}/X) \cong \mathrm{Hilb}_{p^{ne}}(X^{1/p^e}/X)$ that dominates X. We denote the associated structure morphism by $\varphi = \varphi_e \colon \mathrm{FB}_e(X) \to X$.

By definition, $\varphi\colon Z = \mathrm{FB}_e(X) \to X$ is a birational projective morphism such that the torsion-free pullback $\varphi^\star \mathcal{O}_X^{1/p^e} := \varphi^* \mathcal{O}_X^{1/p^e}/\text{torsion}$ is a flat (equivalently, locally free) $\mathcal{O}_Z$-algebra of rank p^{ne}, and $\mathrm{FB}_e(X)$ is universal with respect to this property.

Our main concern is the following question in the surface case $n = 2$.

Question. Let (X, x) be a normal surface singularity over k and let $\mu\colon \widetilde{X} \to X$ be the minimal resolution. When is $\mathrm{FB}_e(X)$ equal to the minimal resolution $\widetilde{X}$?

It is proved that $\mathrm{FB}_e(X) = \widetilde{X}$ for $e \gg 0$ if (X, x) is either a toric singularity [Y1], a tame quotient singularity [TY], or an F-regular double point [HS]. We note that $\mathcal{O}_{X,x}$ is F-regular in all of the above three cases. When $X = S/G$ is a tame quotient of smooth S by a finite group G, the essential part is to prove the isomorphism $\mathrm{FB}_e(X) \cong \mathrm{Hilb}^G(S)$ of the F-blowup with the G-Hilbert scheme [Y1], [TY], since G-Hilb of a tame quotient surface singularity is the minimal resolution by a work of Ishii [I]. In general, an F-regular surface singularity in low characteristic $p > 0$ is not a quotient singularity, so the group G and G-Hilb are not available. However, we can use the rationality of F-regular surface singularities.

A few lemmata on rational surface singularities. Throughout the remainder of this section, we assume that (X, x) is a rational surface singularity and $\mu\colon \widetilde{X} \to X$ is any resolution of the singularity.

It is known that if M is a finitely generated reflexive $\mathcal{O}_X$-module, then its torsion-free pullback $\widetilde{M} = \mu^\star M$ is a μ-generated locally free $\mathcal{O}_{\widetilde{X}}$-module such that $\mu_* \widetilde{M} = M$ and $R^1\mu_* \widetilde{M} = 0$; see e.g., [AV]. Note that $R^1\mu_*\mathcal{F} = 0$ for any μ-generated coherent sheaf on $\widetilde{X}$. This is an easy consequence of the rationality of the singularity (X, x) and the μ-generation of $\mathcal{F}$, which gives rise to a surjection $\mathcal{O}_{\widetilde{X}}^{\oplus m} \to \mathcal{F}$.

In particular, since $\mathcal{O}_X^{1/q}$ is a reflexive $\mathcal{O}_X$-module of rank q^2, we have the following.

Lemma 1.7 ([HS, Corollary 4.6])**.** *The e-th F-blowup $\mathrm{FB}_e(X)$ of a rational surface singularity (X, x) is dominated by the minimal resolution for all $e \geq 0$.*

Lemma 1.8. *Let (X, x) be a rational surface singularity and let $\mu\colon \widetilde{X} \to X$ be any resolution of the singularity. If M is a reflexive $\mathcal{O}_X$-module of rank r, then the natural map*

$$\bigwedge^r M \to \mu_*(\det \mu^\star M)$$

is surjective.

Proof. Let $\widetilde{M} = \mu^\star M$. It is sufficient to show that the natural map $M^{\otimes i} \to \mu_*(\bigwedge^i \widetilde{M})$ is surjective for all $i = 1, \ldots, r$. We factorize this map into two maps, $\alpha_i\colon M^{\otimes i} = (\mu_*\widetilde{M})^{\otimes i} \to \mu_*(\widetilde{M}^{\otimes i})$ and $\beta_i\colon \mu_*(\widetilde{M}^{\otimes i}) \to \mu_*(\bigwedge^i \widetilde{M})$, and prove that α_i and β_i are surjective in two steps.

