F-BLOWUPS OF F-REGULAR SURFACE SINGULARITIES

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Abstract. We prove that F-blowups of any F-regular surface singularity coincide with the minimal resolution.

In [Y1], Yasuda introduced the notion of the $e$th F-blowup, which is a canonical birational modification of varieties in characteristic $p > 0$ defined as a flattening of the direct image of the structure sheaf by the $e$th iterate of the Frobenius morphism. Yasuda also showed the monotonicity of the sequence of F-blowups for F-pure singularities [Y2]. Furthermore, it turns out that the $e$th F-blowup $FB_e(X)$ of a surface singularity $X$ coincides with the minimal resolution for $e \gg 0$ if $X$ is a toric singularity [Y1], a tame quotient singularity [TY] or an F-rational double point [HS].

The above results seem to suggest a connection of F-blowups with F-singularities such as F-purity and F-regularity, which are the concepts from commutative algebra defined via splitting of the Frobenius ring homomorphism in characteristic $p > 0$; see e.g., [HH], [Hu]. Actually, toric and tame quotient singularities are F-regular as well as F-rational double points, while there are non-F-regular surface singularities whose F-blowups do not coincide with the minimal resolution [HS]. Thus it is natural to ask if the F-blowups of F-singularities have particularly nice properties such as the minimal resolution.

In this paper, we give an affirmative answer to the above question:

Theorem 3.1. Let $(X, x)$ be an F-regular surface singularity defined over an algebraically closed field of characteristic $p > 0$. Then the $e$th F-blowup $FB_e(X)$ of $X$ coincides with the minimal resolution of $X$ for $e \gg 0$.

To prove the theorem we first give a characterization of F-regular surface singularities in Theorem 2.1: A surface singularity $(X, x)$ in characteristic $p > 0$ is F-regular if and only if its complete local ring $R = \mathcal{O}_{X, x}$ is a pure subring of a regular local ring that is module-finite over $R$. Actually, toric and tame quotient singularities are F-regular as well as F-rational double points, while there are non-F-regular surface singularities whose F-blowups do not coincide with the minimal resolution [HS]. Thus it is natural to ask if the F-blowups of F-singularities have particularly nice properties such as the minimal resolution.

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To prove the theorem we first give a characterization of F-regular surface singularities in Theorem 2.1: A surface singularity $(X, x)$ in characteristic $p > 0$ is F-regular if and only if its complete local ring $R = \mathcal{O}_{X, x}$ is a pure subring of a regular local ring that is module-finite over $R$. It then follows that if $(X, x)$ is an F-regular surface singularity, then there are only finitely many isomorphism classes of indecomposable reflexive $R$-modules. Once we have established this property for $R$ (which is referred as “finite representation type”), we can directly use the (strong) F-regularity of $R$ to prove that the ring $R^{1/p^e}$ of $p^e$th roots of $R$ (or the $e$th Frobenius direct image $F^e_*\mathcal{O}_X$ of $R = \mathcal{O}_X$) contains all indecomposable reflexive $R = \mathcal{O}_X$-modules as its direct summands for $e \gg 0$ (Theorem 3.3). Thanks to this
key theorem, we are able to prove Theorem 3.1 by the aid of Wunram’s result \[W\] on the correspondence between “special” reflexive modules and exceptional curves of the minimal resolution.

There are some similarities between F-regular surface singularities in characteristic \(p > 0\) and quotient surface singularities in characteristic zero. The local rings of both singularities are pure subrings of a module-finite regular local ring. Also, Corollary 3.6 is considered an analogue of Ishii’s result \[I\] that the G-Hilbert scheme of a complex quotient surface singularity is the minimal resolution. These analogies seem to suggest that for surface singularities, F-regularity is the right notion in characteristic \(p > 0\) that corresponds to quotient singularity in characteristic zero.

1. Preliminaries

F-regularity. The notion of F-regularity was first defined in terms of “tight closure” by Hochster and Huneke. To avoid technicality involving tight closure, we will adopt another version of F-regularity called strong F-regularity, which is known to coincide with the one defined via tight closure for \(\mathbb{Q}\)-Gorenstein rings (and in particular in dimension two). Although it is not known whether or not these two variants of F-regularity coincide in general, we often say just “F-regular” to mean “strongly F-regular,” since we mostly work on surface singularities in this paper.

Definition 1.1 \([HH]\). Let \(R\) be an integral domain of characteristic \(p > 0\) which is F-finite (i.e., the inclusion map \(R \hookrightarrow R^{1/p}\) is module-finite). We say that \(R\) is strongly F-regular if for every nonzero element \(c \in R\), there exists a power \(q = p^e\) such that the inclusion map \(c^{1/q}R \hookrightarrow R^{1/q}\) splits as an \(R\)-module homomorphism.

1.2. We use the following basic properties of (strongly) F-regular rings \([HH]\):

(1) Regular rings are F-regular.

(2) Pure subrings\(^1\) of an F-regular ring are F-regular.

(3) In particular, if \(R\) is a subring of an F-regular ring \(S\) such that \(R\) is a direct summand of \(S\) as an \(R\)-module, then \(R\) is F-regular.

1.3. F-regular vs. splinter. One of the important ring-theoretic properties of F-regularity is that an F-regular domain \(R\) of characteristic \(p > 0\) is a direct summand as an \(R\)-module of every module-finite extension ring; cf. \([HH]\) Theorem 1.7]. Rings having this property are called splinters. It is proved by Singh \([SI]\) that this splinter property characterizes F-regularity for \(\mathbb{Q}\)-Gorenstein rings in characteristic \(p > 0\).

For the reader’s convenience, we give a brief proof to an a priori weaker statement which will be used later.