Step 1. We know that α_1 is surjective. Therefore, let $i \geq 2$ and prove the surjectivity of α_i by induction on i. Let $\mathcal{T}$ be the torsion part of the $\mathcal{O}_{\widetilde{X}}$-module f^*M so that $\widetilde{M} = f^*M/\mathcal{T}$. We fix generators $s_1, \ldots, s_m$ of $M = f_*\widetilde{M}$ and consider the associated surjection $\sigma\colon \mathcal{O}_X^{\oplus m} \to M$. Then we have the following two exact sequences:

$$0 \to \mathcal{S} \to \mathcal{O}_{\widetilde{X}}^{\oplus m} \xrightarrow{\mu^\star\sigma} \widetilde{M} \to 0, \tag{1}$$

$$0 \to \mu^\star\mathrm{Ker}(\sigma) \to \mathcal{S} \to \mathcal{T} \to 0. \tag{2}$$

The long exact sequence of $(1) \otimes \widetilde{M}^{\otimes i-1}$ gives

$$\mu_*(\widetilde{M}^{\otimes i-1})^{\oplus m} \xrightarrow{\sigma'} \mu_*(\widetilde{M}^{\otimes i}) \to R^1\mu_*(\mathcal{S} \otimes \widetilde{M}^{\otimes i-1}) \to 0,$$

where the image of the map $\sigma' = \mu_*(\mu^\star\sigma \otimes \widetilde{M}^{\otimes i-1})$ coincides with the image of the map $\alpha_i\colon M^{\otimes i} \to \mu_*(\widetilde{M}^{\otimes i})$ by induction hypothesis. Hence the surjectivity of α_i would follow if $R^1\mu_*(\mathcal{S} \otimes \widetilde{M}^{\otimes i-1}) = 0$. It then suffices to show that $R^1\mu_*\mathcal{T} = 0$, because $\widetilde{M}^{\otimes i-1}$ is μ-generated and $R^1\mu_*(\mathcal{S} \otimes \widetilde{M}^{\otimes i-1}) = R^1\mu_*(\mathcal{T} \otimes \widetilde{M}^{\otimes i-1})$ by (2). To see this we consider the long exact sequence of $0 \to \mathcal{T} \to \mu^*M \to \widetilde{M} \to 0$:

$$\mu_*\mu^*M \to \mu_*\widetilde{M} \to R^1\mu_*\mathcal{T} \to R^1\mu_*\mu^*M.$$

Here the map $\mu_*\mu^*M \to \mu_*\widetilde{M}$ on the left is surjective since the identity map of M is factorized as $M \to \mu_*\mu^*M \to \mu_*\widetilde{M} = M$, and $R^1\mu_*\mu^*M = 0$ since μ^*M is μ-generated. Thus $R^1\mu_*\mathcal{T} = 0$, as required.

Step 2. Let $I_i = \mathrm{Ker}(\widetilde{M}^{\otimes i} \to \bigwedge^i \widetilde{M})$. If I_i is μ-generated, then $R^1\mu_*I_i = 0$ so that the required surjectivity of β_i follows from the exact sequence

$$\mu_*(\widetilde{M}^{\otimes i}) \to \mu_*(\bigwedge^i \widetilde{M}) \to R^1\mu_*I_i = 0.$$

We assume that X is affine and deduce the global generation of I_2 from that of $\widetilde{M}$. Recall that I_2 is generated by local sections of the form $x \otimes x$, where x is any local section of $\widetilde{M}$. We fix global sections $s_1, \ldots, s_m \in M$ that generate $\widetilde{M}$ and write $x = \sum_{i=1}^m a_i s_i$ with local regular functions $a_1, \ldots, a_m$ on $\widetilde{X}$. Then

$$x \otimes x = \sum_{1 \leq i,j \leq m} a_i a_j (s_i \otimes s_j) = \sum_{i=1}^m a_i^2 (s_i \otimes s_i) + \sum_{1 \leq i < j \leq m} a_i a_j (s_i \otimes s_j + s_j \otimes s_i),$$

from which we see that I_2 is generated by its global sections $s_i \otimes s_i$ and $s_i \otimes s_j + s_j \otimes s_i$ for $1 \leq i, j \leq m$.

Now the global generation of I_i for any $i \geq 0$ follows from that of I_2 and $\widetilde{M}$, since $I = \bigoplus_{i\geq 0} I_i$ is the two-sided ideal of the tensor algebra $T(\widetilde{M}) = \bigoplus_{i\geq 0} \widetilde{M}^{\otimes i}$ generated by all local sections $x \otimes x$ of I_2. $\square$

2. Covering of F-regular surface singularities

This section is devoted to proving the following characterization of F-regular surface singularities.

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a two-dimensional complete local ring with algebraically closed coefficient field $k = R/\mathfrak{m}$ of characteristic $p > 0$. Then R is F-regular if and only if there exists a module-finite extension of local rings $R \subset k[[t, u]]$ via which R is a pure subring of the regular local ring $k[[t, u]]$.*

Proof. The sufficiency is clear from subsection 1.2, so we prove the necessity. Let $X = \operatorname{Spec} R$ and let $\mu\colon \widetilde{X} \to X$ be the minimal resolution of the closed point $x = \mathfrak{m} \in X$. If R is F-regular, then (X, x) is a log terminal singularity and either one of cases (1)–(4) in subsection 1.5 occurs. First suppose that $p \neq 2, 3$. Then there exists a finite covering $(Y, y) \to (X, x)$ with (Y, y) smooth or a rational double point by Kawamata [K], while (Y, y) is covered by a smooth surface germ (S, o) by Artin [Ar2]. Hence we have a module-finite extension $R = \mathcal{O}_{X,x} \subset \mathcal{O}_{S,o} = k[[t, u]]$, via which R is a pure subring of $k[[t, u]]$ by Proposition 1.4. Thus the assertion follows in cases (3) and (4) of subsection 1.5. We consider the remaining cases (1) and (2) separately.

Case (1). Assume that (X, x) is a log terminal singularity of type A. In this case it follows that (X, x) is a toric singularity. We shall sketch the proof given in [BHPV, III.5], since it seems less known in characteristic $p > 0$.[3] Let $E = \bigcup_{i=1}^{s} E_i$ be the exceptional set of μ with irreducible components $E_i \cong \mathbb{P}^1$ and set $a_i = -E_i^2 \geq 2$. By the completeness of R we can choose divisors B_0 and B_{s+1} on $\widetilde{X}$ that intersect E transversally so that B_0, B_{s+1} and the E_i's are arranged as follows:

$$B_0 \text{ — } E_1 \text{ — } E_2 \text{ — } \cdots \text{ — } E_r \text{ — } B_{s+1}.$$

Let n, q be the coprime integers with $0 < q < n$ determined by the continued fraction

$$\frac{n}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_s}}}.$$

Using a recursion formula involving a_i's as in [BHPV, III.5], we can find μ-numerically trivial effective Cartier divisors $Z = nB_0 + qE_1 + (\text{terms of } E_2, \ldots, E_s)$ and $Z' = E_1 + (\text{terms of } E_2, \ldots, E_s, B_{s+1})$ on $\widetilde{X}$ such that $\frac{1}{n}(Z + (n-q)Z')$ is Cartier. Since μ-numerical equivalence of Cartier divisors on $\widetilde{X}$ is μ-linear equivalence by the rationality of (X, x) [Ar1], there exist regular functions f, g, h on $\widetilde{X}$ such that $Z = \operatorname{div}_{\widetilde{X}}(f)$, $Z' = \operatorname{div}_{\widetilde{X}}(g)$ and $h^n = fg^{n-q}$. Thus $\widetilde{X}$ is mapped into the hypersurface W defined by $z^n = xy^{n-q}$.

[3] In characteristic zero, this is a consequence of the well-known "tautness" of quotient surface singularities. On the other hand, the argument in [BHPV, III.5] goes through also in characteristic $p > 0$, although it is stated for complex singularities.