Proposition 1.4. Let \(R\) be an F-finite domain of characteristic \(p > 0\). If \(R\) is strongly F-regular, then \(R\) is a splinter.

Proof. Let \(R \subseteq S\) be any module-finite extension. Then there exists \(\phi \in \text{Hom}_R(S, R)\) such that \(c := \phi(1)\) is a nonzero element of \(R\). Then by the strong F-regularity of \(R\), there exist a power \(q = p^e\) of \(p\) and \(\psi \in \text{Hom}_R(R^{1/q}, R)\) such that \(\psi(c^{1/q}) = 1\).

\(^1\) A ring extension \(R \subseteq S\) is said to be pure if for all \(R\)-modules \(M\), the induced \(R\)-module homomorphism \(M = R \otimes_R M \to S \otimes_R M\) is injective. When \(S\) is module-finite over \(R\), this is equivalent to the condition that the map \(R \hookrightarrow S\) splits as an \(R\)-module homomorphism.
Let $\phi^{1/q} \in \text{Hom}_{R^{1/q}}(S^{1/q}, R^{1/q})$ be the map corresponding to $\phi$. Then $\psi \circ \phi^{1/q} \in \text{Hom}_R(S^{1/q}, R)$ gives a splitting of $R \twoheadrightarrow R^{1/q} \twoheadrightarrow S^{1/q}$. Since this ring extension factors through $S$, the extension $R \twoheadrightarrow S$ also splits, via which $R$ is a direct summand of $S$ as an $R$-module. \hfill $\square$

1.5. F-regular vs. log terminal (H, HW). F-regularity is closely related to log terminal singularity. Namely, we have the implication

F-regular and $\mathbb{Q}$-Gorenstein $\Rightarrow$ log terminal singularity

in arbitrary characteristic. The converse of this implication also holds in characteristic $p \gg 0$. In dimension two, F-regular rings are always $\mathbb{Q}$-Gorenstein, so that the implication “F-regular $\Rightarrow$ log terminal singularity” holds without assuming the $\mathbb{Q}$-Gorensteinness. Let us discuss the case of surface singularities more in detail.

Let $(X, x)$ be a normal surface singularity defined over an algebraically closed field $k$ and let $\mu: \tilde{X} \to X$ be the minimal resolution with irreducible exceptional curves $E_1, \ldots, E_s$. The numerical anti-discrepancy divisor $\Delta$ of $\mu$ is defined to be the $\mu$-exceptional $\mathbb{Q}$-divisor on $\tilde{X}$ such that $\Delta E_i = -K_{\tilde{X}} E_i$ for $1 \leq i \leq s$. Note that $\Delta$ is an effective divisor by the minimality of $(\mu)$. The singularity $(X, x)$ is said to be log terminal if the integral part of $\Delta$ is zero, i.e., $|\Delta| = 0$. Since this is a numerical condition depending only on the intersection matrix $(E_i E_j)_{1 \leq i,j \leq s}$ of the exceptional curves and their genera, log terminal surface singularities are classified in terms of the weighted dual graph $\Gamma$ associated to the exceptional set $E = \bigcup_{i=1}^s E_i$. In particular, if $(X, x)$ is log terminal, then $\Gamma$ is a chain or a star-shaped graph with three branches, and each $E_i \cong \mathbb{P}^1$. In the latter case, we associate to $\Gamma$ a triple $(d_1, d_2, d_3)$ consisting of the absolute values of the determinants of the intersection matrices of the three branches. In this notation, $(X, x)$ is a log terminal singularity if and only if $E_i \cong \mathbb{P}^1$ for $1 \leq i \leq s$ and either one of the following holds:

(A) $\Gamma$ is a chain;
(D) $\Gamma$ is star-shaped with $(d_1, d_2, d_3) = (2, 2, n)$, where $n \geq 2$;
(E_6) $\Gamma$ is star-shaped with $(d_1, d_2, d_3) = (2, 3, 3)$;
(E_7) $\Gamma$ is star-shaped with $(d_1, d_2, d_3) = (2, 3, 4)$;
(E_8) $\Gamma$ is star-shaped with $(d_1, d_2, d_3) = (2, 3, 5)$.

The graphs appearing in this classification are exactly the same as the graphs of quotient surface singularities in characteristic zero, but a log terminal surface singularity in characteristic $p > 0$ is not a quotient singularity in general.

The classification of F-regular surface singularities obtained in [H] is as follows: $(X, x)$ is F-regular if and only if the type of the singularity and characteristic $p > 0$ is either one of the following:

(1) type $A$ (possibly smooth), $p$ is arbitrary;
(2) type $D$, $p \neq 2$;
(3) type $E_6$ or $E_7$, $p > 3$;
(4) type $E_8$, $p > 5$.

In particular, log terminal surface singularities in characteristic $p > 5$ are F-regular.

F-blowups. In what follows, we work over an algebraically closed field $k$ of characteristic $p > 0$. For a variety $X$ over $k$ and an integer $e \geq 0$, let $F_e: X^{(e)} \to X$ be

\footnote{This condition implies that $(X, x)$ is a rational surface singularity from which the $\mathbb{Q}$-Gorensteinness automatically follows.}
the $e$-times iterated $k$-linear Frobenius morphism. We identify this relative Frobenius with the morphism $X^{1/p^e} := \text{Spec}O_X^{1/p^e} \to X$ induced by the inclusion map $O_X \to O_X^{1/p^e}$ into the ring of $p^e$th roots. (We also abuse the absolute and relative Frobenius, since it is harmless under our assumption that $k$ is algebraically closed.)

Now, the fiber $(F_{\text{rel}}^e)^{-1}(x)$ of the $e$-times iterated Frobenius over a smooth (closed) point $x \in X$ is a fat point of $X^{(e)}$ of length $p^{ne}$, where $n = \dim X$. Thus, it is regarded as a reduced point of the Hilbert scheme $\text{Hilb}_{p^{ne}}(X^{(e)})$ of zero-dimensional subschemes of $X^{(e)}$ of length $p^{ne}$.