Consider the ring homomorphism $k[[x, y]] \to R = H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ sending x, y to f, g, respectively, and let

$$\rho\colon \widetilde{X} \xrightarrow{\mu} X \xrightarrow{\gamma} S = \operatorname{Spec} k[[x, y]]$$

be the corresponding morphisms. Since the divisors $Z = \operatorname{div}_{\widetilde{X}}(f)$ and $Z' = \operatorname{div}_{\widetilde{X}}(g)$ intersect exactly on E, we have $\rho^{-1}(o) = E$. Hence $(f, g) \subset R$ is an ideal of finite colength so that γ is a finite morphism. Since γ factors through the normalization $\widetilde{W}$ of W, which is an n-fold covering of S, we would obtain $X \cong \widetilde{W}$ as soon as we know that $\deg \gamma = n$. To see this, let $C_0 \subset X$ and $L_0 = \operatorname{div}_S(x) \subset S$ be the images of the curve $B_0 \subset \widetilde{X}$, and consider the module-finite extension of discrete valuation rings $\mathcal{O}_{S,L_0} \hookrightarrow \mathcal{O}_{X,C_0} \cong \mathcal{O}_{\widetilde{X},B_0}$, via which the regular parameter x of $\mathcal{O}_{S,L_0}$ maps to an element f of order n in $\mathcal{O}_{\widetilde{X},B_0}$. Also, it induces an isomorphism $k((y)) \cong k((g))$ of the residue fields, since g is part of a regular system of parameters at $B_0 \cap E_1$. Hence $\mathcal{O}_{\widetilde{X},B_0}$ is a free $\mathcal{O}_{S,L_0}$-module of rank n so that $\deg \gamma = n$. Consequently, R is isomorphic to the normalization of $\mathcal{O}_{W,o} = k[[x, y, z]]/(z^n - xy^{n-q}) \cong k[[t^n, u^n, tu^{n-q}]] \subset k[[t, u]]$, from which we obtain a desired extension $R \subset k[[t, u]]$.

Case (2). Assume that (X, x) is a log terminal singularity of type D and $p \neq 2$. Let the exceptional curves $E_1, \dots, E_s$ of μ be indexed as

$$\begin{tikzcd}[column sep=small, row sep=tiny]
E_1 \arrow[dr, dash] & & & & \\
& E_3 \arrow[r, dash] & E_4 \arrow[r, dash] & \cdots \arrow[r, dash] & E_s, \\
E_2 \arrow[ur, dash] & & & &
\end{tikzcd}$$

where $E_1^2 = E_2^2 = -2$. Since R is complete, we can choose effective divisors D_i on $\widetilde{X}$ such that $D_i E_j = \delta_{ij}$. Let $B = E_1 + E_2$ and $D = D_1 + D_2 - D_3$. Then $BE_i = -2DE_i$ for $1 \le i \le s$ so that $B \sim -2D$ since (X, x) is a rational singularity [Ar1]. The map $\mathcal{O}_{\widetilde{X}}(D)^{\otimes 2} \cong \mathcal{O}_{\widetilde{X}}(-B) \hookrightarrow \mathcal{O}_{\widetilde{X}}$ gives rise to an $\mathcal{O}_{\widetilde{X}}$-algebra structure on $\mathcal{A} = \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}(D)$, and we have a double cover $\tilde{\pi}\colon \widetilde{Y} = \operatorname{Spec}_{\widetilde{X}} \mathcal{A} \to \widetilde{X}$ branched over $B = E_1 + E_2$. We note that $\widetilde{Y}$ is smooth since B is smooth, and $\tilde{\pi}$ induces a double cover $\pi\colon (Y, y) \to (X, x)$ that fits in the following commutative diagram:

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "g"] \arrow[d, "\tilde{\pi}"'] & Y \arrow[d, "\pi"] \\
\widetilde{X} \arrow[r, "f"] & X
\end{tikzcd}$$