**Definition 1.6** (Yasuda [Y1]). Let $X$ be a variety of dim $X = n$ over $k$. The $e$th $F$-blowup $\text{FB}_e(X)$ of $X$ is defined to be the closure of the subset

$$\{(F_{\text{rel}}^e)^{-1}(x) \mid x \in X(k) \text{ smooth}\} \subseteq \text{Hilb}_{p^{ne}}(X^{(e)}).$$

It is proved in [Y1] that it is birational and projective over $X$. Indeed, $\text{FB}_e(X)$ is isomorphic to the irreducible component $\text{Hilb}_{p^{ne}}(X^{(e)}/X)^0$ of the relative Hilbert scheme $\text{Hilb}_{p^{ne}}(X^{(e)}/X) \cong \text{Hilb}_{p^{ne}}(X^{1/p^e}/X)$ that dominates $X$. We denote the associated structure morphism by $\varphi = \varphi_e: \text{FB}_e(X) \to X$.

By definition, $\varphi: Z = \text{FB}_e(X) \to X$ is a birational projective morphism such that the torsion-free pullback $\varphi^*O_X^{1/p^e} := \varphi^*O_X^{1/p^e}/\text{torsion}$ is a flat (equivalently, locally free) $O_Z$-algebra of rank $p^{ne}$, and $\text{FB}_e(X)$ is universal with respect to this property.

Our main concern is the following question in the surface case $n = 2$.

**Question.** Let $(X, x)$ be a normal surface singularity over $k$ and let $\mu: \tilde{X} \to X$ be the minimal resolution. When is $\text{FB}_e(X)$ equal to the minimal resolution $\tilde{X}$?

It is proved that $\text{FB}_e(X) = \tilde{X}$ for $e \gg 0$ if $(X, x)$ is either a toric singularity [TY], a tame quotient singularity [TY], or an $F$-regular double point [HS]. We note that $O_{X,x}$ is $F$-regular in all of the above three cases. When $X = S/G$ is a tame quotient of smooth $S$ by a finite group $G$, the essential part is to prove the isomorphism $\text{FB}_e(X) \cong \text{Hilb}^G(S)$ of the $F$-blowup with the $G$-Hilbert scheme [TY], since $G$-Hilb of a tame quotient surface singularity is the minimal resolution by a work of Ishii [I]. In general, an $F$-regular surface singularity in low characteristic $p > 0$ is not a quotient singularity, so the group $G$ and $G$-Hilb are not available. However, we can use the rationality of $F$-regular surface singularities.

**A few lemmata on rational surface singularities.** Throughout the remainder of this section, we assume that $(X, x)$ is a rational surface singularity and $\mu: \tilde{X} \to X$ is any resolution of the singularity.

It is known that if $M$ is a finitely generated reflexive $O_X$-module, then its torsion-free pullback $\tilde{M} = \mu^*M$ is a $\mu$-generated locally free $O_{\tilde{X}}$-module such that $\mu_*\tilde{M} = M$ and $R^1\mu_*\tilde{M} = 0$; see e.g., [AV]. Note that $R^1\mu_*\mathcal{F} = 0$ for any $\mu$-generated coherent sheaf on $\tilde{X}$. This is an easy consequence of the rationality of the singularity $(X, x)$ and the $\mu$-generation of $\mathcal{F}$, which gives rise to a surjection $O_{\tilde{X}}^{\oplus m} \to \mathcal{F}$.

In particular, since $O_X^{1/q}$ is a reflexive $O_X$-module of rank $q^2$, we have the following.

**Lemma 1.7** ([HS Corollary 4.6]). The $e$-th $F$-blowup $\text{FB}_e(X)$ of a rational surface singularity $(X, x)$ is dominated by the minimal resolution for all $e \geq 0$. 

Lemma 1.8. Let \((X, x)\) be a rational surface singularity and let \(\mu : \tilde{X} \to X\) be any resolution of the singularity. If \(M\) is a reflexive \(\mathcal{O}_X\)-module of rank \(r\), then the natural map

\[
\bigwedge^r M \to \mu_*(\det \mu^* M)
\]

is surjective.

Proof. Let \(\tilde{M} = \mu^* M\). It is sufficient to show that the natural map \(M^\otimes i \to \mu_*(\bigwedge^i \tilde{M})\) is surjective for all \(i = 1, \ldots, r\). We factorize this map into two maps, \(\alpha_i : M^\otimes i = (\mu_* M)^\otimes i \to \mu_*(\bigwedge^i \tilde{M})\) and \(\beta_i : \mu_*(\bigwedge^i \tilde{M}) \to \mu_*(\bigwedge^i M)\), and prove that \(\alpha_i\) and \(\beta_i\) are surjective in two steps.

Step 1. We know that \(\alpha_1\) is surjective. Therefore, let \(i \geq 2\) and prove the surjectivity of \(\alpha_i\) by induction on \(i\). Let \(T\) be the torsion part of the \(\mathcal{O}_{\tilde{X}}\)-module \(f^* M\) so that \(\tilde{M} = f^* M/T\). We fix generators \(s_1, \ldots, s_m\) of \(M = f_* \tilde{M}\) and consider the associated surjection \(\sigma : \mathcal{O}_{\tilde{X}}^\oplus m \to M\). Then we have the following two exact sequences:

\[
\begin{align*}
(1) \quad & 0 \to S \to \mathcal{O}_{\tilde{X}}^\oplus m \xrightarrow{\mu^* \sigma} \tilde{M} \to 0, \\
(2) \quad & 0 \to \mu^* \text{Ker}(\sigma) \to S \to T \to 0.
\end{align*}
\]

The long exact sequence of \((1) \otimes \tilde{M}^{\otimes i-1}\) gives

\[
\mu_*(\tilde{M}^{\otimes i-1})^{\otimes m} \xrightarrow{\mu^*(\tilde{M}^{\otimes i})} R^1 \mu_*(S \otimes \tilde{M}^{\otimes i-1}) \to 0,
\]

where the image of the map \(\mu^*(\tilde{M}^{\otimes i}) \to R^1 \mu_*(S \otimes \tilde{M}^{\otimes i-1})\) coincides with the image of the map \(\alpha_i : M^\otimes i \to \mu_*(\tilde{M}^{\otimes i})\) by induction hypothesis. Hence the surjectivity of \(\alpha_i\) would follow if \(R^1 \mu_*(S \otimes \tilde{M}^{\otimes i-1}) = 0\). It then suffices to show that \(R^1 \mu_* T = 0\), because \(\tilde{M}^{\otimes i-1}\) is \(\mu^\bullet\)-generated and \(R^1 \mu_*(S \otimes \tilde{M}^{\otimes i-1}) = R^1 \mu_*(T \otimes \tilde{M}^{\otimes i-1})\) by \((2)\).

To see this we consider the long exact sequence of \(0 \to T \to \mu^* M \to \tilde{M} \to 0\):

\[
\mu_* \mu^* M \to \mu_* \tilde{M} \to R^1 \mu_* T \to R^1 \mu_* \mu^* M.
\]

Here the map \(\mu_* \mu^* M \to \mu_* \tilde{M}\) on the left is surjective since the identity map of \(M\) is factorized as \(M \to \mu_* \mu^* M \to \mu_* \tilde{M} = M\), and \(R^1 \mu_* \mu^* M = 0\) since \(\mu^* M\) is \(\mu^\bullet\)-generated. Thus \(R^1 \mu_* T = 0\), as required.

Step 2. Let \(I_i = \text{Ker}(\tilde{M}^{\otimes i} \to \bigwedge^i \tilde{M})\). If \(I_i\) is \(\mu^\bullet\)-generated, then \(R^1 \mu_* I_i = 0\) so that the required surjectivity of \(\beta_i\) follows from the exact sequence

\[
\mu_*(\tilde{M}^{\otimes i}) \to \mu_*(\bigwedge^i \tilde{M}) \to R^1 \mu_* I_i = 0.
\]

We assume that \(X\) is affine and deduce the global generation of \(I_2\) from that of \(\tilde{M}\). Recall that \(I_2\) is generated by local sections of the form \(x \otimes x\), where \(x\) is any local section of \(M\). We fix global sections \(s_1, \ldots, s_m \in M\) that generate \(\tilde{M}\) and write \(x = \sum_{i=1}^m a_i s_i\) with local regular functions \(a_1, \ldots, a_m\) on \(\tilde{X}\). Then

\[
x \otimes x = \sum_{1 \leq i, j \leq m} a_i a_j (s_i \otimes s_j) = \sum_{i=1}^m a_i^2 (s_i \otimes s_i) + \sum_{1 \leq i < j \leq m} a_i a_j (s_i \otimes s_j + s_j \otimes s_i),
\]

from which we see that \(I_2\) is generated by its global sections \(s_i \otimes s_i\) and \(s_i \otimes s_j + s_j \otimes s_i\) for \(1 \leq i, j \leq m\).
Now the global generation of $I_i$ for any $i \geq 0$ follows from that of $I_2$ and $\tilde{M}$, since $I = \bigoplus_{i \geq 0} I_i$ is the two-sided ideal of the tensor algebra $T(\tilde{M}) = \bigoplus_{i \geq 0} \tilde{M}^\otimes i$ generated by all local sections $x \otimes x$ of $I_2$.

2. Covering of F-regular surface singularities

This section is devoted to proving the following characterization of F-regular surface singularities.

**Theorem 2.1.** Let $(R,m)$ be a two-dimensional complete local ring with algebraically closed coefficient field $k = R/m$ of characteristic $p > 0$. Then $R$ is F-regular if and only if there exists a module-finite extension of local rings $R \subset k[[t,u]]$ via which $R$ is a pure subring of the regular local ring $k[[t,u]]$.

**Proof.** The sufficiency is clear from subsection 1.2, so we prove the necessity. Let $X = \text{Spec } R$ and let $\mu: \tilde{X} \to X$ be the minimal resolution of the closed point $x = m \in X$. If $R$ is F-regular, then $(X,x)$ is a log terminal singularity and either one of cases (1)–(4) in subsection 1.5 occurs. First suppose that $p \neq 2,3$. Then there exists a finite covering $(Y,y) \to (X,x)$ with $(Y,y)$ smooth or a rational double point by Kawamata [K], while $(Y,y)$ is covered by a smooth surface germ $(S,a)$ by Artin [Ar2]. Hence we have a module-finite extension $R = \mathcal{O}_{X,x} \subset \mathcal{O}_{S,a} = k[[t,u]]$, via which $R$ is a pure subring of $k[[t,u]]$ by Proposition 1.4. Thus the assertion follows in cases (3) and (4) of subsection 1.5. We consider the remaining cases (1) and (2) separately.

Case (1). Assume that $(X,x)$ is a log terminal singularity of type $A$. In this case it follows that $(X,x)$ is a toric singularity. We shall sketch the proof given in [BHPV III.5], since it seems less known in characteristic $p > 0$.