Here $g\colon \widetilde{Y} \to Y$ is a proper birational morphism, and it is easy to see that its exceptional set $g^{-1}(y) = \bigcup F_j$ consists of smooth rational curves $F_1, \dots, F_{2s-3}$ arranged as follows:

$$\begin{tikzcd}[column sep=small, row sep=tiny]
F_1 \arrow[dr, dash] & & F_4 \arrow[r, dash] & \cdots \arrow[r, dash] & F_s \\
& F_3 \arrow[ur, dash] \arrow[dr, dash] & & & \\
F_2 \arrow[ur, dash] & & F_{s+1} \arrow[r, dash] & \cdots \arrow[r, dash] & F_{2s-3},
\end{tikzcd}$$

where $F_1^2 = F_2^2 = -1$, $F_3^2 = 2E_3^2$ and $F_i^2 = F_{s-3+i}^2 = E_i^2$ for $4 \le i \le s$. Contracting the (-1)-curves F_1 and F_2, we obtain a chain of $2s-5$ rational curves, which contracts to (Y, y). Thus (Y, y) is of type A, which admits a covering $(S, o) \to (Y, y)$ from a smooth surface germ as we have seen in Case (1). Composing this with the double cover π, we obtain a desired covering $(S, o) \to (X, x)$. $\square$

It follows from Theorem 2.1 that a two-dimensional F-regular local ring $R = \mathcal{O}_{X,x}$ has *finite representation type*, that is, there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules. (Note that maximal Cohen–Macaulay modules over a two-dimensional ring are the same as the reflexive modules.)

Corollary 2.2 (cf. [Aus, Proposition 2.1])**.** *Let $(R, \mathfrak{m})$ be a two-dimensional complete local ring with algebraically closed coefficient field $k = R/\mathfrak{m}$ of characteristic $p > 0$. If R is F-regular, then R has finite representation type.*

Proof. By Theorem 2.1, R is a pure subring of a complete regular local ring $k[[x, y]]$ that is module-finite over R. We abuse the notation to denote $S = k[[x, y]]$.

Let M be any indecomposable reflexive R-module and let $M^\vee = \mathrm{Hom}_R(M, R)$ be its R-dual. By the purity of the ring extension $R \subset S$, the monomorphism of R-modules $\mathrm{Hom}_R(M^\vee, R) \to \mathrm{Hom}_R(M^\vee, S)$ splits. Since $\mathrm{Hom}_R(M^\vee, S)$ is a reflexive module over a two-dimensional regular local ring S, it is a free S-module so that $M = \mathrm{Hom}_R(M^\vee, R)$ is a direct summand as an R-module of a finite sum $S^{\oplus n}$. We note that the category of R-modules is a Krull-Schmidt category because R is complete. Hence the fact that M is indecomposable and a direct summand of $S^{\oplus n}$ implies that M is a direct summand of S.

It follows that S is a *full* R-module; that is, every indecomposable reflexive module is isomorphic to a direct summand of the R-module S. Since S is module-finite over R, the conclusion follows. $\square$

Question. We may ask if Theorem 2.1 and Corollary 2.2 remain true in the absence of F-regularity. Let (X, x) be any log terminal surface singularity. Then:

(1) Does there exist a finite covering $S \to X$ from a smooth surface germ (S, o)?
(2) Does $R = \mathcal{O}_{X,x}$ have finite representation type?

It is known that (1) and (2) are affirmative for rational double points [Ar2], [AV].

3. F-blowups of F-regular surface singularities

Throughout this section we work under the following notation: Let (X, x) be a normal surface singularity defined over an algebraically closed field of characteristic $p > 0$. Since our problem is local, we will presumably put $X = \mathrm{Spec}\, R$, where $R = \mathcal{O}_X = \mathcal{O}_{X,x}$.

The purpose of this section is to prove the following theorem conjectured in [HS].

Theorem 3.1. *If (X, x) is an F-regular surface singularity, then its e-th F-blowup $\mathrm{FB}_e(X)$ coincides with the minimal resolution of X for $e \gg 0$.*

This generalizes Proposition 4.9 of [HS], which is proved for (F-)rational double points only. This earlier result is based on an explicit description of the McKay correspondence involving a covering of the rational double point by a smooth surface ([AV], [GSV]). We give here a simpler approach that directly uses strong F-regularity without referring to any explicit description of a covering.

Proof. To begin with, let $\hat{R}$ be the $\mathfrak{m}_x$-adic completion of R and let $\hat{X} = \operatorname{Spec} \hat{R}$. Then the F-regularity of R inherits to $\hat{R}$, and $\mathrm{FB}_e(\hat{X}) \cong \mathrm{FB}_e(X) \times_X \hat{X}$ by [Y1, Proposition 2.11]. Since (X, x) is an isolated singularity, we may assume without loss of generality that R is a complete local ring to prove Theorem 3.1.

Thanks to Corollary 2.2, the F-regularity of R implies that it has finite representation type. Let $M_1, \dots, M_n$ be the (isomorphism classes of) indecomposable reflexive $\mathcal{O}_{X,x}$-modules. $\square$

The next few results may hold in arbitrary dimension. We first show an easy lemma.

Lemma 3.2. *Let R be an integral domain and let $M_1, \dots, M_n$ be finitely generated torsion-free R-modules with* rank $M_i = r_i$. *Then there is a non-zero element $c \in R$ such that for all $i = 1, \dots, n$, the multiplication map by c: $M_i \to M_i$; $m \mapsto cm$, factors through the free module of rank r_i as $M_i \to R^{\oplus r_i} \to M_i$.*

Proof. It is easy to see that we can choose $0 \neq c_i \in R$ such that the multiplication by c_i factors as $M_i \to R^{\oplus r_i} \to M_i$. Then the product $c = c_1 \cdots c_n$ satisfies the required property. $\square$

The following is our key technical result; cf. Corollary 2.2.

Theorem 3.3. *Let $R = \mathcal{O}_X$ be a strongly F-regular complete local ring with only finitely many isomorphism classes of indecomposable reflexive R-modules $M_1, \dots,$ M_n. Then the R-module $R^{1/p^e} \cong F^e_* \mathcal{O}_X$ is full for $e \gg 0$, that is, M_i is isomorphic to a direct summand as an R-module of R^{1/p^e}.*

Proof. Choose c as in Lemma 3.2 and let $M_i^\vee = \operatorname{Hom}_R(M_i, R)$ be the R-dual of M_i. Then, since R is strongly F-regular, there exists $q = p^e$ such that the monomorphism of R-modules

$$M_i = \operatorname{Hom}_R(M_i^\vee, R) \xrightarrow{c^{1/q}} \operatorname{Hom}_R(M_i^\vee, R^{1/q}) \tag{3}$$

splits.[4] Since $\operatorname{Hom}_R(M_i^\vee, R^{1/q}) \cong \operatorname{Hom}_{\mathcal{O}_X}(M_i^\vee, F^e_* \mathcal{O}_X)$ is a reflexive module over $R^{1/q} \cong F^e_* \mathcal{O}_X$, it is isomorphic to a direct sum of copies of $F^e_* M_i$ $(i = 1, \dots, n)$: $\operatorname{Hom}_R(M_i^\vee, R^{1/q}) \cong \bigoplus_{j=1}^n F^e_* M_j^{\oplus a_{ij}}$. Thus by Lemma 3.2, the multiplication map by $c^{1/q} = F^e_*(c)$ on $\operatorname{Hom}_R(M_i^\vee, R^{1/q}) \cong \operatorname{Hom}_{\mathcal{O}_X}(M_i^\vee, F^e_* \mathcal{O}_X)$ factors through the free module $(R^{1/q})^{\oplus r_i} \cong F^e_* \mathcal{O}_X^{\oplus r_i}$, where $r_i = \sum_{j=1}^n a_{ij} r_j$. Hence the monomorphism (3) is factorized as

$$M_i \to \operatorname{Hom}_R(M_i^\vee, R^{1/q}) \to (R^{1/q})^{\oplus r_i} \to \operatorname{Hom}_R(M_i^\vee, R^{1/q}),$$

and it has a splitting as an R-module homomorphism. Thus M_i is isomorphic to a direct summand of $(R^{1/q})^{\oplus r_i}$. This implies that M_i is a direct summand of $R^{1/q}$, since the category of R-modules is a Krull-Schmidt category. $\square$

Corollary 3.4. *In the situation of Theorem 3.3, if we write $R^{1/p^e} = \bigoplus_{i=1}^n M_i^{\oplus a_i^{(e)}}$, then the limit*

$$\lim_{e \to \infty} \frac{a_i^{(e)}}{p^{2e}}$$

exists and is a positive rational number for all $i = 1, \dots, n$.