Let $E = \bigcup_{i=1}^r E_i$ be the exceptional set of $\mu$ with irreducible components $E_i \cong \mathbb{P}^1$ and set $a_i = -E_i^2 \geq 2$. By the completeness of $R$ we can choose divisors $B_0$ and $B_{s+1}$ on $\tilde{X}$ that intersect $E$ transversally so that $B_0, B_{s+1}$ and the $E_i$’s are arranged as follows:

$$B_0 - E_1 - E_2 - \cdots - E_r - B_{s+1}.$$  

Let $n,q$ be the coprime integers with $0 < q < n$ determined by the continued fraction

$$\frac{n}{q} = a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_s}}}.$$  

Using a recursion formula involving $a_i$’s as in [BHPV III.5], we can find $\mu$-numerically trivial effective Cartier divisors $Z = nB_0 + qE_1 + (\text{terms of } E_2, \ldots, E_s)$ and $Z' = E_1 + (\text{terms of } E_2, \ldots, E_s, B_{s+1})$ on $\tilde{X}$ such that $\frac{1}{n}(Z + (n-q)Z')$ is Cartier. Since $\mu$-numerical equivalence of Cartier divisors on $\tilde{X}$ is $\mu$-linear equivalence by the rationality of $(X,x)$ [Ar1], there exist regular functions $f,g,h$ on $\tilde{X}$ such that $Z = \text{div} \varphi(f)$, $Z' = \text{div} \varphi(g)$ and $h^n = fg^{n-q}$. Thus $\tilde{X}$ is mapped into the hypersurface $W$ defined by $z^n = xy^{n-q}$.

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3In characteristic zero, this is a consequence of the well-known “tautness” of quotient surface singularities. On the other hand, the argument in [BHPV III.5] goes through also in characteristic $p > 0$, although it is stated for complex singularities.
Consider the ring homomorphism \( k[[x, y]] \to R = H^0(\overline{X}, \mathcal{O}_{\overline{X}}) \) sending \( x, y \) to \( f, g \), respectively, and let

\[
\rho: \overline{X} \xrightarrow{\mu} X \xrightarrow{\gamma} S = \text{Spec } k[[x, y]]
\]

be the corresponding morphisms. Since the divisors \( Z = \text{div}_X(f) \) and \( Z' = \text{div}_X(g) \) intersect exactly on \( E \), we have \( \rho^{-1}(o) = E \). Hence \( (f, g) \subset R \) is an ideal of finite colength so that \( \gamma \) is a finite morphism. Since \( \gamma \) factors through the normalization \( \overline{W} \) of \( W \), which is an \( n \)-fold covering of \( S \), we would obtain \( X \cong \overline{W} \) as soon as we know that \( \deg \gamma = n \). To see this, let \( C_0 \subset X \) and \( L_0 = \text{div}_S(x) \subset S \) be the images of the curve \( B_0 \subset \overline{X} \), and consider the module-finite extension of discrete valuation rings \( \mathcal{O}_{S,L_0} \hookrightarrow \mathcal{O}_{X,C_0} \cong \mathcal{O}_{\overline{X},B_0} \) via which the regular parameter \( x \) of \( \mathcal{O}_{S,L_0} \) maps to an element \( f \) of order \( n \) in \( \mathcal{O}_{\overline{X},B_0} \). Also, it induces an isomorphism \( k((y)) \cong k((g)) \) of the residue fields, since \( g \) is part of a regular system of parameters at \( B_0 \cap E_1 \). Hence \( \mathcal{O}_{\overline{X},B_0} \) is a free \( \mathcal{O}_{S,L_0} \)-module of rank \( n \) so that \( \deg \gamma = n \). Consequently, \( R \) is isomorphic to the normalization of \( \mathcal{O}_{W,o} = k[[x, y, z]]/(z^n - xy^{n-q}) \cong k[[t^n, u^n, tu^{n-q}]] \subset k[[t, u]] \), from which we obtain a desired extension \( R \subset k[[t, u]] \).

**Case (2).** Assume that \((X, x)\) is a log terminal singularity of type \( D \) and \( p \neq 2 \). Let the exceptional curves \( E_1, \ldots, E_s \) of \( \mu \) be indexed as

\[
\begin{array}{c c c c c c c}
E_1 & E_2 & E_3 & E_4 & \cdots & E_s, \\
\end{array}
\]

where \( E_1^2 = E_2^2 = -2 \). Since \( R \) is complete, we can choose effective divisors \( D_i \) on \( \overline{X} \) such that \( D_i E_j = \delta_{ij} \). Let \( B = E_1 + E_2 \) and \( D = D_1 + D_2 - D_3 \). Then \( BE_1 = -2DE_i \) for \( 1 \leq i \leq s \) so that \( B \sim -2D \) since \((X, x)\) is a rational singularity [Ar1]. The map \( \mathcal{O}_{\overline{X}}(D)^{\otimes 2} \cong \mathcal{O}_{\overline{X}}(-B) \hookrightarrow \mathcal{O}_{\overline{X}} \) gives rise to an \( \mathcal{O}_{\overline{X}} \)-algebra structure on \( A = \mathcal{O}_{\overline{X}} \oplus \mathcal{O}_{\overline{X}}(D) \), and we have a double cover \( \overline{\pi}: \overline{Y} = \text{Spec}_{\overline{X}} A \to \overline{X} \) branched over \( B = E_1 + E_2 \). We note that \( \overline{Y} \) is smooth since \( B \) is smooth, and \( \overline{\pi} \) induces a double cover \( \pi: (Y, y) \to (X, x) \) that fits in the following commutative diagram:

\[
\begin{array}{c c c c c c c}
\overline{Y} & \xrightarrow{g} & Y \\
\overline{\pi} \downarrow & & \downarrow \pi \\
\overline{X} & \xrightarrow{f} & X \\
\end{array}
\]