[4] This monomorphism is exactly equal to the bidual of the map $M_i = M_i \otimes_R R \xrightarrow{c^{1/q}} M_i \otimes_R R^{1/q}$.

Proof. The existence and positivity of the limit follows from Smith–Van den Bergh [SVdB, Proposition 3.3.1] and the rationality from Seibert [Sei, Lemma 2.4]. $\square$

Remark 3.5. (1) We note the difference between Corollary 3.4 and [SVdB, Proposition 3.3.1]. In our argument, $M_1, \ldots, M_n$ represent all isomorphism classes of indecomposable reflexive R-modules, whereas they are just those which appear as a direct summand of R^{1/p^e} for some $e \geq 0$ in [SVdB, Proposition 3.3.1].

(2) When a rational double point R is a pure subring of $S = k[[x, y]]$, if we write $S = \bigoplus_{i=1}^n M_i^{\oplus b_i}$ as an R-module, then the above limit is described as

$$\lim_{e\to\infty} \frac{a_i^{(e)}}{p^{2e}} = \frac{b_i}{r} \in \frac{1}{r}\mathbb{Z},$$

where $r = \sum_{i=1}^s b_i r_i = \mathrm{rank}_R S$; see [HS, Lemma 4.10].

To complete the proof of Theorem 3.1, we introduce some additional notation:
$\mu\colon \widetilde{X} \to X$: the minimal resolution of (X, x) with $\mathrm{Exc}(\mu) = \bigcup_{i=1}^s E_i$,
$\varphi = \varphi_e\colon \mathrm{FB}_e(X) \to X$: the eth F-blowup.

Note that the F-regularity of $R = \mathcal{O}_X$ implies that (X, x) is a rational singularity. Then by Wunram [W, Main Result (a)],[5] part of indecomposable reflexive $\mathcal{O}_X$-modules called *special* reflexives are in one-to-one correspondence with the irreducible exceptional curves $E_1, \ldots, E_s$ of the minimal resolution. We reorder the indecomposable reflexive $\mathcal{O}_X$-modules $M_1, \ldots, M_n$ so that the first s of them, $M_1, \ldots, M_s$ $(s \leq n)$, are special. Then one has

$$c_1(\mu^\star M_i)E_j = \delta_{ij} \tag{4}$$

for $1 \leq i, j \leq s$.

Now, with the aid of Wunram's result, the argument for F-rational double points in [HS, Proposition 4.9] works for F-regular surface singularities as well.

Proof of Theorem 3.1 *continued.* Let the reflexive $\mathcal{O}_X$-module $\mathcal{O}_X^{1/p^e}$ be generated by m elements and pick a surjection $\mathcal{O}_X^{\oplus m} \to \mathcal{O}_X^{1/p^e}$. Since $\mu^\star\mathcal{O}_X^{1/p^e}$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module of rank p^{2e} ([AV]), the induced surjection $\mathcal{O}_{\widetilde{X}}^{\oplus m} \to \mu^\star\mathcal{O}_X^{1/p^e}$ gives rise to a morphism $\Phi_e\colon \widetilde{X} \to \mathbb{G}$ over X to the Grassmannian $\mathbb{G} = \mathrm{Grass}(p^{2e}, \mathcal{O}_X^{\oplus m})$ such that $\mu^\star\mathcal{O}_X^{1/p^e}$ is isomorphic to the pullback of the universal quotient bundle of $\mathbb{G}$.

Similarly, since the torsion-free pullback $\varphi^\star\mathcal{O}_X^{1/p^e}$ to $Z = \mathrm{FB}_e(X)$ is locally free, the surjection $\mathcal{O}_Z^{\oplus m} \to \varphi^\star\mathcal{O}_X^{1/p^e}$ gives rise to a morphism $Z = \mathrm{FB}_e(X) \to \mathbb{G}$ over X, through which Φ_e factors as