Here \( g: \overline{Y} \to Y \) is a proper birational morphism, and it is easy to see that its exceptional set \( g^{-1}(y) = \bigcup F_j \) consists of smooth rational curves \( F_1, \ldots, F_{2s-3} \) arranged as follows:

\[
\begin{array}{c c c c c c c}
F_1 & F_2 & F_3 & F_4 & \cdots & F_s \\
\end{array}
\]

\[
\begin{array}{c c c c c c c}
F_2 & F_3 & F_4 & \cdots & F_{s+1} & \cdots & F_{2s-3}, \\
\end{array}
\]
where \( F_2^2 = F_2^3 = -1, F_3^2 = 2E_2^2 \) and \( F_s^2 = F_{s-3}^2 = E_i^2 \) for \( 4 \leq i \leq s \). Contracting the (-1)-curves \( F_1 \) and \( F_2 \), we obtain a chain of \( 2s - 5 \) rational curves, which contracts to \( (Y, y) \). Thus \( (Y, y) \) is of type \( A \), which admits a covering \( (S, o) \rightarrow (Y, y) \) from a smooth surface germ as we have seen in Case (1). Composing this with the double cover \( \pi \), we obtain a desired covering \( (S, o) \rightarrow (X, x) \). \( \Box \)

It follows from Theorem 2.1 that a two-dimensional F-regular local ring \( R = \mathcal{O}_{X, x} \) has finite representation type, that is, there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay \( R \)-modules. (Note that maximal Cohen–Macaulay modules over a two-dimensional ring are the same as the reflexive modules.)

**Corollary 2.2** (cf. [Aus, Proposition 2.1]). Let \( (R, m) \) be a two-dimensional complete local ring with algebraically closed coefficient field \( k = R/m \) of characteristic \( p > 0 \). If \( R \) is F-regular, then \( R \) has finite representation type.

**Proof.** By Theorem 2.1, \( R \) is a pure subring of a complete regular local ring \( k[[x, y]] \) that is module-finite over \( R \). We abuse the notation to denote \( S = k[[x, y]] \).

Let \( M \) be any indecomposable reflexive \( R \)-module and let \( M' = \text{Hom}_R(M, R) \) be its \( R \)-dual. By the purity of the ring extension \( R \subset S \), the monomorphisms of \( R \)-modules \( \text{Hom}_R(M', R) \rightarrow \text{Hom}_R(M', S) \) splits. Since \( \text{Hom}_R(M', S) \) is a reflexive module over a two-dimensional regular local ring \( S \), it is a free \( S \)-module so that \( M = \text{Hom}_R(M', R) \) is a direct summand as an \( R \)-module of a finite sum \( S \oplus n \).

We note that the category of \( R \)-modules is a Krull-Schmidt category because \( R \) is complete. Hence the fact that \( M \) is indecomposable and a direct summand of \( S \oplus n \) implies that \( M \) is a direct summand of \( S \).

It follows that \( S \) is a full \( R \)-module; that is, every indecomposable reflexive module is isomorphic to a direct summand of the \( R \)-module \( S \). Since \( S \) is module-finite over \( R \), the conclusion follows. \( \Box \)

**Question.** We may ask if Theorem 2.1 and Corollary 2.2 remain true in the absence of F-regularity. Let \( (X, x) \) be any log terminal surface singularity. Then:

1. Does there exist a finite covering \( S \rightarrow X \) from a smooth surface germ \( (S, o) \)?
2. Does \( R = \mathcal{O}_{X, x} \) have finite representation type?

It is known that (1) and (2) are affirmative for rational double points [Ar2], [AV].

### 3. F-blowups of F-regular surface singularities

Throughout this section we work under the following notation: Let \( (X, x) \) be a normal surface singularity defined over an algebraically closed field of characteristic \( p > 0 \). Since our problem is local, we will presumably put \( X = \text{Spec} R \), where \( R = \mathcal{O}_X = \mathcal{O}_{X, x} \).

The purpose of this section is to prove the following theorem conjectured in [HS].

**Theorem 3.1.** If \( (X, x) \) is an F-regular surface singularity, then its \( e \)-th F-blowup \( \text{FB}_e(X) \) coincides with the minimal resolution of \( X \) for \( e \gg 0 \).

This generalizes Proposition 4.9 of [HS], which is proved for (F-)rational double points only. This earlier result is based on an explicit description of the McKay correspondence involving a covering of the rational double point by a smooth surface ([AV], [GSV]). We give here a simpler approach that directly uses strong F-regularity without referring to any explicit description of a covering.
Proof. To begin with, let \( \hat{R} \) be the \( m_\ast \)-adic completion of \( R \) and let \( \hat{X} = \text{Spec} \hat{R} \). Then the F-regularity of \( R \) inherits to \( \hat{R} \), and \( \text{FB}_{\bullet} (\hat{X}) \cong \text{FB}_{\bullet} (X) \times_X \hat{X} \) by [Y1 Proposition 2.11]. Since \((X, x)\) is an isolated singularity, we may assume without loss of generality that \( R \) is a complete local ring to prove Theorem 3.3.

Thanks to Corollary 2.2, the F-regularity of \( R \) implies that it has finite representation type. Let \( M_1, \ldots, M_n \) be the (isomorphism classes of) indecomposable reflexive \( O_{X,x} \)-modules.

The next few results may hold in arbitrary dimension. We first show an easy lemma.

**Lemma 3.2.** Let \( R \) be an integral domain and let \( M_1, \ldots, M_n \) be finitely generated torsion-free \( R \)-modules with rank \( M_i = r_i \). Then there is a non-zero element \( c \in R \) such that for all \( i = 1, \ldots, n \), the multiplication map by \( c \): \( M_i \to M_i; \ m \mapsto cm \), factors through the free module of rank \( r_i \) as \( M_i \to R^{\oplus r_i} \to M_i \).

**Proof.** It is easy to see that we can choose \( 0 \neq c_i \in R \) such that the multiplication by \( c_i \) factors as \( M_i \to R^{\oplus r_i} \to M_i \). Then the product \( c = c_1 \cdots c_n \) satisfies the required property. \( \square \)

The following is our key technical result; cf. Corollary 2.2.

**Theorem 3.3.** Let \( R = O_X \) be a strongly F-regular complete local ring with only finitely many isomorphism classes of indecomposable reflexive \( R \)-modules \( M_1, \ldots, M_n \). Then the \( R \)-module \( R^{1/p^e} \cong F_*^e O_X \) is full for \( e \gg 0 \), that is, \( M_i \) is isomorphic to a direct summand as an \( R \)-module of \( R^{1/p^e} \).

**Proof.** Choose \( c \) as in Lemma 3.2 and let \( M_i^\vee = \text{Hom}_R (M_i, R) \) be the \( R \)-dual of \( M_i \). Then, since \( R \) is strongly F-regular, there exists \( q = p^e \) such that the monomorphism of \( R \)-modules

\[
M_i = \text{Hom}_R (M_i^\vee, R) \overset{c^i_{1/q}}\longrightarrow \text{Hom}_R (M_i^\vee, R^{1/q})
\]

splits. Since \( \text{Hom}_R (M_i^\vee, R^{1/q}) \cong \text{Hom}_{O_X} (M_i^\vee, F_*^e O_X) \) is a reflexive module over \( R^{1/q} \cong F_*^e O_X \), it is isomorphic to a direct sum of copies of \( F_*^e M_i \) \((i = 1, \ldots, n)\): \( \text{Hom}_R (M_i^\vee, R^{1/q}) \cong \bigoplus_{j=1}^n F_*^e M_j^{\oplus r_j} \). Thus by Lemma 3.2, the multiplication map by \( c^i_{1/q} = F_*^e (c) \) on \( \text{Hom}_R (M_i^\vee, R^{1/q}) \cong \text{Hom}_{O_X} (M_i^\vee, F_*^e O_X) \) factors through the free module \( (R^{1/q})^{\oplus r_i} \cong F_*^e O_X^{\oplus r_i} \), where \( r_i = \sum_{j=1}^n a_{ij} r_j \). Hence the monomorphism (3) is factorized as

\[
M_i \to \text{Hom}_R (M_i^\vee, R^{1/q}) \to (R^{1/q})^{\oplus r_i} \to \text{Hom}_R (M_i^\vee, R^{1/q}),
\]

and it has a splitting as an \( R \)-module homomorphism. Thus \( M_i \) is isomorphic to a direct summand of \( (R^{1/q})^{\oplus r_i} \). This implies that \( M_i \) is a direct summand of \( R^{1/q} \), since the category of \( R \)-modules is a Krull-Schmidt category. \( \square \)

**Corollary 3.4.** In the situation of Theorem 3.3, if we write \( R^{1/p^e} = \bigoplus_{i=1}^n M_i^{\oplus a_i^{(e)}} \),

then the limit

\[
\lim_{e \to \infty} \frac{a_i^{(e)}}{p^{e}}
\]

exists and is a positive rational number for all \( i = 1, \ldots, n \).

---

4This monomorphism is exactly equal to the bidual of the map \( M_i \cong M_i \otimes_R R^{1/q} \to M_i \otimes_R R^{1/q} \).
Proof. The existence and positivity of the limit follows from Smith–Van den Bergh [SVdB Proposition 3.3.1] and the rationality from Seibert [Sc3 Lemma 2.4]. □

Remark 3.5. (1) We note the difference between Corollary 3.4 and [SVdB Proposition 3.3.1]. In our argument, $M_1, \ldots, M_n$ represent all isomorphism classes of indecomposable reflexive $R$-modules, whereas they are just those which appear as a direct summand of $R^{1/p^e}$ for some $e \geq 0$ in [SVdB Proposition 3.3.1].

(2) When a rational double point $R$ is a pure subring of $S = k[[x, y]]$, if we write $S = \bigoplus_{i=1}^n M_i^{\oplus b_i}$ as an $R$-module, then the above limit is described as

$$\lim_{e \to \infty} \frac{a_i^{(e)}}{p^{2e}} = \frac{b_i}{r} \in \frac{1}{r} \mathbb{Z},$$

where $r = \sum_{i=1}^s b_ir_i = \text{rank}_R S$; see [HS Lemma 4.10].

To complete the proof of Theorem 3.1 we introduce some additional notation:

$\mu: \bar{X} \to X$: the minimal resolution of $(X, x)$ with $\text{Exc} (\mu) = \bigcup_{i=1}^s E_i$;

$\varphi = \varphi_\mu: FB_\mu(X) \to X$: the $\mu$-F-blowup.

Note that the F-regularity of $R = \mathcal{O}_X$ implies that $(X, x)$ is a rational singularity. Then by Wunram [W, Main Result (a)] part of indecomposable reflexive $\mathcal{O}_X$-modules called special reflexives are in one-to-one correspondence with the irreducible exceptional curves $E_1, \ldots, E_s$ of the minimal resolution. We reorder the indecomposable reflexive $\mathcal{O}_X$-modules $M_1, \ldots, M_n$ so that the first $s$ of them, $M_1, \ldots, M_s$ ($s \leq n$), are special. Then one has

$$c_1(\mu^* M_i) E_j = \delta_{ij}$$

for $1 \leq i, j \leq s$.

Now, with the aid of Wunram’s result, the argument for F-rational double points in [HS Proposition 4.9] works for F-regular surface singularities as well.