$$\Phi_e\colon \widetilde{X} \to \mathrm{FB}_e(X) \to \mathbb{G}$$

by Lemma 1.7. Composing Φ_e with the Plücker embedding $\mathbb{G} \hookrightarrow \mathbb{P}$ into the projective $N = (\binom{m}{p^{2e}} - 1)$-space $\mathbb{P} = \mathbb{P}_X^N$ over X, we obtain a morphism induced by the surjection $\mathcal{O}_{\widetilde{X}}^{\oplus N+1} = \bigwedge^{p^{2e}} \mathcal{O}_{\widetilde{X}}^{\oplus m} \to \det \mu^\star\mathcal{O}_X^{1/p^e}$, which coincides with the morphism

$$\Phi_{|L|}\colon \widetilde{X} \to \mathrm{FB}_e(X) \to \mathbb{P}$$

given by the *complete* linear system associated to the μ-generated line bundle $L = c_1(\mu^\star\mathcal{O}_X^{1/p^e})$ by Lemma 1.8. Now if we write $F_*^e\mathcal{O}_X = \mathcal{O}_X^{\oplus a_0} \oplus \bigoplus_{i=1}^n M_i^{\oplus a_i^{(e)}}$, then

[5] Part (a) of the main results of [W] is verified to be true for rational surface singularities in arbitrary characteristic, without any change of the proof. See also [AV].

the intersection number of L with each exceptional curve E_i is $L \cdot E_i \geq a_i^{(e)}$ by (4), so that L is μ-very ample for $e \gg 0$ by Corollary 3.4. It follows that Φ_e is a closed immersion for $e \gg 0$ so that $\widetilde{X} \cong \mathrm{FB}_e(X)$. □

Corollary 3.6. *Let (X, x) be an F-regular surface singularity and let $\pi\colon (S, o) \to (X, x)$ be a finite covering from a smooth surface germ (S, o) as in Theorem 2.1. Let $\mathrm{Hilb}_d(S/X)^\circ$ be the irreducible component of the relative Hilbert scheme $\mathrm{Hilb}_d(S/X)$ that dominates X, where $d = \deg \pi$. Then*

$$\widetilde{X} \cong \mathrm{Hilb}_d(S/X)^\circ \cong \mathrm{FB}_e(X)$$

for $e \gg 0$.

Proof. Since $\mu^\star \pi_* \mathcal{O}_S$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module of rank d ([AV]), we have a morphism $\widetilde{X} \to \mathrm{Hilb}_d(S/X)^\circ$ over X. On the other hand, since the ring homomorphism $\mathcal{O}_X \to \pi_* \mathcal{O}_S$ splits as an $\mathcal{O}_X$-module homomorphism by Proposition 1.4, we have $\mathrm{Hilb}_d(S/X)^\circ \to \mathrm{FB}_e(X)$ by [HS, Proposition 4.2]. Thus the conclusion follows from Theorem 3.1. □

There exist non-F-regular surface singularities whose F-blowups are not smooth.

Example 3.7 ([HS, Example 4.4])**.** Let $X = \mathrm{Spec}\, k[[x^p, x^p y - x y^p, y^p]]$, where k is a field of characteristic p, and let $g\colon X' \to X$ be the weighted blowup. The exceptional set of g is a single $\mathbb{P}^1$, on which X' has $p+1$ A_{p-1}-singularities. Resolving these A_{p-1}-singularities, we obtain the minimal resolution $f\colon \widetilde{X} \to X$. (So in case $p = 2$, X has a rational double point of type D_4.) It follows that the torsion-free pullback $g^\star \mathcal{O}_X^{1/p^e}$ is a flat $\mathcal{O}_{X'}$-module so that $\mathrm{FB}_e(X)$ is dominated by X' for all $e \geq 0$. Thus the F-blowups of X do not coincide with any resolution of X.

The behavior of F-blowups for non-F-regular surface singularities is a mystery yet. We pose here two extremal questions in opposite directions.

Questions.
1. Let (X, x) be a log terminal surface singularity in characteristic $p > 0$. Does $\mathrm{FB}_e(X)$ coincide with the minimal resolution of X for $e \gg 0$ only if (X, x) is F-regular?
2. Let (X, x) be a normal surface singularity defined over $\mathbb{Q}$. Does the F-blowup of reduction (X_p, x_p) modulo p of (X, x) coincide with the minimal resolution for $p \gg 0$ (or infinitely many p)?

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