Proof of Theorem 3.1 continued. Let the reflexive $\mathcal{O}_X$-module $\mathcal{O}_X^{1/p^r}$ be generated by $m$ elements and pick a surjection $\mathcal{O}_X^{\oplus m} \to \mathcal{O}_X^{1/p^r}$. Since $\mu^* \mathcal{O}_X^{1/p^r}$ is a locally free $\mathcal{O}_X$-module of rank $p^{2e}$ (AV), the induced surjection $\mathcal{O}_X^{\oplus m} \to \mu^* \mathcal{O}_X^{1/p^r}$ gives rise to a morphism $\Phi_\mu: \bar{X} \to \mathbb{G}$ over $X$ to the Grassmannian $\mathbb{G} = \text{Grass}(p^{2e}, \mathcal{O}_X^{\oplus m})$ such that $\mu^* \mathcal{O}_X^{1/p^r}$ is isomorphic to the pullback of the universal quotient bundle of $\mathbb{G}$.

Similarly, since the torsion-free pullback $\varphi^* \mathcal{O}_X^{1/p^r}$ to $Z = FB_\mu(X)$ is locally free, the surjection $\mathcal{O}_Z^{\oplus m} \to \varphi^* \mathcal{O}_X^{1/p^r}$ gives rise to a morphism $Z = FB_\mu(X) \to \mathbb{G}$ over $X$, through which $\Phi_\mu$ factors as

$$\Phi_\mu: \bar{X} \to FB_\mu(X) \to \mathbb{G}$$

by Lemma 1.7. Composing $\Phi_\mu$ with the Plücker embedding $\mathbb{G} \hookrightarrow \mathbb{P}$ into the projective $N = (\binom{m}{p^{2e}} - 1)$-space $\mathbb{P} = \mathbb{P}^{N}$ over $X$, we obtain a morphism induced by the surjection $\mathcal{O}_X^{N+1} = \bigwedge^{p^{2e}} \mathcal{O}_{\bar{X}}^{\oplus m} \to \det \mu^* \mathcal{O}_X^{1/p^r}$, which coincides with the morphism

$$\Phi_{|L_1|}: \bar{X} \to FB_\mu(X) \to \mathbb{P}$$

given by the complete linear system associated to the $\mu$-generated line bundle $L = c_1(\mu^* \mathcal{O}_X^{1/p^r})$ by Lemma 1.8. Now if we write $F^*_\mu \mathcal{O}_X = \mathcal{O}_X^{\oplus m} \oplus \bigoplus_{i=1}^n M_i^{\oplus s_i}$, then

\footnotesize

\begin{itemize}
\item Part (a) of the main results of [AV] is verified to be true for rational surface singularities in arbitrary characteristic, without any change of the proof. See also [AV].
\end{itemize}
the intersection number of $L$ with each exceptional curve $E_i$ is $L \cdot E_i \geq a_{1e}$ by (4), so that $L$ is $\mu$-very ample for $e \gg 0$ by Corollary 3.4. It follows that $\Phi_e$ is a closed immersion for $e \gg 0$ so that $\tilde{X} \cong \text{FB}_e(X)$. \hfill \square

**Corollary 3.6.** Let $(X, x)$ be an F-regular surface singularity and let $\pi: (S, o) \rightarrow (X, x)$ be a finite covering from a smooth surface germ $(S, o)$ as in Theorem 2.1. Let $\text{Hilb}_d(S/X)^o$ be the irreducible component of the relative Hilbert scheme $\text{Hilb}_d(S/X)$ that dominates $X$, where $d = \deg \pi$. Then

$$\tilde{X} \cong \text{Hilb}_d(S/X)^o \cong \text{FB}_e(X)$$

for $e \gg 0$.

**Proof.** Since $\mu^* \pi_* \mathcal{O}_S$ is a locally free $\mathcal{O}_\tilde{X}$-module of rank $d$ ([AV]), we have a morphism $\tilde{X} \rightarrow \text{Hilb}_d(S/X)^o$ over $X$. On the other hand, since the ring homomorphism $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_S$ splits as an $\mathcal{O}_X$-module homomorphism by Proposition 1.4, we have $\text{Hilb}_d(S/X)^o \rightarrow \text{FB}_e(X)$ by [HS] Proposition 4.2. Thus the conclusion follows from Theorem 3.1. \hfill \square

There exist non-F-regular surface singularities whose F-blowups are not smooth.

**Example 3.7 ([HS] Example 4.4).** Let $X = \text{Spec } k[[x^p, x^p y - xy^p, y^p]]$, where $k$ is a field of characteristic $p$, and let $g: X' \rightarrow X$ be the weighted blowup. The exceptional set of $g$ is a single $P^1$, on which $X'$ has $p+1 A_{p-1}$-singularities. Resolving these $A_{p-1}$-singularities, we obtain the minimal resolution $f: \tilde{X} \rightarrow X$. (So in case $p = 2$, $\tilde{X}$ has a rational double point of type $D_4$.) It follows that the torsion-free pullback $g^* \mathcal{O}_X^{1/p^e}$ is a flat $\mathcal{O}_{X'}$-module so that $\text{FB}_e(X)$ is dominated by $X'$ for all $e \geq 0$. Thus the F-blowups of $X$ do not coincide with any resolution of $X$.

The behavior of F-blowups for non-F-regular surface singularities is a mystery yet. We pose here two extremal questions in opposite directions.

**Questions.**

1. Let $(X, x)$ be a log terminal surface singularity in characteristic $p > 0$. Does $\text{FB}_d(X)$ coincide with the minimal resolution of $X$ for $e \gg 0$ only if $(X, x)$ is F-regular?

2. Let $(X, x)$ be a normal surface singularity defined over $\mathbb{Q}$. Does the F-blowup of reduction $(X_p, x_p)$ modulo $p$ of $(X, x)$ coincide with the minimal resolution for $p \gg 0$ (or infinitely many $p$)?

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**References**


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